The Mathematics Consortium

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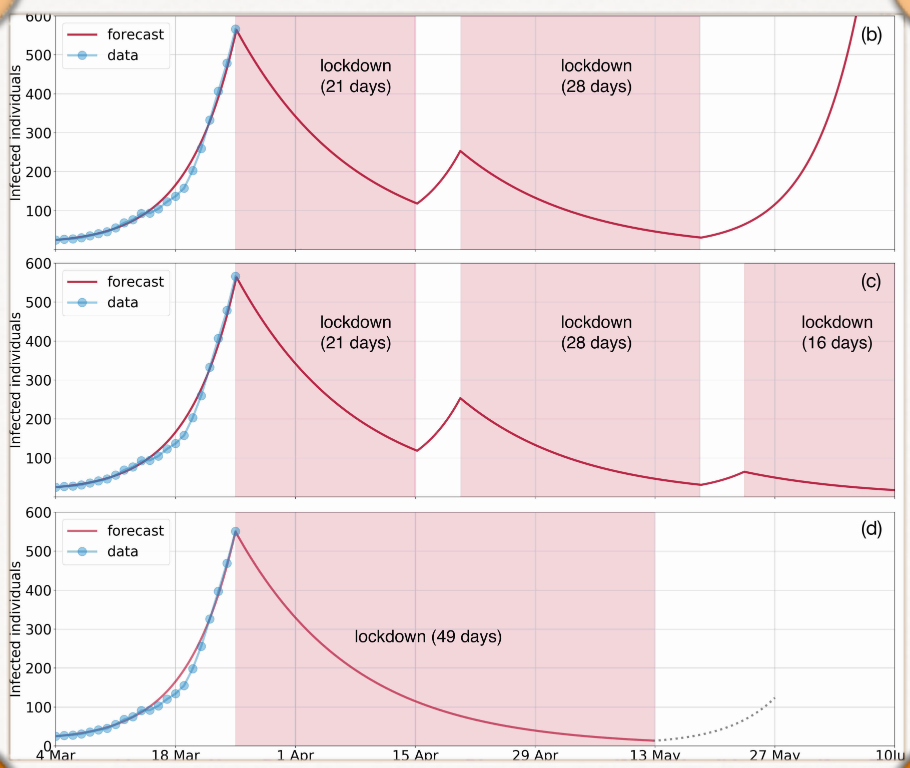
COVID-19 Response

SIR Model with effectiveness of Lockdown



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Contents

About the cover-page: The figure contains a part of the figure 4 in the following article: Rajesh Singh and R. Adhikari, Age-structured impact of social distancing on the COVID-19 epidemic in India, pre-print featured on the COVID-19 research community page in March 2020. (Updates available at *https://github.com/rajeshrinet/pyross*). We Thank the authors for giving consent to reuse the figure in this issue.

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From Editors' Desk

Before three months, who could have imagined that COVID-19 would become a global pandemic straining health system in many countries? It made us realize that, uncertainty is a part and parcel of human life. Such uncertainties can arise in any magnitude and are completely unpredictable. Currently, the world is struggling to contain the Corona virus infection. Lakhs of people have been already infected with the virus and about 75000 people have already died. Every country is trying to fight this menace in its own way. In the absence of a COVID-19 vaccine, people are trying to find the ways of preventing, or at least delaying possible corona-attacks. For example, Ayurveda recommends drinking a lot of warm water after every 3-4 hours, and intake of turmeric, ginger, garlic etc. in warm water, or tea, or milk, or honey, as immunity boosters. However, most common and effective approach is reducing the contact rates in the population so as to reduce the transmission of the deadly corona virus. Implementation of such measures is really challenging and has long term implications on the economy of the nations. Hence, critical evaluation of effectiveness of such measures is essential for their justification.

Both these tasks are difficult due to inadequate theoretical knowledge of this virus and one has to look for data driven inference for forecasting of corona virus infections. In this connection, the article entitled "Nonstandard statistical distributions: Perspectives and future scopes" by Prof. Muralidharan and Dr. Pratima is most relevant from statistical modeling and prediction perspective. Like many examples stated in the article, in this context also, one can contemplate an example on infected patient's life time data as a non-standard probability model. Really, there is a lot of scope for this kind of study.

Government also realizes the importance of developing mathematical models to study the rate of spread of COVID-19 among the population. Keeping this in view, recently, Science and Engineering Research Board of GOI, has announced one year projects in relevant areas, with a fixed grant of Rs. 5 lakh plus overheads, under its MATRICS program. Researchers are requested to keep a watch on similar announcements in future.

In the first article, Prof. Anupam Singh and Dr. Yash Arora are dealing with the question of short presentations of finite simple groups. This way of defining groups is useful in solving various combinatorial problems for such groups, using excellent mathematical packages such as GAP, MAGMA, SAGEmath etc. The authors give an account of several results on two-generation and (2,3)-generation of finite simple groups and their impact on the subject of computational group theory.

In the third article, Prof. Inder K. Rana discusses about the Maharashtra Government's successful project for training the Mathematics teachers at secondary level with a view to improve the quality of mathematics teaching and learning.

Dr. D. V. Shah writes about important events which occurred in the Mathematics world during last three months and pays tribute to those Mathematicians who left us in recent past. Specifically, Prof. M. H. Vasavada pays tribute to one of the leading Mathematicians from Gujarat, late Prof. Subhash J. Bhatt.

We specially congratulate Prof. A. Raghuram from IISER, Pune, for publishing a monograph jointly with Prof. Günter Harder in Annals of Mathematics studies. We appreciate that Dr. Jagnnath Nagorao Salunke, has worked out the correct solution to the problem which was posed in our previous issue and that Mr. Dhruv Bhasin, an Integrated Ph. D. student at IISER Pune, has posed a new problem in this issue. In the end Prof. Katre gives an account of TMC activities in 2019-20 and Dr. Ramesh Kasilingum gives calendar of Academic events which are planned during June-August, 2020.

We are really happy to bring out the fourth issue of Volume 1 of the Bulletin in April, 2020. We thank all the authors who have contributed articles for this issue, all the editors, our designers Mrs. Prajkta Holkar and Dr. R. D. Holkar and all those who have directly or indirectly helped us in bringing out this issue on time.

Chief Editor, TMC bulletin.

1. Two-generation of finite simple groups

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Abstract. This expository article revolves around the question of finding short presentations of finite simple groups. This subject is one of the most active research areas of group theory in recent times. We bring together several known results on two-generation and (2, 3)-generation of finite simple groups and their impact on the subject of computational group theory.

1.1 Introduction

The groups usually arise from symmetries of an object. One of the ways groups arise naturally is while studying topological invariants, e.g., the fundamental group, homotopy groups etc. The groups, in this situation, naturally come with generators and relations. Often the groups are realised as a quotient of certain infinite groups, namely the free groups. This gives rise to a completely new way of looking at groups (as opposed to the definition and examples given to us in our undergraduate classes), and is studied under the subject of combinatorial group theory and geometric group theory. This is quite different from the usual notion of groups, where we know all of the elements, and how to multiply them.

In the modern times, this way of defining groups has gained more importance due to the subject of computational group theory (see [19]). Due to advancements in computational power, it is natural to expect that one can use computers to solve various mathematical problems. There are excellent mathematical packages such as GAP, MAGMA, SAGEmath, to name a few, where one can work with groups. Usually several well known groups, for example, symmetric groups, finite simple groups, matrix groups, etc. are implemented in these packages. Each of these packages allows one to do further computations within those groups. Thus from the point of view of implementation, it is not efficient to define a group on the computer using all of its elements along with its multiplication table. Thus, a "small" presentation is a better way to implement a group. This brings us to two different points of view of looking at these packages: how to develop better algorithms to implement these groups, and another, how to use these packages to work with these groups and compute further within them. In this article we keep our focus on the family of finite simple groups, more specifically, the finite classical groups.

Definition 1 *A non-trivial group G is said to be simple if it has no non-trivial proper normal subgroup, i.e., the only normal subgroup of G are $\{e\}$ and G.*

While studying finite groups it is natural to ask if we can classify all finite simple groups. The answer is in affirmative and we briefly recall the classification of finite simple groups (CFSG), and refer the reader to an excellent book by Wilson [41] on this subject. The classification of finite simple group is one of the main achievements of the last century. We all should aspire to know at least the statement. The finite simple groups FSG (see [41] page 3 for details) are broadly in four families:

FSG1: Cyclic groups of prime order $\mathbb{Z}/p\mathbb{Z}$ (these are the only Abelian groups).

FSG2: Alternating groups A_n for $n \geq 5$.

FSG3: Finite groups of Lie type: Classical types and exceptional types.

FSG3a: Classical groups of Lie type:

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A_l type: The projective special linear groups, $PSL_{l+1}(q)$, for $l \geq 1$ except $PSL_2(2)$ and $PSL_2(3)$.

B_l type: The projective quotient of the commutator of orthogonal groups, $P\Omega_{2l+1}(q)$, for $l \geq 3$ and q odd.

C_l type: The projective symplectic groups, $PSp_{2l}(q)$, for $l \geq 2$ except $PSp_4(2)$.

D_l type: The projective quotient of commutator of orthogonal groups, $P\Omega^+_{2l}(q)$, for $l \geq 4$.

${}^2A_l$ type: The projective special unitary groups, $PSU_{l+1}(q)$, for $l \geq 2$ except $PSU_3(2)$.

${}^2D_l$ type: The projective quotient of commutator of orthogonal groups, $P\Omega^-_{2l}(q)$, for $l \geq 4$.

FSG3b: Exceptional groups of Lie type:

(i) $G_2(q), q \geq 3$; $F_4(q)$; $E_6(q)$; $E_7(q)$; $E_8(q)$; ${}^2E_6(q)$; ${}^3D_4(q)$.

(ii) ${}^2B_2(2^{2n+1}), n \geq 1$; ${}^2G_2(3^{2n+1}), n \geq 1$; ${}^2F_4(2^{2n+1}), n \geq 1$; ${}^2F_4(2)'$.

FSG4: The 26 sporadic groups named as follows:

Mathieu groups: $M_{11}, M_{12}, M_{22}, M_{23}, M_{24}$;

Leech lattice groups: $Co_1, Co_2, Co_3, McL, HS, Suz, J_2$;

Fischer groups: $Fi_{22}, Fi_{23}, Fi'_{24}$;

Monster groups: $\mathbb{M}, \mathbb{B}, Th, HN, He$;

Pariahs: $J_1, J_2, J_3, O'N, Ly, Ru$.

In all of the above q is a prime power p^a, which indicates the size of underlying finite field F_q. There is a very small number of duplication in the above list.

In this article, by simple groups we always mean non-abelian simple groups. We begin with the basic idea of groups given by generators and relations. A broad general problem is to determine which of the (finite) groups are generated by 2 of its elements. Such groups would be a quotient of the free group on two generators $\mathcal{F}_2$. However, this could be a wild problem, thus one restricts to understand which of the finite simple groups are two-generated. This has been very well studied over the years, and reasonably good answers are known. While working with finite simple groups, it was noted that many of these groups can be generated by two elements, one of order 2 and another of order 3. Thus, the question is to further determine, which of the finite simple groups are $(2,3)$-generated, i.e., generated by an element of order 2 and another of order 3. This problem can be also thought of as determining the quotients of the free product group $C_2 \star C_3$. In this article we briefly present some of the work done in this direction.

Acknowledgments

The authors would like to thank Professor B. Sury for his lecture on this topic in the "Workshop on Group Theory 2019" held at IISER Pune. We also thank Dr Uday Bhaskar Sharma and Professor Marco Antonio Pellegrini for their feedback on this article. We thank the referee(s) for suggestions which improved the readability of this paper.

1.2 Generators and relations for a group

Let G be a group.

Definition 2 *A presentation of the group G is*

$$G = \langle S \mid R \rangle, \tag{1.1}$$

where S is a subset of G which generates G, and R is a set of words on S called relations, i.e., $G \cong F(S)/N(R)$ where $F(S)$ is the free group on S and $N(R)$ is the normal subgroup of $F(S)$ generated by the set of relations R.

The presentation 1.1 is said to be a finite presentation if both S and R are finite. In this paper, we discuss only finite presentations, even though the groups may be finite or infinite. Let us begin with recalling some examples of presentations.

Example 1.2.1 *The symmetric group S_n has a Coxeter presentation given by,*

$$S_n = \langle s_1, s_2, \ldots, s_{n-1} \mid s_i^2, (s_i s_{i+1})^3, (s_i s_j)^2, \ 1 \leq i < j \leq n-1, |i-j| \geq 2 \rangle.$$

Here we can identify s_i with the transposition $(i, i+1)$ to get the isomorphism.

The reflection groups are defined to be certain subgroups of the orthogonal group $O_n(\mathbb{R})$ generated by some order 2 elements. There is a more general theory of Coxeter groups, and we refer an interested reader to the book by Humphreys [20] on this subject.

Example 1.2.2 $S_n = \langle (1,2), (1,2,\cdots,n) \rangle$. *Thus, S_n is a quotient of $C_2 \star C_n$. In fact, the presentation with respect to this generating set is as follows:*

$$S_n = \left\langle x, y \mid x^2, y^n, (xy)^{n-1}, (xy^{-1}xy)^3, (xy^{-j}xy^j)^2, 2 \leq j \leq \lfloor n/2 \rfloor \right\rangle.$$

Example 1.2.3 (Dihedral group) *The group of symmetries of a regular n-gon in the plane, is the Dihedral group D_n with $2n$ elements. Its presentations are as follows:*

$$D_n = \langle r, s \mid r^n, s^2, (sr)^2 \rangle = \langle s_1, s_2 \mid s_1^2, s_2^2, (s_1 s_2)^n \rangle.$$

The infinite dihedral group is

$$D_\infty = C_2 \star C_2 = \langle x, y \mid x^2, y^2 \rangle.$$

It has another presentation $D_\infty = \langle r, s \mid s^2, (rs)^2 \rangle$. Using this, one can easily see that the finite dihedral group D_n is a quotient of D_∞.

Example 1.2.4 (Modular group) *The group $PSL_2(\mathbb{Z}) = SL_2(\mathbb{Z})/\{\pm I\}$ is called the modular group, where*

$$SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}.$$

It is an infinite group, and has the presentation: $PSL_2(\mathbb{Z}) \cong C_2 \star C_3 = \langle x, y \mid x^2, y^3 \rangle$. We refer the reader to an article by Conrad [9] for the proof of this interesting fact.

Example 1.2.5 (Hurwitz groups) *The group $\Delta = \langle x, y \mid x^2, y^3, (xy)^7 \rangle$ is called a $(2,3,7)$-triangular group, and any finite quotient of this group is called a Hurwitz group. These groups arise in the study of automorphisms of Riemann surfaces. We refer the reader to an article by Conder [10] for further study.*

1.2.1 Word problem

When a group G is given by generators S and relations R, we can write a random element of the group G as a word in the generators. However, several words might represent the same group element. Thus, it is an important problem to decide when these words represent the same group element, which also amounts to finding words that represent the identity. If we have an algorithm to decide if a word represents the identity of the group G, then we say that we have a solution to the word problem in G.

1.2.2 Cayley graph

Let G be a group and S a generating set of G. We assume $1 \notin S$. The Cayley graph $\Gamma(G,S)$, of the group G with respect to S, is defined as follows.

Definition 3 *The vertex set $V(\Gamma)$ for the graph is given by the elements of the group G. The edge set $E(\Gamma) = \{(g, sg) \mid g \in G, s \in S\}$, i.e., there is an edge from g_1 to g_2 if $g_2 = sg_1$ for some $s \in S$.*

This graph is usually directed. However, when the set S is symmetric (i.e. $S = S^{-1}$) the graph is undirected. The Cayley graph of finite simple groups provide examples of expander family of graphs (see [24, 3, 2]) thus playing an important role in this subject. In this article we will not go into this aspect, instead we refer an interested reader to the book by Tao [37].

1.2.3 Computational group theory

When a group is defined by generators and relations on the computer, it becomes important to have a fast algorithm which produces a random element of the group. Often groups are defined as a subgroup of the symmetric group or matrix groups, as these groups are easier to implement. We refer the reader to look the book [19] on this subject. This subject has given rise to several interesting research problems and associated projects to solve these problems such as "group recognition project".

1.3 Two generation problem for finite simple groups

Let G be a group. Do there exist two elements in G, such that the group G is generated by those two elements? A further question is if we can put restriction on the order of elements, for example, can we have one of these elements of order 2.

Definition 4 *A group G is said to be two generated if it has two elements r, s in G such that $G = \langle r, s\rangle$.*

A two generated group is a quotient of the free group on two generators $\mathcal{F}_2$. Classifying two generated finite groups would be a very general problem (for all groups), thus a restricted questions is, to determine which of the finite simple groups are two generated. This problem is often referred to as the "Two-generation problem". We mostly focus on the family of classical groups (FSG3a in our notation).

1.3.1 Chevalley-Steinberg generators for the groups of Lie type

One of the largest family of finite simple groups, is of the groups of Lie type. Chevalley (see [8]), and his work extended by Steinberg (see [35]), gave a uniform method, starting from simple Lie algebras over $\mathbb{C}$, to construct these groups over any field, by providing an explicit set of generators. We briefly explain this process here, and refer to the book by Carter [7] on this subject for further details.

Let $\mathfrak{L}$ be a simple Lie algebra over $\mathbb{C}$. Let Φ be the corresponding reduced root system of $\mathfrak{L}$. Chevalley proved that there exists a basis of $\mathfrak{L}$ such that all the structure constants are integers. Such a basis is called a Chevalley basis, and it essentially means that $\mathfrak{L}$ can be defined over $\mathbb{Z}$. Let k be a field. For each $r \in \Phi$ and $t \in k$, there are certain automorphisms $x_r(t)$ of the Lie algebra $\mathfrak{L}$. Let G be the group generated by these elements $x_r(t)$. These generators are called Chevalley generators of the corresponding group. Using the adjoint representation, Chevalley and Steinberg proved that the groups thus obtained are simple groups, and over the finite field $k = \mathbb{F}_q$, they give the family of finite simple groups (FSG3 in our notation). Let us understand this process through some examples.

Example 1.3.1 *In the group $SL_n(\mathbb{F}_q)$, the elements $x_{i,j}(t) = I + te_{i,j}$ for $1 \leq i \neq j \leq n$, where $t \in \mathbb{F}_q$ and $e_{i,j}$ is the matrix with 1 at the ij^{th} place and 0 elsewhere, are called Chevalley generators. The Gaussian elimination algorithm, using row-column operations, provides an algorithmic proof that these elements generate the group $SL_n(\mathbb{F}_q)$. In fact, a smaller subset of this set,*

$$\{x_{i,i+1}(t), x_{i+1,i}(s) \mid t, s \in \mathbb{F}_q, 1 \leq i \leq n-1\},$$

generates the group $SL_n(\mathbb{F}_q)$. This is because we can get all of the remaining Chevalley generators by taking commutators of these ones.

Example 1.3.2 *Let q be odd. Let $J = \begin{pmatrix} 0 & I_l \\ -I_l & 0 \end{pmatrix}$, and $Sp_{2l}(\mathbb{F}_q) = \{X \in GL_{2l}(\mathbb{F}_q) \mid {}^tXJX = J\}$ be the symplectic group. Following [7], we use the index set for the matrix of size $2l$ as $1, \ldots, l, -1, \ldots, -l$. The group $Sp_{2l}(\mathbb{F}_q)$ is generated by the following set of Chevalley generators:*

$$\{x_{i,j}(t), i \neq j; x_{i,-j}(t), i < j; x_{-i,j}(t), i < j, x_{i,-i}(t), x_{-i,i}(t) \mid t \in \mathbb{F}_q\},$$

where $x_{i,j}(t) = I + t(e_{i,j} - e_{-j,-i})$; $x_{i,-j}(t) = I + t(e_{i,-j} + e_{j,-i})$; $x_{-i,j}(t) = I + t(e_{-i,j} + e_{-j,i})$; $x_{i,-i}(t) = I + te_{i,-i}$; and $x_{-i,i}(t) = I + te_{-i,i}$. One can further compute that the simple generators (corresponding to simple roots) with their negative counterparts are enough to generate $Sp_{2l}(\mathbb{F}_q)$. The remaining generators can be produced using the commutators of these. For example, if we work with $Sp_6(\mathbb{F}_q)$, the set

$$\{x_{1,2}(t), x_{2,3}(t), x_{3,-1}(t), x_{2,1}(t), x_{3,2}(t), x_{-1,3}(t) \mid t \in \mathbb{F}_q\}$$

generates this group.

We refer the reader to [30, 4] for analogue of Gaussian elimination algorithm in orthogonal groups, symplectic groups and unitary groups, which provides an algorithmic proof of generation of the corresponding groups using their Chevalley generators. Notice that the number of Chevalley generators is usually large, as it varies with the field size and the matrix size. However, if we restrict to the base field $\mathbb{F}_p$, this is still an interesting generating set.

Example 1.3.3 *The group $SL_2(\mathbb{F}_p)$ is generated by the two elements, $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. The diameter of the Cayley graph of this is studied by Larsen in [25] to resolve certain conjectures of Luboztky.*

1.3.2 2-Generation of finite simple groups

In 1930, Brahana [5] noticed that several known finite simple groups are 2-generated. Following that, Albert and Thompson (see [1]) proved the same for projective linear groups. Eventually, Steinberg in [34], proved that all the finite groups of Lie type are two-generated. There are several other results, case-by-case, on this subject, which we do not go into at the moment. The need of doing this case-by-case is because of the more general $(2,3)$-generation problem explained in the following section. Steinberg, following the general Chevalley-Steinberg construction, gave explicit generators in each case. For the purpose of demonstration we give an example here.

Example 1.3.4 *Let q be an odd prime power and $n \geq 2$. Fix a cyclic generator of the finite field, say, $\mathbb{F}_q^* = \langle \zeta \rangle$. Steinberg proved that $SL_n(\mathbb{F}_q) = \langle r, s \rangle$ where*

$$r = \begin{pmatrix} \zeta & 0 & 0 \\ 0 & \zeta^{-1} & 0 \\ 0 & 0 & I_{n-2} \end{pmatrix}, s = \begin{pmatrix} 1 & 0 & \cdots & 0 & (-1)^{n-1} \\ 1 & 0 & \cdots & 0 & 0 \\ & \ddots & \ddots & & \vdots \\ & & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

Notice that $s = x_{1,2}(1)m$, *where*

$$m = \begin{pmatrix} 0 & 0 & \cdots & 0 & (-1)^{n-1} \\ 1 & 0 & \cdots & 0 & 0 \\ & \ddots & \ddots & & \vdots \\ & & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

Let $H = \langle r, s \rangle$ *be the subgroup of* $SL_n(\mathbb{F}_q)$ *generated by r and s. We need to prove that* $H = SL_n(\mathbb{F}_q)$. *The steps are as follows:*

1. *First we compute the commutator* $[srs^{-1}, r] = \begin{pmatrix} 1 & -(\zeta-\zeta^{-1})^2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{n-2} \end{pmatrix}$, *which is in H. Clearly* $\zeta - \zeta^{-1} \neq 0$, *thus we get a non-trivial unipotent* $x_{1,2}(t)$ *in H where* $t = -(\zeta-\zeta^{-1})^2$. *By taking powers of r, and doing similar computations, we prove that* $x_{1,2}(t) \in H$ *for all* $t \in \mathbb{F}_q$.
2. *Thus,* $x_{1,2}(-1).s = m$ *is in H.*
3. *Now compute* $mx_{1,2}(t)m^{-1} = x_{2,3}(t)$, *and inductively check that* $mx_{i-1,i}(t)m^{-1} = x_{i,i+1}(t)$ *for all* $1 \leq i \leq n-1$.
4. *Now,* $mx_{n-1,n}(t)m^{-1} = x_{2,1}(t)$ *is in H, and inductively check that* $mx_{i,i-1}(t)m^{-1} = x_{i+1,i}(t)$ *for all* $2 \leq i \leq n-1$.

The proof is complete, combined with the fact in Example 1.3.1.

Example 1.3.5 *Suppose q is an odd prime power. The group* $Sp_6(\mathbb{F}_q)$ *is generated by the two of its elements r and s. The element* $r = diag(\zeta, \zeta^{-1}, 1, \zeta^{-1}, \zeta, 1)$, *where* ζ *is a cyclic generators of* $\mathbb{F}_q^*$, *and*

$$s = x_{1,2}(1).w = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix},$$

where $w = \begin{pmatrix} 0 & 1 \\ -I_5 & 0 \end{pmatrix}$. *Let* $H = \langle r, s \rangle$ *be the subgroup of* $Sp_6(\mathbb{F}_q)$, *generated by r and s. To prove equality, we prove that all the Chevalley generators are in the subgroup H, and hence* $H = Sp_6(\mathbb{F}_q)$. *The steps to prove this are as follows:*

1. $x_{1,2}(\delta)$, *where* $\delta = (\zeta^2-1)(\zeta^{-1}-1)$, *is the commutator* $[srs^{-1}, r]$, *which is in H. By varying the powers of r, and multiplying the* $x_{1,2}(d)$ *thus obtained, we prove that* $x_{1,2}(t)$ *in H, for all* $t \in \mathbb{F}_q$.
2. *Now* $w = x_{1,2}(-1).s$ *is in the group H.*
3. $wx_{1,2}(t)w^{-1} = x_{2,3}(t)$ *is in H. Further,* $wx_{2,3}(t)w^{-1} = x_{3,-1}(t)$ *is in H.*
4. $wx_{3,-1}(t)w^{-1} = x_{2,1}(-t)$, $wx_{2,1}(t)w^{-1} = x_{3,2}(t)$, *and* $wx_{3,2}(t)w^{-1} = x_{-1,3}(t)$.

Combined with the fact in Example 1.3.2, the proof is complete,

At this point, we refer the reader to survey articles by Di Martino and Tamburini [14] and Wilson [42] on this subject. The work on $(2,3)$-generation, discussed in the next section, naturally contains information about the two-generation problem. In our discussion we have not taken into account how many relations are required. We fast forward to the recent breakthrough in 2011 on this problem, by Guralnick, Kantor, Kassabov and Lubotzky (see Theorem A [17]).

Theorem 1 *Every finite quasi-simple (the groups which are perfect and simple modulo its center) group of Lie type, except the Ree groups* ${}^2G_2(3^{2e+1})$, *have presentations with 2 generators and 51 relations.*

We urge the readers to take a moment to soak into this glorious theorem by reminding themselves that the size of finite groups of Lie type, such as the classical groups FSG3a, vary with two variables n and q.

1.3.3 Two generation over $\mathbb{Z}$

In [18], Gow and Tamburini proved that the group $SL_n(\mathbb{Z})$, for $n \neq 4$, is generated by two matrices $x = I + \sum_{i=1}^{n} e_{i,i+1}$ and $y = I + \sum_{i=1}^{n} e_{i,i-1}$. Kassabov in [21] has proved that the matrix ring $M_n(\mathbb{Z})$, for $n \geq 2$, has a ring presentation by 2 generators and 3 relations as follows:

$$M_n(\mathbb{Z}) = \langle x, y \mid x^n,\ y^n,\ xy + y^{n-1}x^{n-1} - 1 \rangle,$$

given by associating $x = \sum_{i=1}^{n} e_{i,i+1}$ and $y = \sum_{i=1}^{n} e_{i,i-1}$. There is a lot of literature on generating dense subgroups in an arithmetic group, but we won't delve in that direction. We refer the reader to a survey article by Tamburini [37] on this subject, and a paper by Vsemirnov [40] for further reading.

1.3.4 Standard generators used in MAGMA

The generators for finite groups of Lie type used in MAGMA are not Chevalley generators or Steinberg 2-generators mentioned above. These are, what is called "Standard generators" used for the classical groups. These were originally proposed by Costi [12], and later adopted by Leedham-Green and O'Brien [27, 28]. These are at most 8 in number for all classical groups.

1.4 The $(2,3)$-generation problem

Let G be a group.

Definition 5 *A two generated group G is said to be (2, 3)-generated if one of the generators is of order 2 and other of order 3. That is $G = \langle r, s \rangle$ where $r^2 = 1$ and $s^3 = 1$.*

The modular group $PSL_2(\mathbb{Z})$ 1.2.5 is a free product of C_2 and C_3 thus freest possible (2, 3)-group. Thus, any (2, 3)-generated group is a quotient of the modular group $PSL_2(\mathbb{Z})$. Since finite simple group have an element of order 2, the question here is to classify which finite simple groups are (2, 3)-generated, i.e., which finite simple groups are quotients of $PSL_2(\mathbb{Z})$. Historically, the known answers indicated that almost all of them (see below for further details) are quotients, and thus it led to the belief that $PSL_2(\mathbb{Z})$ is the mother of almost all finite groups.

Several results have been obtained for the $(2,3)$-generation of groups of Lie type: by Tamburini and Wilson [38, 39] for classical groups when n is large enough, Di Martino and Vavilov [15, 16] for $SL(n,q)$, Lubeck and Malle [26] for exceptional groups, Pellegrini [32], Malle, Saxl and Weigel [31] and others. In most of these results an explicit generating set is exhibited and often they prove that these groups are $(2,3)$-generated. However, there are some groups of Lie type which are not $(2,3)$-generated (see [40]). We refer the reader to the survey article on this subject by Pellegrini and Tamburini [33] and by Burness [6].

A more general problem is the (a,b)-generation problem, where a and b are given positive integers. The problem asks what are all possible finite simple groups which are quotients of the group $C_a \star C_b$. King [22] has proved that every non-abelian finite simple group is $(2,p)$-generated, where p is a prime. We remark that in this result p depends on the group. More generally, King is [23] has classified which finite simple groups are (a,b)-generated.

Clearly, for a finite simple group G to be $(2,3)$-generated, it is necessary that G should possess an element of order 2, and an element of order 3. It is well known that non-abelian simple groups contain elements of order 2. However, Suzuki groups, (see FSG4, Suz) do not contain an element of order 3 and hence are not $(2,3)$-generated. These are the only non-abelian simple groups which do not contain an element of order 3. We can ask further question that if a finite simple group contains an element of order 3, is it always $(2,3)$-generated? The answer is no! Among the classical groups $PSp(4,2^k)$ and $PSp(4,3^k)$ are not $(2,3)$-generated (see [29], Theorem 1.6), although they contain elements of order 3. For the recent update on this problem please see the survey article by [6], for example, Theorem 2.4.

1.4.1 Probabilistic Two-generation

The subject of probabilistic group theory is a less explored domain. We refer a reader to the beautiful survey article by Dixon [13] to get a feel of the subject. To give some idea of the questions dealt with in this subject, we mention some recent results proved by Liebeck and Shalev [29]. Let G be a finite simple group. The question is to understand if the probability that two randomly chosen elements of G generate G, tends to 1 as $|G| \to \infty$. This question is further refined with stricter conditions that one of the two randomly chosen elements is an involution. This is Conjecture 2 in [29], which they prove for classical and alternating groups. The Conjecture 3 in [29], due to Di Martino, Vavilov, Wilson and others, is that all finite simple classical groups (with some small exception in low rank and low characteristic) are $(2,3)$-generated. Liebeck and Shalev (see Theorem 1.4 and 1.5 in [29]) proved that if G is a finite simple classical or alternating group, except $PSp_4(q)$, then the probability that a randomly chosen order 2 element and a randomly chosen order 3 element of G generates G, tends to 1 as $|G| \to \infty$. Thus they establish that $PSp_4(q)$ is not $(2,3)$-generated, and almost all finite simple classical groups are $(2,3)$-generated.

1.4.2 $(2,3,7)$-generation

A group is said to be $(2,3,7)$-generated if it is generated by two elements x, y, where $x^2 = 1, y^3 = 1$, and the product $(xy)^7 = 1$. Thus such groups are a quotients of the $(2,3,7)$-triangular group Δ (see Example 1.2.5). The question is to determine all Hurwitz groups. In particular, determine which finite simple groups are $(2,3,7)$-generated, i.e., which finite simple groups are Hurwitz groups. We refer the reader to survey articles by Conder [10, 11] on this subject.

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2. Nonstandard statistical distributions: Perspectives and future scopes

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Abstract: Nonstandard distributions are generally used for modeling inconsistent and unusual observations which are termed as inliers. Inliers (instantaneous or early failures) are natural occurrences of a life test, where some of the items fail instantaneously or within a short span of the life test, and thus produces inconsistent observations. The inconsistency of such situation is modeled using a nonstandard mixture of distributions; where degeneracy can happen either at a discrete mass point zero or at two discrete points zero and one and so on. In this article, we discuss the inliers prone situation and models that are under challenges for the researchers. Some examples are also discussed.

Keywords: *early failures; failure time distribution; inliers; instantaneous failures.*

2.1 Introduction

Uncertainties are a part and parcel of human life. They can arise in any size and amount and is completely unpredictable. They often occur in experimental science. One of the aims of science is to predict and describe real world situations which are generally subject to some uncertainties. Probability models (or probability distributions) are often used to describe uncertainties pertaining to a situation. The probability models are constructed based on probability laws. And these laws are perceived through random experiments of situations, where, we are often interested in the behavior of certain quantities that take different values in different outcomes of the experiments. These quantities are called random variables. Suppose Ω be the set of all outcomes of an experiment, $\mathcal{A}$ be the collection of subsets, then for a given probability space $(\Omega, \mathcal{A}, P[.])$, a random variable, denoted by X or $X(.)$, is a function with domain Ω and range the real line. The function X(.) must be such that the set $\mathcal{A}r$, defined by $\mathcal{A}r = \{\omega : X(\omega) \leq r\}$, belongs to $\mathcal{A}$ for every real number r. Here the probability function $P[.]$ is a set function having domain $\mathcal{A}$ and range $[0,1]$. A random variable X will be called discrete if the range of X is countable. A random variable X is said to be continuous if there exists a function $f_x(.)$ such that $P(A_x) = Fx(x) = \int_{-\infty}^{x} f_X(u)du$ for every real number x.

The distribution of a discrete variable is called a discrete probability distribution and the distribution corresponds to a continuous variable is called continuous probability distribution. Some of the most frequently used discrete probability distributions are Bernoulli distribution, Poisson distribution, Geometric distribution, Hyper Geometric distribution, rectangular distribution etc. And some of the continuous uncertainty models are exponential distribution, Normal distribution, Gamma distribution, Weibull distribution, Triangular distribution, Uniform distribution etc. There are about two hundred statistical distributions now available in the literature to model uncertainties in the world. Besides, Statisticians have also contributed a number of modified, generalized, multivariate, and extended distributions to the statistics literature for describing simple and complex real-world situations.

2.2 Complete (or finite) mixture of probability models

Mixture of distributions, in particular normal, have been extensively used as model in a wide variety of important situations where data can be viewed as arising from two or more populations mixed in varying proportions (Pearsons, 1894). The mixture distribution represent a useful way of describing heterogeneity in the distribution of a variable. Let $X_1, X_2 \cdots X_n$ be independent, identically distributed p- dimensional observations from a distribution with probability density function

$$f(x;\pi) = \Sigma_{k=1}^{g} \pi_k f_k(x), \tag{2.1}$$

where π_k represents the k^{th} mixing proportion or the probability that the observation X_i belongs to the k^{th} subpopulation with corresponding density $f_k(x)$ called the k^{th} mixing or component density. Here, g represents the total number of components with $\pi = (\pi_1, \pi_2, \cdots, \pi_g)$ lying in the (g-1) dimensional simplex, i.e. $0 \leq \pi_k \leq 1 \, \forall \, k = 1, 2, \cdots, g$ and $\Sigma_{k=1}^{g} \pi_k = 1$. This is the most general form of a mixture: usually f_k's are assumed to be of parametric form i.e. $f_k(x) = f_k(x; \vartheta_k)$, where the functional form of $f_k(.;.)$ is completely known, but for the parametrizing vector ϑ_k. Thus, 2.1 can be written in the form

$$f(x;\vartheta) = \Sigma_{k=1}^{g} \pi_k f_k(x;\vartheta_k) \tag{2.2}$$

We refer to $f(x;\vartheta)$ as a finite mixture model density with parameter vector ϑ where $\vartheta = (\pi', \vartheta_1', \vartheta_2', \cdots, \vartheta_g')$. When the number of mixture components g, is also known, only ϑ has to be estimated. When g is not provided, we have to additionally estimate the number of components in the mixture. Finite mixture models made their first recorded appearance in the modern statistical literature in the nineteenth century in a paper by Newcomb (1886) who used it in the context of modeling outliers. A few years later, Pearson (1894) used a mixture of two univariate Gaussian distributions to analyze a dataset containing ratios of forehead to body lengths for 1000 crabs, using the method of moments (MOM) to estimate the parameters in the model. One may refer to Titterington et al. (1985), McLachlan and Basford (1988) and McLachlan and Peel (2000) for a comprehensive survey on the history and applications of finite mixture models. Other helpful resources on the theory, applications and developments in the field are Bohning and Schlattmann (1998), Fruhwirth-Schnatter (2006), Lindsay (1983, 1995) and so on. The problems and applications of mixtures also appear in the literature associated with the term heterogeneity (Keyfitz, 1984).

The challenge here is to estimate the unknown parameters of the model and resulting fit of the mixture for the parametric forms adopted for its component densities. In both the situations, getting closed form estimating equations and mathematical functions for fitting the model is really difficult. With the advent of computing technology, these issues are somewhat resolved. It is interesting to note that, many Big Data applications are now modeled through mixtures of distributions, as there is every chance to have more than one mode in the underlying distribution.

2.3 Nonstandard mixture models

There is a plethora of examples of phenomena concerning nature, life and human activities where the real data do not conform to the standard distributions. In such cases, we either use mixtures of standard distributions of similar types (i.e. either all discrete mixtures or all continuous mixtures) or nonstandard mixtures of degenerate distribution and a standard distribution which may be again a discrete or continuous one. The literature contains many papers that deal with mixtures of distributions of similar types, such as the mixture of normal distributions, the mixture of chi-square distributions, the mixture of exponential distributions, and the mixture of binomial distributions and so on (Robbins and Pitman, 1949). However, the literature contains very few papers that provide and deal with special "nonstandard" mixture that mix discrete (degenerate) and continuous distributions as emphasized in this article.

Nonstandard mixture of distributions generally contains inliers, where Inliers is an observation (or a group of observations) sufficiently small relative to the rest of the observations, which appears to be inconsistent with the remaining dataset. They are either the results of instantaneous failures, or early failures, experienced in life testing experiments, survival studies, clinical trials and many other application areas. The test items that fail at time 0 are called *instantaneous failures*, and the test items that fail prematurely are called early failures. In reliability studies, instantaneous failures may be attributable to inferior quality or faulty manufacturing, where as in clinical trials these events may manifest due to adverse reactions to treatments, clinical definitions of outcomes or due to no response to the treatments etc.

Kale and Muralidharan (2000) have introduced the term inliers in connection with the estimation of (π, θ) of early failure model with modified failure time distribution being an exponential distribution with mean θ assuming π known. Glaser (1995) indicated that early failures can be regarded as premature samples, which never need to be recorded. Consequently, the unreported early failures are regarded as the samples coming from a truncated distribution. Some other studies related to early failures in the literature are Hsieh (1996), Lee et al. (2011) and references contained therein.

In situations above, the failure observations usually discard the assumption of a single mode distribution and hence the usual method of modeling and inference procedures may not be accurate in practice. Usually, these situations are handled by suitably modifying commonly used parametric models incorporating inconsistent observations, and hence, the model becomes nonstandard mixture of distributions. Following are some of the practical contexts, where inliers can arise as natural occurrences of the specific situations involved, and degeneracy can happen at one or more discrete points and a positive distribution for the remaining lifetimes.

1. In an audit report, we have two pieces of information, namely, the booking amount (recorded) and the audited amount (correct). The difference between the two is called the error amount. Here some population elements contain no error, whereas other population elements contain error of varying amounts. The distribution of errors can, therefore, be viewed as two distinguishable distributions, one with a discrete probability mass at 'zero' (no error) and the other a continuous distribution of non-zero positive and/or negative error amounts.
2. In the mass production of technological components of hardware, intended to function over a period of time, the failure rate is initially relatively high, and then actually decreases with increasing age. The high failure rate either results in zero life time or marginally small life times, otherwise, the life time will be of any positive number (usually continuous). Thus, the overall distribution of lifetimes may be represented by using a nonstandard mixture of distribution.
3. In studies of tooth decay, the numbers of surfaces in a mouth which are filled, missing or decayed are scored to produce a decay index. Healthy teeth are scored 0 for no evidence of decay. Thus, the distribution is a mixture of a mass point at 0 and a nontrivial continuous distribution of decay score. The problem could be further complicated if the decay score is expressed as a percentage of damage to measured teeth. The distribution should then be a mixture of a discrete random variable (0-for healthy teeth, 1-for all missing teeth) with a nonzero probability of both outcomes and a continuous random variable (amount of decay in the (0, 1) interval).
4. In the studies of genetic birth defects, children can be characterized by two variables, a discrete variable to indicate if one is affected and born dead and a continuous variable measuring the survival time of affected children born alive. We may consider here a mixture of the mass point at 0 and a nontrivial continuous distribution of survival time of affected children.
5. In studies of methods for removing certain behaviors (e.g., predatory, behavior of salt consumption), the amount of the behavior which is exhibited at a certain point in time may be measured. In this context, the complete absence of the target behavior may represent a different result than would a reduction from a baseline level of the behavior. Thus, one would model the distribution of activity levels as a mixture of a discrete value of zero and a continuous random level.
6. The first response time of patients during a medical operation: Consider measurements of physical performance scores of patients with a debilitating disease such as multiple sclerosis. There will be frequent zero measurements from those giving no performance and many observations with graded positive performance.
7. In studying the human smoking behavior, two variables of interest are smoking status- - 'ever smoked' and 'never smoked' and a score on a 'pharmacological scale' of people who have smoked. Here the data is modeled with two discrete value points, namely 0 (for never smoked), 1 (for ever smoked) and a "pharmacological score" in continuous measurement. A nontrivial conditional distribution of the second variate can be defined only in association with the first outcome of the first variate.
8. Machines and software's are tested for their correctness and perfectness or reliability. Bugs (or errors) in such situation are important to assess the durability and credibility of machines and programs. Zero defects or zero bugs are considered to be good in such situations. If there are bugs, then it can be measured in terms of some discrete measurements. Data on this example contains many zeros (no bugs), few countable bugs, and large number of bugs constituting a mixture data.
9. In a community, a particular service, such as a specific medical care, may not be utilized by all families in the community. There may be a substantial portion of non-takers of such a service. Those families who subscribe to it do so in varying amounts. Thus, the distribution of consumption of service may be represented by a mixture of zeros and positive values.

10. The size of tumor lesions is of interest to treat Hematologic malignancy patients. The measurement effect is zero who have lesions absent (or due to disappearance of tumor during treatment), though who have lesions present at baseline that are evaluable but do not meet the definitions of measurable disease may be considered as measurement 1. Otherwise lesions can be accurately measured as longest diameter to be recorded in at least one dimension by chest x-ray, with CT scan or with calipers by clinical examination. Similarly, in studies like Bone lesions, leptomeningeal disease, ascites, pleural/pericardial effusions, lymphangitis cutis/pulmonitis, inflammatory breast disease, and abdominal masses, either the effect is absent or present, are considered as non-measurable otherwise accurately measurable on continuous scale.

One can contemplate many such examples in practical situations involving degeneracy at one or more points and positive configurations of observations. A univariate probability model or a complete (finite) mixture of distributions may not work in those situations. Hence, probability modeling in such situations may require special attention and treatment. From the above examples, it is seen that the values including zeros and close to zeros are important as well as significant in most cases. For instance, zero errors in an audit report, zero tooth decay, and zero bugs in a computer program or electronic machine are all good to judge the efficiency and reliability of the situation, and hence they are significant. Similarly, zero lifetime, zero rainfall (dry day) etc., are all practically not good but again significant as per the conditions prevailing. Thus, inliers are more natural than outliers as per various contexts discussed above. As a consequence, the modeling of inliers distribution is more important than its detection for statistical decision making. Below, we introduce various inliers prone models (nonstandard distributions).

2.3.1 *Instantaneous failure model*

The model $F = \{F(x;\theta), x \geq 0, \theta \in \Theta\}$, where $F(x;\theta)$ is a continuous failure time distribution function with $F(0) = 0$ is to be suitably modified as a non-standard mixture of distribution by mixing a singular distribution at zero to accommodate instantaneous failures. The modified model is represented as

$$G(x;\pi,\theta) = \begin{cases} 1-\pi, & x = 0 \\ 1-\pi+\pi F(x;\theta), & x > 0 \end{cases} \tag{2.3}$$

with respect to a measure μ which is the sum of Lebesgue measure on $(0,\infty)$ and a singular unit measure at the origin; and $0 < \pi < 1$. Aitchison (1955) was the first to discuss the inference problem of instantaneous failures in life testing. Other authors who have studied this kind of model are Kleyle and Dahiya (1975), Jayade and Prasad (1990), Vannman (1991, 1995), Kale (1998, 2003), Muralidharan (1999), Muralidharan and Kale (2002), Muralidharan and Lathika (2005, 2006), Kale and Muralidharan (2006, 2007), Adlouni et al. (2011) and so on.

2.3.2 *Early failure model-1*

If it is assumed that $\lambda(x) = \lambda = \frac{1}{\theta}$ for all x from an exponential distribution, then the assumption of an exponential density is equivalent to the assumption of a constant failure rate. Under this setup, Miller (1960), proposes the early failure model as

$$\lambda(x) = \begin{cases} \lambda_1, & 0 \leq x < T_0 \\ \lambda_2, & T_0 \leq x \end{cases} \tag{2.4}$$

where $\lambda_1 > \lambda_2$. The probability density correspond to this failure rate is

$$f_x(x;\lambda_1,\lambda_2) = \begin{cases} \lambda_1 e^{-\lambda_1 x}, & 0 \leq x < T_0 \\ \lambda_2 e^{-\lambda_1 T_0 - \lambda_2 (x-T_0)}, & T_0 \leq x. \end{cases} \tag{2.5}$$

The justification follows from the fact that when a component is put on the test (or experiment), it is not known whether it is an 'early failure' or a 'standard' item. Since some will be early failures, the failure rate on the average will be high at the start, but if an item has survived for a certain period of time T_0, then it cannot be an 'early failure' so its failure rate will be lower for the succeeding time period. The above model can also be viewed as a model for a shift in the hazard function of exponential distribution.

2.3.3 *Early failure model-2*

To accomodate early failures, the family the family $\mathscr{F}_2$ is modified to $\mathscr{G}_1 = \{Gg_1,\ x \geq 0,\ \theta \,\epsilon\, \Theta,\ 0 < \pi < 1\}$, where the CDF corresponding to $g_1 \,\epsilon\, \mathcal{G}_1$ is given by

$$G(x;\pi,\theta) = (1-\pi)H(x) + \pi F(x;\theta). \tag{2.6}$$

Here $H(x)$ is a CDF with $H(\delta) = 1$ for δ sufficiently small, assumed known and specified in advance. Then the modified family $\mathcal{G}_1$ has a PDF with respect to measure μ, which is the sum of Lebesgue measure on $(\delta,\ \infty)$ and a singular measure at δ as

$$g_1(x;\pi,\theta) = \begin{cases} 0, & x < \delta \\ 1-\pi+\pi F(\delta;\theta), & x = \delta \\ \pi F(x;\theta), & x = \delta. \end{cases} \tag{2.7}$$

Some of the references which treats early failure analysis are Kale and Muralidharan (2000, 2006, 2007), Kale (2003), Muralidharan and Lathika (2006), Muralidharan and Arti (2008, 2013), Muralidharan (2010), Muralidharan and Bavagosai (2016a) and the references contained therein. These authors treated early failures as inliers using the sample configurations from parametric models including exponential, Weibull, Pareto, Normal and Gompertz distribution.

The model in equations (2.3) and (2.7) can be combined to form the CDF as

$$G(x;\pi,\theta) = \begin{cases} 0, & x < \delta \\ 1-\pi+\pi F(x;\ \theta), & x \geq \delta. \end{cases} \tag{2.8}$$

having the corresponding probability density function (pdf) as given in equation (2.7). If $d = 0$ the model reduces to the instantaneous failures case and if $d > 0$, it reduces to the case of early failures. One may also see Lai et al. (2007) for a complete mixture model, where they have treated the instantaneous part based on Dirac delta function and a probability distribution for the positive observations.

2.3.4 *Model with inliers at zero and one*

In some of the examples discussed above (e.g. 3, 4, 7, 8, 10) the observations 0 and 1 become a natural occurrence with other positive observations. If these observations are treated as inliers, then, the distribution function of such models can be written as

$$H(x;\pi_1,\pi_2,\theta) = \begin{cases} 0, & x < 0 \\ \pi_1, & 0 \leq x < 1 \\ \pi_1+\pi_2, & x = 1 \\ \pi_1+\pi_2+(1-\pi_1-\pi_2)\frac{F(x;\,\theta)-F(1;\,\theta)}{1-F(1;\,\theta)}, & x \geq 1 \end{cases} \tag{2.9}$$

where π_1 and π_2 are the propotion of 0 and 1 observation. The model was first studied by Muralidharan and Pratima (2017, 2018a, b) with $F(x;\theta)$ as exponential, Pareto and Weibull and applied on rainfall data, NEFT data, NFHS survey data, surgical data and so on. One can also use other probability models for $F(x;\theta)$.

2.4 Data Analysis

In this section, we consider a dataset to illustrate the usefulness and effectiveness of the proposed models in identifying the number of inliers and the estimators of the parameters on the distribution. The detailed description regarding the dataset and estimates of model parameters are given below. To estimate the number of inliers, say 'k' in the data set, maximize the likelihood function (say $L_k(\underset{\sim}{x}) = \prod_{i=1}^{n} f(x_i)$) for fixed k successively and then determine $\max_{0 \le x \le n} lnL_k(\underset{\sim}{x})$ and consider $\hat{k}$ to be that where the likelihood is maximum. Another way of estimating the value of k is by testing hypothesis about the number of inliers. Readers are advised to refer to various papers of the authors of this article for more details.

The dataset is based on Vanmann (1991) experiment in which a batch of wooden boards is dried by a particular chemical process and the object of the experiment is to compare two processes as regards the extent of deformation of boards due to checking. The measure of damage to the board is the checking area $x = \frac{I\bar{d}}{hI_o} \times 100$, where I is the length of the check, $\bar{d}$ is the mean depth of the check, h is the thickness of the board area and I_o is the length of the board.

	Estimator	Estimates for	
		Instantaneous Failure model d=0	Early Failure Model d=1.0
$\alpha unknown$	MLE of π& $S\hat{E}(\hat{\pi})$	0.64865(0.07845)	0.54995(0.09689)
	MLE of θ& $S\hat{E}(\hat{\theta})$	0.17304(0.05629)	0.11770(0.05099)
	MLE of α& $S\hat{E}(\hat{\alpha})$	0.03733(0.05203)	0.08128(0.05736)
	95% CI for π	(0.49482, 0.80247)	(0.36002, 0.73988)
	95% CI for θ	(0.06272, 0.28336)	(0.01776, 0.21764)
	95% CI for α	(0.00000, 0.13930)	(0.00000, 0.19370)
$\alpha = 0.02$	MLE of p& $S\hat{E}(\hat{\pi})$	0.64865 (0.07848)	0.57678 (0.10006)
	MLE of θ& $S\hat{E}(\hat{\theta})$	0.18785 (0.03834)	0.16855 (0.03972)
	95% CI for π	(0.49482, 0.80247)	(0.37957, 0.77398)
	95% CI for θ	(0.11270, 0.26300)	(0.09070, 0.24640)

Table 2.1: Summary of estimates for Vanmann's dataset

Thus x is the check area measured as percentage of the board area. The boards are dried at the same time under different schedule and under same climate conditions. When drying boards not all of them will get the checks and a typical sample of wood contain several observations with $x_i = 0$ or $x_i > 0$ but relatively small compared to the rest of the checks. These observations will correspond to instantaneous failures or early failures. We reproduce Vanmanns dataset from Schedule 1of Experiment 2 on one batch of 37 boards. There are 13 instantaneous failures in this Schedule. The other positive observations arranged in increasing order of their magnitude are 0.08, 0.32, 0.38, 0.46, 0.71, 0.82, 1.15, 1.23, 1.40, 3.00, 3.23, 4.03, 4.20, 5.04, 5.36, 6.12, 6.79, 7.90, 8.27, 8.62, 9.50, 10.15, 10.58 and 17.49.

To understand the correct model for the positive observations, we have first fitted a number of distributions to the dataset by using various information criteria's, like Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Kolmogorov-Smirnov (K-S) Statistic, Cramer-Von Mises (CVM) statistics, Anderson-Darling (AD) statistic. One of the appropriate models is the Gompertz distribution with $f(x; \alpha, \theta) = \theta e^{\alpha x} e^{-\frac{\theta}{\alpha}(e^{\alpha x}-1)}$, $x > 0$. The Gompertz distribution is applied many fields including reliability and life testing studies, epidemiological and biomedical studies and actuarial sciences. The number of inliers estimated in this data set is 9. The summary of estimates of model parameters for some selected values for the above dataset is shown in the Table 2.1 above.

2.5 Non standard distributions: Issues and problems

In literature, many authors have studied their problems taking inliers into account in various perceptions. For example, the failure rate function of electronic products is characterized by the bathtub curve, where at the beginning; the failure rate is at high level and then decreases with time in the early-failure period. After that, the failure rate remains at a stable level for a period of time, which is called the useful-life period. Then, the failure rate gradually increases again in the wear-out period. Chenga and Sheu (2015) attempt to solve the problem of early failures occurrences in conducting parameter estimation for the Weibull distribution in a constant stress partially accelerated life tests (CS-PALTs) scheme. In a constant stress partially accelerated life test (CS-PALT) experiment, the total test units are first divided into two groups. The items of one of the groups are allocated to a normal condition, and the items of the other group are allocated to a stress condition. Each unit is run at a constant level of stress until the unit fails or is censored. In literature CS-PALT models with various types of distributions have been proposed, such as Weibull, Burr XII, Rayleigh, Exponential, and Inverted Weibull distributions.

In clinical trials, issues such as consent withdrawal, procedure misfit etc., would rightly attribute to instantaneous failures. Survival data with instantaneous events is not uncommon in epidemiological and clinical studies, and for this reason, herein a general methodology under the proportional hazards (PH) model is developed for the analysis of interval-censored data subject to instantaneous failures. To analyze time to event data arising from clinical trials and longitudinal studies, etc, the event time of interest is not directly observed but is known relative to periodic examination times; i.e., practitioners observe either current status or interval-censored data. In some such studies the observed data also consists of instantaneous failures; i.e., the event times for several study units coincide exactly with the time at which the study begins. In light of these difficulties, Gamage et al. (2018) focused on developing a mixture model under the proportional hazards (PH) assumptions, which can be used to analyze interval-censored data subject to instantaneous failures.

When the failure times are exactly observed, as is the case in reliability studies, it is common to incorporate instantaneous failures through a mixture of parametric models, with one being degenerate at time zero; e.g., see Muralidharan (1999), Kale and Muralidharan (2002), Murthy et al (2004), Muralidharan and Lathika (2006), Pham and Lai (2007), and Knopik (2011). In the case of interval-censored data, seen commonly in epidemiological studies and clinical trials, accounting for instantaneous failures becomes a more tenuous task, with practically no guidance available in the existing literature. Arguably, in the context of interval-censored data, one could account for instantaneous failures by introducing an arbitrarily small constant for each as an observation time, and subsequently treat the instantaneous failures as left-censored observations. For the analysis of interval-censored data subject to instantaneous failures a new mixture model is proposed, which is a generalization of the semi-parametric PH model studied in Wang et al (2016).

Burn-in has been widely used as an effective procedure for screening out failed electronic products in the early-failure period, before shipment to the customers. That is burn-in forces electronic products to pass the early-failure period before they are sold so that products already possess a stable quality when sent to the customers. Inliers prone model is very useful to decide optimal burn-in time and hence to obtain the burn-in cost and warranty cost.

Financial institutions are at the core of the functioning of the economy, hence, to quantify their exposure to operational risk, banks should take into account the fact that there are huge differences between the behavior of the central part and the tail of the distribution of losses. This is especially true in the case of losses characterized by the so-called low frequency-high severity losses. Adlouni et al. (2011) used instantaneous failure model to estimate the extent of the exposure of a Moroccan bank to operational risk when zero losses are recorded.

Long series of days or decades without rainfall allow one to determine the probabilities of adverse developments in agriculture (droughts). This could be the basis for forecasting crop yields in the future. Instantaneous failure model describing the amount of precipitation and taking into account periods without rain is studied by Bojar et al. (2014).

2.6 Future Scope

Since inliers models are nonstandard and incomplete mixtures, closed-form expressions for descriptive statistics are not available in general and hence, the usual inferences are not smooth. Judging a good inliers model itself can pose many problems of completeness and identifiability of the model. In some cases a specific model for positive observations also poses challenges for estimation in flexible form. This kind of problems are also common in testing of hypothesis concerning the parameters and detection of number of inliers present in the model. In such cases, we resort to numerical evaluation of the likelihood equations.

One of the motivating factors of pursuing further study in this kind of research is that, there are many practical situations where inliers are occurring in a natural way. The examples range from almost all fields of statistical applications. Therefore, constructing appropriate statistical /mathematical model is still an area for exploration.

As in discrete mixture models, where, data points are organized in a frequency distribution in varying proportions, it is not easy to present nonstandard mixtures of data in a unified way. This is the major difference between truncated and inflated models as seen in literature, and hence is a big challenge for modeling inliers type situations.

As such the literature does not include many studies on inliers prone models. The least we can find in the literature are some study on lower outliers (inliers) detection related studies. They are also not articulated in a proper statistical way. The testing procedures when number of instantaneous failures unknown as well as there are more than three or more inliers also pose lots problems in terms of getting the distribution of the test statistics. For more number of inliers the existing procedures may not work well for because of masking and swamping effects. This is really a challenge to work with.

A very important limitation of inliers study is that, if the number of discrete points (zeros or ones) exceeds more than half of the total number of observations, then the estimation can create problems and in case estimators are available, then those estimators may not be sensible enough to interpret. Although we have not come across such practical situations, we are not ruling out.

Another scope for research is to study the generalization of inliers prone model when there are more than two discrete mass and continuous measurements. We strongly believe that, there are practical situations where at many discrete points there can be discrete mass points and continuous measures.

There is a potential scope for inferences based on Bayesian approach for constructing computationally strong model for inliers estimation and detection. Even censoring concepts need a relook into this kind of studies.

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3. Quality Improvement in Math Education - an experiment

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The aim of this article is to present a methodology to improve the quality of mathematics teaching and learning. Though this project was carried out successfully for mathematics teachers at the secondary school level in the state of Maharashtra, it has universal appeal, for all science subjects at all levels.

3.1 Introduction

George Polya on aims of educations:
One of the aims of education is to turn out employable adults, -adults who can fill a job. But a higher aim is to develop all the resources of the growing child in order that he can fill in the job for which he is best fitted. So the higher aim is to develop all the inner resources of the child.

Mathematics is an essential part of education to achieve these aims. Mathematics is called the Queen of all sciences. Despite of the vast potential, students start getting Math Phobia at an early stage of education. What is the reason? Is it the biased opinion perpetuated by the parents? Is there something wrong with the curriculum? Are our mathematics teachers to blame? Unfortunately our education planners though aware of this, are doing little to remedy the situation. In any case, mathematics teachers have a great role to play. An innovative method is always appreciated by the students and leaves long lasting impression on the young mind. The technological developments have opened a host of possibilities to enhance teaching.

In 2015 when IIT Bombay was approached by the Education Department of Maharshtra state to suggest measures that can improve the quality of math education, the “Quality Improvement in Mathematics Education (QIME) project” was conceived with the aim: to change the mind-set of teachers from “Behaviourism” to “Constructivism”. The task of engaging all teachers in the state directly being almost impossible, then it was decided to select about 300 teachers from 39 districts of the state and train them in face to face workshops. These trained teachers will be called “Master Trainers” and after the completion of the training, they will train other teachers in their districts. Subgroups of these Master trainers will also develop resources on topics difficult to teach and learn. These teachers will also be trained to encourage and prepare bright students for exams like PISA and Olympiad. It is aimed to make resources available both online and off line. Creation of DVDs and mobile apps will be an objective of the project. With an aim to integrate the official bodies like DIECPDs, mathematics faculties of these bodies will also be trained.

3.2 Methodology of training

The training was developed as per the following three steps:

Step1: Observe and Explore:
The aim is to develop the pedagogical and content skill that are necessary for effectiveness as a mathematics teacher. Start your lesson with an activity related to the topic. It can be a video, a puzzle, a game, or a hands-on activity. This should motivate initiating "Math Talk" in the classroom. Asking leading questions, soliciting students' responses, and discussions should lead to related material, the mathematical content, you intend to teach.

Step 2: Define and Prove:
This step relates to transacting the main part of the curriculum. Once again, as and when a concept is introduced, there should be an effort to incorporate "inquiry" and "discussion". While proving theorems, ideas of "reasoning and proof" should be incorporated. A list of "frequently asked doubts" with explanations be prepared for each topic. Technological tools should be integrated topic wise to strengthen day today teaching.

Step 3: Apply and Evaluate:
This part relates to applying the concepts of a topic/subtopic to problems. It will be a good idea to incorporate a list of "non-routine" problems that challenge students thinking. Application of the concepts developed to real world problems should also be included. These will also help to evaluate the effectiveness of teaching.

3.3 Activity snapshot

3.3.1 *Selection and training*

- To select master trainers, a voluntary test was conducted in all the districts. Around 1000 teachers appeared in the test and on the basis of merit, and to give reasonable representation to all the districts, 335 teachers were selected.
- A series of 10 of face-to-face workshops were conducted for the training the Master Trainers. The 18 topics were taken from the curriculum and on the three step methodology, inputs were given.
- Three workshops were conducted on integrating GeoGebra in all these topics.
- Master trainers, in groups, prepared contents on all topics in the above format, of the curriculum. These were translated both in English and Marathi.
- Resources prepared have been uploaded on the server with URL: *http://teachertraining.math.iitb.ac.in* so that all the master trainers can access these.

3.3.2 *Evaluation and certification*

- To test the presentation capabilities as a teacher, each Master Trainer was asked to prepare a 30-minute video presentation, which was evaluated by independent experts. These are available at *https://www.youtube.com/channel/UC7e0EHwwELWkK7RHWy8tMzw*.
- Two written Examinations were conducted to test the mathematical competency of each Master Trainer.
- Out of all, 173 teachers have been declared as Master Trainer. Remaining are put on probation.
- All the Master Trainers were administered an oath to spread the knowledge they had gained.

3.4 Post certification

- From 15 September to 15 December, 2019, About 12,000 teachers across all the districts of Maharshtra were given face-to-face training in 3 day long workshops by the master trainers.
- Efforts are being made to make these available on apps.
- Second batch of 270 teachers are undergoing similar training to be Master trainers.

□□□

4. What is happening in the Mathematical world?

Devbhadra V. Shah

4.1 Rainbow proof shows Graphs have uniform parts

Mathematicians have found a colorful way to show how smaller components can perfectly tile, or cover, a graph. On January 8, 2020, mathematicians *Richard Montgomery* of the University of Birmingham, *Alexey Pokrovskiy* of Birkbeck College, University of London and *Benny Sudakoy* of the Swiss Federal Institute of Technology Zurich posted a proof of a nearly 60-year-old problem in combinatorics called *Ringel's* conjecture.

In 1963, a German mathematician named *Gerhard Ringel* posed a simple but broad question about the relationship between complete graphs and trees. He said: First imagine a complete graph containing $2n + 1$ vertices. Then think about every possible tree you can make using $n + 1$ vertices-which is potentially a lot of different trees. Now, pick one of those trees and place it so that every edge of the tree aligns with an edge in the complete graph. Then place another copy of the same tree over a different part of the complete graph. Ringel predicted that if you keep going, assuming you started in the right place, you'll be able to tile the complete graph perfectly. This means that every edge in the complete graph is covered by an edge in a tree, and no copies of the tree overlap each other. Where the first tile is placed, matters.

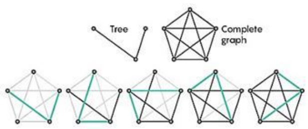

Figure 1

Finally, Ringel predicted that the tiling works regardless of which of the many different possible trees you use to perform it. Figure 1, shows the tiling of complete graph with 5 vertices by a tree having 3 vertices. Ringel's conjecture applies equally to complete graphs with 11 trillion and 1 vertices. And as the complete graphs get bigger, the number of possible trees you can draw using $n + 1$ vertices also rise steeply. How could each and every one of those trees perfectly tile the corresponding complete graph? But there were reasons to think Ringel's conjecture might be true. Mathematicians identified a piece of evidence that suggested the conjecture was at least feasible, and it set in motion a chain of discoveries that eventually led to a proof.

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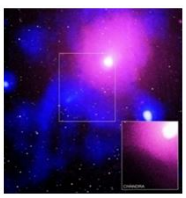
4.2 Astronomers spot the largest explosion since the Big Bang



Galaxy clusters like Ophiuchus are some of the largest objects in the universe and contain thousands of individual galaxies, dark matter, and hot gas. At the heart of the Ophiuchus cluster is a giant galaxy that contains a super-massive black hole with a mass equivalent to that of around 10 million suns. This black hole punched a dent of the size of 15 Milky Ways in the surrounding space, when it erupted and caused the largest known explosion since the Big Bang.

The explosion-captured using four telescopes; NASA's Chandra X-ray Observatory, ESA's XMM-Newton, the Murchison Widefield Array (MWA) in Western Australia and the Giant Metrewave Radio Telescope (GMRT) in India-occurred 390 million light-years from Earth. It was so powerful that it punched a cavity in the cluster plasma, the super-hot gas surrounding the black hole. The find was made with Phase 1 of the MWA, when the telescope had 2048 antennas pointed towards the sky. Scientists dismiss the idea that it could be caused by an energetic outburst, as it was too large. The explosion was initially recorded by the Chandra Observatory in 2016, however, the results were dismissed as astronomers believed a cavity of that magnitude was impossible.

Lead author of the study *Dr. Simona Giacintucci* from the Naval Research Laboratory in the US said the blast was similar to the 1980 eruption of Mount St. Helens, which ripped the top off the mountain. The difference is that one could fit 15 Milky Way galaxies in a row into the crater this eruption punched into the cluster's hot gas. The study was published in *The Astrophysical Journal*.
Source: *https://www.zmescience.com/ science/largest-explosion-since-big-bang-054235423/?utm_source =feedburner&utm_medium=email&utm_campaign=Feed%3A+zmescience+%28ZME+Science%29*

4.3 STATICA: A novel processor that solves a notoriously complex mathematical problem

Prof. Masato Motomura, along with the team of researchers from Tokyo Institute of Technology and in collaboration with Hitachi Hokkaido University Laboratory and the University of Tokyo has now designed a novel processor architecture to specifically solve combinatorial optimization problems expressed in the form of an Ising model.

Combinatorial optimization consists of locating an optimal object or solution in a finite set of possible ones. Such problems manifest shows up in finance as portfolio optimization, in logistics as the well-known traveling salesman problem, in machine learning, and in drug discovery. However, current computers cannot cope with these problems when the number of variables is high, making specialized processor architectures very important.

The Ising model was originally used to describe the magnetic states of atoms (spins) in magnetic materials. However, this model can be used as an abstraction to solve combinatorial optimization problems because the evolution of spin, which tends to reach the so-called lowest energy state, mirrors how an optimization algorithm searches for the best solution.

The proposed processor architecture, called STATICA, is fundamentally different from existing processors that calculate Ising models, called annealers which only consider spin interactions between neighboring particles. This allows for faster calculation, but limits their possible applications. In contrast, STATICA is fully connected and all spin-to-spin interactions are considered. While STATICA's processing speed is lower than those of similar annealers, its calculation scheme is better, as it uses parallel updating using what is known as stochastic cell automata. Instead of calculating spin states using the spins themselves, STATICA creates replicas of the spins and spin-to-replica interactions are used, allowing for parallel calculation. This saves a tremendous amount of time due to the reduced number of steps needed.

STATICA offers reduced power consumption, higher processing speed, and better accuracy than other annealers. Further refinements will make STATICA an attractive choice for combinatorial optimization.
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2. *https://www.eurekalert.org/pub_releases/2020-02/tiot-san022620.php*

4.4 A Monograph by Anantharam Raghuram and Günter Harder published in the Annals of Mathematics Studies:

A Monograph titled **A. Eisenstein Cohomology for GL_N and the Special Values of Rankin-Selberg L-Functions by Anantharam Raghuram and Günter Harder (AMS-203)** has been published in the Annals of Mathematics Studies, by the Princeton University Press, Princeton NJ, 2020 (xi + 220 pp). (See Photo on inner back cover page)

The Annals of Mathematics Studies is one of the oldest and most prestigious series, and some of the greatest mathematical tracts of the last century have been published there. Prof. A. Raghuram from IISER, Pune, is only the fourth person from India or of Indian origin to publish in that series. The previous ones were: the late Prof. K. Chandrasekharan in 1950 (*Fourier Transforms (AMS-19), with Salomon Bochner*), the late Prof. Shreeram Abhyankar from Purdue University in 1959 (*Ramification Theoretic Methods in Algebraic Geometry (AMS-43)*), and Prof. Gopal Prasad from University of Michigan in 2016 (*Classification of Pseudo-reductive groups, (AMS-191) with Brian Conrad*).

This book by A. Raghuram and G. Harder studies the interplay between the geometry and topology of locally symmetric spaces, and the arithmetic aspects of the special values of L-functions. The authors study the cohomology of locally symmetric spaces for GL(N) where the cohomology groups are with coefficients in a local system attached to a finite-dimensional algebraic representation of GL(N). The image of the global cohomology in the cohomology of the Borel-Serre boundary is called Eisenstein cohomology, since at a transcendental level the cohomology classes may be described in terms of Eisenstein series and induced representations. However, because the groups are sheaf-theoretically defined, one can control their rationality and even integrality properties. A celebrated theorem by Langlands describes the constant term of an Eisenstein series in terms of automorphic L-functions. A cohomological interpretation of this theorem in terms of maps in Eisenstein cohomology allows the authors to study the rationality properties of the special values of Rankin-Selberg L-functions for GL(n) x GL(m), where $n + m = N$. The authors carry through the entire program with an eye towards generalizations. This book should be of interest to advanced graduate students and researchers interested in number theory, automorphic forms, representation theory, and the cohomology of arithmetic groups.

While giving his impressions on this monumental work Prof. Raghuram said: "Prof. Harder is one of the towering mathematicians of our times. He is often credited with establishing the subjects of Algebraic Geometry and Automorphic Forms in post-war Germany. So, I was invited to Harder's 70th birthday conference at Oberwolfach, Germany, where to my great delight, he showed some interest in whatever I was doing at that point of time on the special values of L-functions. We talked extensively; especially during the mid-conference-hike in the Black Forest of Germany, which is probably the most memorable and influential walk of my life! It became clear that there was something we could attempt. I met him again many times at Max Planck Institute in Bonn, and also at the Erwin Schrödinger Institute in Vienna. In about a year after the Oberwolfach meeting, a program of thought was becoming clear in our heads. It then took us another 7 years of real hard work to nail down all the details. The manuscript took about 2+ years of refereeing. Harder keeps telling me that the longest journey starts with but the first step. Anyway, I am thrilled that an arduous decade-long journey will be culminating in a volume in the Annals of Mathematics Studies".

Source: *https://press.princeton.edu/books/hardcover/9780691197883/eisenstein-cohomology-for-gln-and-the-special-values-of-rankin.*

4.5 Awards

4.5.1 Abel Prize 2020 awarded to Hillel Furstenberg and Gregory Margulis

On March 18, the Norwegian Academy of Science and Letters named two mathematicians Hillel Furstenberg, 84, and Gregory Margulis, 74, (in top and bottom photograph, respectively) as the winners of the 2020 Abel Prize, named after the Norwegian mathematician Niels Henrik Abel. The citation for the prize lauds the two mathematicians for their fundamental contributions playing a far-reaching role in diverse areas like number theory, combinatorics, geometry, group theory and probability theory. The two awardees split the prize amount of 7.5 million Norwegian kroner, equivalent to over US $ 7,00,000.

Dr. Furstenberg was born in Berlin in 1935, and graduated from Yeshiva University, New York, in 1955. He received his Ph.D. in 1958 from Princeton University, for his thesis on Prediction theory, written under the supervision of Salomon Bochner, involving a novel approach via ergodic theory. His work unraveled an intricate relationship between behaviour of random walks on a group, with the structure of the group, and gave rise to the notion of the 'Furstenberg boundary'. After working for some years in the US, in 1965 he settled at the Hebrew University of Jerusalem, Israel.

Furstenberg's ground-breaking contributions include: The notion of "disjointness" of ergodic systems, akin to the concept of coprimality in the case of integers; showing that the classical horocycle flow associated with a compact surface of constant negative curvature admits a unique invariant measure, up to scaling, which led to enormous follow up activity in the ensuing decades, including in India by S. G. Dani and later by Nimish Shah; a strikingly novel, and substantially simpler, proof of a renowned theorem of Szemeredi (Abel Prize laureate of 2012), which states that any subset of the integers having positive upper density contains arbitrarily long arithmetic progressions. Furstenberg has also received many honors including the Israel Prize and the Wolf Prize in mathematics, and is an elected member of the Israel Academy and the American Academy of Arts and Science.

Dr. Margulis, who shares the Abel Prize, was born in Moscow in 1946. He attended Moscow State University, and received his Ph.D. in 1970 under the supervision of Yakov Sinai (Abel Prize laureate of 2014). His dissertation revealed a particularly original mind: he constructed a measure-now called the Bowen-Margulis measure-which has enabled discovery of new properties in geometry of hyperbolic spaces. He worked at the Institute for Problems in Information Transmission, in Moscow, until 1991, when he settled at Yale University, New Haven, USA. He was awarded the Fields Medal in 1978, at the age of 32 for his work on lattices in Lie groups, notably his arithmeticity and superrigidity theorems. The superrigidity proof demonstrated novel applications of ergodic theory, establishing powerful new methods that became very influential in many fields. Margulis produced the first known systematic construction of expander graphs using ideas from group representation theory. This ground-breaking discovery has had many applications in the design of computer networks, as well as in areas such as error correction algorithms, random number generators and cryptography. He is renowned for proving the Oppenheim conjecture, going back to 1929, on values of quadratic forms at integral points, following an idea proposed by M. S. Raghunathan, in 1984. His subsequent work on related problems, in which S. G. Dani (an editor of the Bulletin) also contributed, has now given rise to a flourishing new area, known as homogeneous dynamics; the works of three recent Fields medalists-Elon Lindenstrauss, Maryam Mirzakhani and Akshay Venkatesh-all build on Margulis's earlier ideas. Margulis is also a winner of the Lobachevsky Prize and the Wolf Prize, and an elected member of the U.S. National Academy of Sciences.

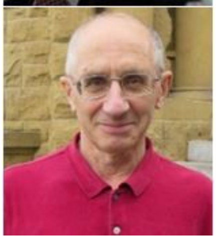
Furstenberg and Margulis have made a huge impact on mathematics of our time, and they continue to be much sought after for inspiration and guidance. It is also a pleasure to mention here that both of them have had considerable contact with mathematics from India, and in particular have visited the TIFR, Mumbai and Furstenberg also gave a plenary lecture, on the work of Elon Lindenstrauss, one of the Fields Medal awardees at the Congress, in ICM-2010 at Hyderabad, India.

Sources:

1. *https://www.abelprize.no/c76018/binfil/download.php?tid=76027*
2. *https://www.abelprize.no/c76018/binfil/download.php?tid=76026*
3. *https://www.google.com/url?rct=j&sa=t&url= https://www. nytimes.com/2020/03 / 18/ science / abelprize -mathematics. html&ct=ga&cd = CAEYACoUMTIzNTQ2NDE2MzQ1MDE1NjM1 OD EyGm ZlMDg 1ZDc0NjJiMTgzYmY6Y29tOmVuOlVT&usg = AFQjCNGBH9V gycQT2H TF 30rKmvh _Zmlx5w*

4.5.2 Wolf Prize 2020 awarded to Simon Donaldson and Yakov Eliashberg

Along with the Fields Medal and Abel Prize, the Wolf Prize is considered one of the most prestigious awards in mathematics. *Sir Simon Donaldson*(top) and *Yakov Eliashberg*(bottom) will receive the Wolf Prize 2020 for Mathematics for their contribution to differential geometry and topology. They will share the $1,00,000 award prize money. The prize is awarded in Israel each year to outstanding scientists and artists from around the world for "achievements in the interest of mankind and friendly relations among people".



Donaldson is renowned for his work on the topology of smooth (differentiable) four-dimensional manifolds and the Donaldson-Thomas theory. His research includes a unique combination of novel ideas in global non-linear analysis, topology, algebraic geometry, and theoretical physics. As a graduate student at the University of Oxford, Donaldson made a discovery that earned him international esteem and stunned the mathematical world. He showed that there are phenomenon in 4-dimensions which have no counterpart in any other dimension. This went against the accepted understanding at the time. He is renowned for applying physics to solve mathematical problems, a remarkable reversal of the usual interaction of these two subjects. Many of his techniques are now prevalent throughout many branches of modem mathematics and physics.

Eliashberg is one of the founders of symplectic and contact topology, a field that arose in part from the study of various classical phenomena in physics that involve the evolution of mechanical systems, such as springs and planetary systems. The emergence of symplectic and contact topology has been one of the most striking long-term advances in mathematical research over the past four decades. Eliashberg's work played a key role in this evolution.

Sources:

1. *https://www.imperial.ac.uk/news/194737/simon-donaldson-awarded-wolf-prize-mathematics/*
2. *https://news.stanford.edu/2020/01/17/yakov-eliashberg-awarded-wolf-prize-mathematics/*

4.6 Obituary

4.6.1 Abel Prize winner & great mathematician Louis Nirenberg passes away at the age of 94

Louis Nirenberg, a mathematician who puzzled out equations describing the vibrating of strings, the flow of heat and the movement of water commonly used by physicists and engineers, and who shared the 2015 Abel Prize died on Jan. 26, 2020 at the age of 94.

One of the world's most cited and productive mathematicians, Nirenberg was also among the most collaborative. His work continued to make waves until he was well into his eighties, and reshaped how mathematicians understand and study dynamical systems, from cells to markets. He shared the 2015 Abel Prize with John Nash, (whose life was depicted in the movie "*A Beautiful Mind*"). Nirenberg spent an illustrious seven decades at New York University (NYU) and transformed the field of partial differential equation (PDE). One famous and still unresolved questions in which Nirenberg's insights have been significant asks whether the equations governing the movement of water from a given initial state are always compatible with a smooth flow.

An expert of approximation, Nirenberg was renowned for manipulating inequalities that govern the properties of unknown functions. His works catalyzed large bodies of research, from general relativity to biology. He gained a reputation for his exceptional insight and taste as a poser of problems that stretched the limits of research in mathematics and beyond.

His career awards included the first Crafoord Prize in 1982, the first Chern Medal of the International Mathematical Union in 2010 and the American Mathematical Society's Bôcher Prize in 1959.

Sources:

1. *https://www.nature.com/articles/d41586-020-00449-y*

2. *https://www.nytimes.com/2020/01/31/science/louis-nirenberg-dead.html*

4.6.2 Katherine Johnson, pioneering NASA mathematician passes away at the age of 101

Katherine G. Johnson, a human 'computer' who broke colour barriers and was essential to early space flight, was a NASA mathematician who helped to send the first American into space and the first astronauts into space. She played a key role in numerous NASA missions during the Space Race, perhaps most notably calculating the trajectory needed to get the Apollo 11 mission to the moon and back.

Johnson's passion was geometry, which was useful for calculating the trajectories of spacecraft. For NASA's 1961 Mercury mission, she knew that the trajectory would be a parabola. Subsequent orbital missions were more complicated, with more variables involving the position and rotation of the Earth, so Johnson used a celestial training device to perform her calculations.

Johnson was tasked with calculating the trajectory for Alan Shepard's historic flight, during which he became the first American to reach space. She also confirmed the trajectory to send the first American into orbit around the Earth. The next challenge was to send humans to the moon, and Johnson's calculations helped sync the Apollo 11 lunar lander with the moon-orbiting command and service module to get the astronauts back to Earth. She also proved invaluable on the Apollo 13 mission, providing backup procedures that helped ensure the crew's safe return after their craft malfunctioned.

Her role in space history was highlighted by Hidden Figures, a book by Margot Lee Shetterly that was made into a 2016 Oscar-nominated film. The release of *Hidden Figures* made Johnson one of the most celebrated black women in space science and a hero for those calling for action against sexism and racism in science and engineering. She later helped to develop the space shuttle program and Earth resources satellite, and she co-authored 26 research reports before retiring in 1986.

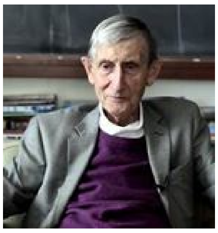
In 2015, President Barack Obama awarded Johnson the Presidential Medal of Freedom, America's highest civilian honor. And in 2016, the NASA Langley facility at which Johnson worked renamed a building in her honor: the Katherine G. Johnson Computational Research Facility. Johnson died on Feb. 24, 2020, at the age of 101.

Sources:

1. *https://www.space.com/katherine-johnson.html*
2. *https://www.ft.com/content/8727b506-5a09-11ea-abe5-8e03987b7b20*

4.6.3 Venerated mathematical physicist Freeman J. Dyson passes away at the age of 96

Freeman J. Dyson, one of the last great theoretical physicists of the World War II era and a mathematical prodigy who left his mark on subatomic physics before turning to messier subjects like Earth's environmental future and the morality of war, died on Feb 28, 2020 at the age of 96. He was the super specialist who applied his mathematical brain to nuclear magic, quantum physics, space travel, and many more branches of science.



He earned a ton of awards, which included the Max Planck Medal (1969), the Harvey Prize (1977) and the Wolf Prize in Physics (1981) and was a professor emeritus at Princeton, and a member of various scientific organizations.

His biggest contribution was uniting mathematical formulations describing interactions of subatomic particles with the wavy lines of Feynman diagrams. He also came up with the additive number theory technique dubbed Dyson's transform, star-harvesting Dyson spheres that featured in *Star Trek*, and more. But it was as a writer and technological visionary that he gained public renown. He imagined exploring the solar system with spaceships propelled by nuclear explosions and establishing distant colonies nourished by genetically engineered plants.

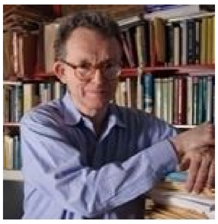
As a young graduate student at Cornell in 1949, Dyson wrote a landmark paper-"The Radiation Theories of Tomonaga, Schwinger, and Feynman"-which depended the understanding of how light interacts with matter to produce the physical world. The theory introduced in the paper, called quantum electrodynamics, or QED, ranks among the great achievements of modern science. Dyson was also a fierce advocate for nuclear disarmament and dedicated to finding progressive usage of nuclear energy.

Sources:

1. *https://www.dailyprincetonian.com / article/ 2020/ 03/ venerated- mathematical- physicist- and -technology -visionary-freeman-j-dyson-dies-at-96*
2. *https://www.avpress.com/news/mathematical-prodigy-freeman-dyson-dies-at/article_d3b0cf28-5ab5-11ea-b1a7-d7fdafd5e1ee.html*
3. *https://www.theregister.co.uk/2020/02/28/freeman_dyson/*

4.6.4 Richard Dudley, professor emeritus of mathematics passes away at the age of 81

Richard Mansfield Dudley, longtime MIT professor Emeritus of Mathematics who strongly influenced the fields of probability, statistics, and machine learning, died on January 19, 2020 at the age of 81. Dudley served on the MIT mathematics faculty from 1967 until 2015, when he officially retired. Over the course of those 48 years, during which he published over 100 articles as well as numerous books and monographs. He made fundamental breakthroughs in the theory of stochastic processes and the general theory of weak convergence.



Dudley made highly influential contributions to the theory of Gaussian processes and empirical processes. What is now widely known as "Dudley's entropy bound" has become a standard tool of modern research in probability, statistics, and machine learning. Dudley's work also had a transformative impact on the theory of empirical processes.

Dudley was always highly regarded as a graduate mentor throughout his career. He advised 33 Ph. D. candidates (32 at MIT), yielding some 105 academic "descendants". Dudley served the scholarly community as an associate editor (1972-78) and then chief editor (1979-81) of Annals of Probability. He gave a number of distinguished research talks, including an invited talk at the 1974 International Congress of Mathematicians.

Among his honors, Dudley was an Alfred P. Sloan Research fellow from 1966-68 and Guggenheim Foundation Fellow in 1991. In 1993, Dudley was elected a fellow of the American Statistical Association. He was elected fellow of the Institute of Mathematical Statistics, the American Association for the Advancement of Science, and the American Mathematical Society and was selected to be a member of the International Statistical Institute.

Source: *http://news.mit.edu/2020/richard-dudley-mit-mathematics-professor-emeritus-dies-0218*

4.6.5 Henry C. Wente, known for soap bubble curvature research passes away at the age of 83

Henry C. Wente, an American mathematician and University of Toledo's distinguished professor of mathematics who won international renown for his 1986 discovery of the "Wente torus"-an immersed constant-mean-curvature surface whose existence disproved a conjecture of Heinz Hopf, died on January 20, 2020 at the age of 83.

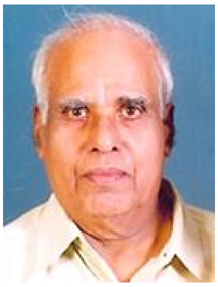
His research focused on the mathematics of soap films, liquid drops in equilibrium, and capillary theory. He received wide attention starting in 1984 when he discovered counterexamples to the then 40-year-old Hopf Conjecture. Wente's counterexample was a torus, which he described as a bulging doughnut shape, that could be stretched, intersecting itself, while keeping a constant mean curvature. Mathematician Heinz Hopf early in the 20^{th} century had said that only spheres-soap bubbles-could keep a constant mean curvature.

Wente worked on the problem for years, largely in secret. He received University of Toledo's first outstanding faculty research award. In 2013 he was named an inaugural fellow of the American Mathematical Society.

Source: *https://www.toledoblade.com/news/deaths/2020/01/25/toledo-blade-obituaries-henry-wente- mathematics -professor/stories/20200124145*

4.6.6 Renowned Mathematician K. S. S. Nambooripad passes away at the age of 84

K. S. S. Nambooripad who has made fundamental contributions to the structure theory of regular



semigroups and instrumental in popularizing the TeX software in India and also in introducing and championing the cause of the free software movement in India passed away on Jan 4, 2020 at the age of 84.

He was with the Department of Mathematics, University of Kerala, since 1976. He served the Department as its Head from 1983 until his retirement from University service in 1995. After retirement, he was associating with the academic and research activities of the Center for Mathematical Sciences, Thiruvananthapuram in various capacities.

Nambooripad axiomatically characterized the structure of the set of idempotents in a regular semigroup. He called a set having this structure a biordered set. The axioms defining a biordered set are quite complicated. However, considering the general nature of semigroups, it is rather surprising that such a finite axiomatization is even possible. He also developed an alternative approach to describe the structure of regular semigroups. This particular work utilizes the abstract theory of cross-connections to provide a useful framework for studying various classes of regular semigroups.

TeX was introduced into Kerala by Nambooripad after a visit to the United States in early 1990s. Nambooripad was the prime catalyst for the formation of Indian TeX Users Group in 1998. He was the inaugural Chairman of the Group.

Source: *https://en.wikipedia.org/wiki/K._S._S._Nambooripad*

□ □ □

5. A tribute to Subhashbhai

M. H. Vasavada
Retired Professor and Head, Dept. of Mathematics. Sardar Patel University, V. V. Nager.

A leading Mathematician from Gujarat, Dr. Subhashchandra Jayantilal Bhatt (popularly known

as Subhashbhai), retired Professor and Head, Department of Mathematics, Sardar Patel University, Vallabh Vidyanagar, passed away on 26th February, 2020, at the age of 71. Subhashbhai met with an accident while driving a scooter on 20th February causing a spinal injury. He went through a successful surgery on 21st; however, a sudden cardiac arrest on 22nd turned out to be fatal and he could not be saved despite untiring efforts of doctors in Zydus hospital, at Anand. Subhashbhai breathed his last in the afternoon of 26th.

Subhashbhai came from a poor family and had his school education in a small village of South Gujarat. But these apparent handicaps did not stop him from developing keen interest in Mathematics and becoming an erudite scholar. He obtained his B.Sc. and M.Sc. degrees from South Gujarat University, Surat, and a Ph.D. degree from S. P. University, Vallabh Vidyanagar.

His commitment to Mathematics was total. For him, doing Mathematics was nothing less than worshipping God. He was an excellent research worker and had published more than one hundred research papers, in highly reputed international journals. His main work was on Banach and Topological Algebras, Operator Algebras and their Applications, and Algebras of Unbounded Operators. He also worked in Non-commutative Geometry, Abstract Harmonic Analysis, Mathematical Physics and Financial Mathematics.

Subhashbhai worked on several collaborative research programmes with experts from IIT, Chennai, Fukuoka University & Fukuoka Institute of Technology, Japan, Ecole Normale Superior, Rabat, Morocco, Mat. Inst. Leipzig University, Germany, and University of Athens, Greece. He delivered lectures based on his research work in several national and foreign institutes of repute.

He bagged several research awards, including Ramanujan Prize-2004 of University of Madras, A Narasingh Rao Gold medal-1980 and Prof. Hansraj Gupta Memorial Award-2010 of Indian Mathematical Society, and Prof. A. R. Rao Foundation Research Award-2013. He also bagged Hari Om Ashram Prerit Bhaikaka Research Prize of Sardar Patel University for several years. He was elected Fellow of Gujarat Science Academy in 1997 and Fellow of Indian Academy of Sciences, Bangalore in 2009.

After receiving Ph.D. degree in 1979, he joined the Department of Mathematics, Sardar Patel University, V. V. Nagar in 1980, became Professor in 1997 and headed the department from 2001 to 2011-till the time of his retirement.

After retirement, from 2011-2017, at one time or other, he held positions of NBHM Visiting Professor, Retired Scientist, DST-PURSE program and UGC Emeritus fellow in Sardar Patel University. His research work continued almost till the end of his life, ending with three communicated papers and two under preparation.

Besides being a dedicated Mathematician, Subhashbhai was an excellent human being who never cared for fame, publicity or money. He was a simple man, simple to the core. Surprisingly, despite his passion for and total involvement in Mathematics, he was never short of time for helping, encouraging and counselling others-be it his friends, students or fellow mathematicians. Spiritually oriented Subhashbhai followed Gita's dictum of "Karmanyevadhikaraste ..." in his life.

He was an obedient son, dedicated husband and a loving father. He is survived by his wife Dr. Sujataben, who retired as a professor of Biology from SPU in 2018 and a daughter Shreema who hopes to follow her father's footprints for her career.

Subhashbhai's departure has been as shocking as it was sudden. It is a great loss to the Mathematics community. In him, we have lost a committed research mathematician, a devoted teacher, a good administrator and a man of impeccable honesty. His memory shall always remain in our hearts.

□ □ □

6. Problem Corner

Vinaykumar Acharya and Udayan Prajapati

Solution to the problem posed in Problem corner of January issue of the TMC Bulletin by Dr. Jagnnath Nagorao Salunke, Former Professor, School of Mathematical Sciences, Swami Ramanand Teerth Marathwada University, Nanded (Maharashtra). M. No. 9420389908 and Email: drjnsalunke@gmail.com

Problem Statement: Find all distinct positive integers $l, m, n > 1$ and $t \epsilon \mathbf{N}$ such that

$$\text{l}! + \text{m}! + \text{n}! = 6^{\text{t}}. \tag{6.1}$$

Solution: Let $l < m < n$ in $\mathbf{N} - \{1\}$, i.e. $2 \leq l < m < n$ and $t \epsilon \mathbf{N}$.
Then $m,\ n \epsilon\ \{3, 4, 5, \cdots\}$ with $m < n$ and both $m!, n!$ are divisible by 6, hence $l \leq 3$.
If $l \geq 5$ then LHS of 6.1 is divisible by 5. But RHS of 6.1 is not divisible by 5, hence $l \leq 4$.
Thus $l = 3$ *or* 4

Case 1: Suppose $l = 3$. Then by 6.1,

$$l! + m! + n! = 6[1 + (m! + n!)/6] = 6^t \tag{6.2}$$

Moreover, $n > m > l = 3$ and $(m! + n!)/6$ is even (divisible by 4) and so $[1 + (m! + n!)/6]$ is an odd integer ≥ 25 Hence, $l \neq 3$.

Case 2: Suppose $l = 4$. Then $n > m \geq 5$ and by 6.1,

$$l! + m! + n! = 24[1 + (m! + n!)/24] = 6^t \tag{6.3}$$

(i) If $m > 5$ then $[1 + (m! = n!)/24]$ is an add integer≥ 241 which contradicts 6.3 for $t\epsilon\mathbf{N}$, since $24 \times p = 6^t$, for odd $p \,\epsilon\, \mathbf{N} \Rightarrow p = 9$ only.

(ii) Thus for $l = 4$, we have $m = 5$ and $n \geq 6$ and by 6.1,

$$l! + m! + n! = 24[1 + 5 + n!/24] = 6^t \tag{6.4}$$

Since $n \geq 6$, $n!/24$ is a multiple of 10 and hence it is a positive integer having last digit 0. Therefore, $[1+5+n!/24]$ is a positive integer having last digit 6 and hence $24[1+5+n!/24]$ is a positive integer having last digit 4, where as 6^t for any $t\epsilon\mathbf{N}$, is a positive integer having last digit 6. This contradicts 6.4. Hence, $m \neq 5$.

From above we can conclude that there does not exist distinct positive integer $l, m, n > 1$ and $t \,\epsilon\, \mathbf{N}$ such that $l! + m! + n! = 6^t$.
Note: $2! + 2! + 2! = 6^1, 3! + 3! + 4! = 6^2$.
We congratulate Prof. Jagnnath Salunke for giving an elegant solution to the above problem.

Problem for this issue- Posed by Mr. Dhruv Bhasin Integrated Ph. D. student IISER Pune.

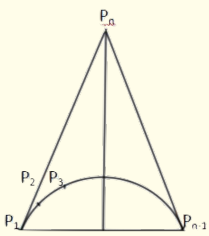


Figure - 1

Suppose n points are arranged as in the given Figure-1, $n-1$ points are equally placed on an arc of a circle of radius r, of length strictly less than πr and n^{th} point is on the perpendicular bisector of the base line. It is placed high so that whole of the arc is inside the triangle formed by the 1st point, the $(n-1)^{\text{th}}$ point and the n^{th} point. What is the number of n-sided simple polygons that can be constructed using the above configuration? A solution, for the case $n = 4$, where we get 3 simple polygons, is given in Figure-2, as an illustration.

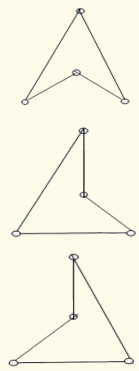


Figure- 2

□□□

7. Opportunities and Academic events

Ramesh Kasilingam

7.1 International Calendar of Mathematical events

June 2020

1. POPL 2021 Call for Papers-deadline Thursday, July 9th, AoE (Michael Greenberg) More information about the research project and the details of the position can be found here: *https://www.universiteitleiden.nl/en/ vacancies / 2020/q1/ 20-150-teaching-phd- position- in- quantitative-systems-and-reasoning-methods.*

2. 8-12 June, 2020, SDIDE 2020-6th Workshop on Stability and Discretization Issues in Differential Equations, Budapest, Hungary. *http://sdide2020.elte.hu/index.html.*

3. 15-24 June, 2020, FoCM 20 Foundations of Computational Mathematics (FoCM) 2020, Vancouver, BC, Canada. *http:://focm-society.org/2020/index.html.*

July 2020

1. July 6-18, 2020, *5th EU/US Summer School + Workshop on Automorphic Forms and Related Topics*, Sarajevo, Bosnia and Herzegovina.

2. July 13-17, 2020, *Number Theory Conference in Honour of Kálmán Győry-János Pintz-András Sárközy*, Debrecen, Hungary.

3. July 20-25, 2020, *19th International Fibonacci Conference*, University of Sarajevo, Bosnia and Herzegovina.

4. July 28-29, 2020 "2^{nd} International Conference on Big Data, Data Mining and machine learning". Berlin, Germany Gavin Conferences, 5911 Oak Ridge Way, Lisle, IL 60532, USA. *datamining.gavinconferences.com/.*

August 2020

1. August 02-06, 2020, CCP 2020-XXXII IUPAP Conference on Computational Physics, Coventry, United Kingdom. *http://ccp2020.complexity-coventry.org/.*

2. August 20-21, 2020, Connections for Women: Decidability, definability and computability in number theory, Mathematical Sciences Research Institute, Berkeley, California. *www.msri.org/workshops/913.*

3. August 24-28, 2020, *ELAZ conference on elementary and analytic number theory (ELAZ 2020)*, 2020, Adam Mickiewicz University, Poznań, Poland.

4. August 24-28, 2020, Introductory Workshop: Decidability, definability and computability in number theory, Mathematical Sciences Research Institute, Berkeley, California. *www.msri.org/workshops/914.*

5. August 25-28, 2020, 9th International Eurasian Conference on Mathematical Sciences and Applications, International Balkan University, Skopje, North Macedonia. *iecmsa.org.*

6. August 26-28, 2020, The 7th International Conference on Control and Optimization with Industrial Applications-COIA-2020 Baku, Azerbaijan. *www.coia-conf.org/en/.*

7. August 26-29, 2020, The 3rd Mediterranean International Conference of Pure & Applied Mathematics and Related Areas (MICOPAM 2020)-Akdeniz University, Antalya, TURKEY. *micopam.akdeniz.edu.tr/.*

8. August 28-29, 2020, 2nd International Conference on AI and Robotics-Holiday Inn Atrium, Singapore. *larixconferences.com/ai-robotics/.*

9. August 30-September 2, 2020, Computational Techniques & Applications Conference CTAC)-UNSW Sydney, Australia. *www.ctac2020.unsw.edu.au/ https://www.newton.ac.uk /event/kahw03.*

7.2 Opportunities

1. Chair Professor Position in Statistics, Academy of Mathematics and Systems Science, Chinese Academy of Sciences Beijing, China *https://jobs.imstat.org/job/chair-professor-position-in statistics/53542035/?utm_source = JobFlash & utm_medium = Email&utm_campaign=JobFlash-3%2B31%2C%2B2020.*

2. SCIENCE AND ENGINEERING RESEARCH BOARD (SERB) announces short-term projects in the following areas, preferably with multidisciplinary efforts under its MATRICS program: (1) Mathematical Modeling of COVID-19 Spread. (2) Statistical Machine Learning, Forecasting and Inferences from Pandemic Data. (3) Focused Algorithms for Infectious Disease Modeling. (4) Quantitative Social Science Approaches for Epidemiological Models.

   Project duration would be one year with a fixed grant of Rs. 5 lakh plus overhead. Proposals should be submitted through SERB online portal (www.serbonline.in) in MATRICS format.

   Project proposals will be evaluated on first-come basis, with a last date of submission as April 30, 2020.

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8. TMC Activities

Communicated by Prof. S. A. Katre

The Regional Programmes in 2019-20: In all 14 Regional Programmes were held in different parts of the country during April 2019 to March 2020, by TMC in collaboration with regional Institutions and societies. The details of 4 of these programmes were given in Vol. 1, Issue 2 and 3 of TMC Bulletin. We give below brief reports of the rest of these programmes, details of which are available on TMC website *www.themathconsortium.in*.

1. The Mathematics Consortium and Dayalbagh Educational Institute, Agra (Institutional Member of TMC) organised a workshop in Mathematics entitled "Advances in Mathematical Sciences and Applications" during 18-20 April, 2019 at DEI, Agra. The Workshop was attended by 75 college teachers and students in Mathematics. Lectures were held on Modelling, Optimization, Algebraic Geometry, Topology and Ancient Indian Mathematics.

2. National Level Workshop on Advanced Mathematics (WAM-2019) for College and University Teachers and Research Scholars was organized in collaboration with TMC by Calcutta Math. Society (a Society Member of TMC), during June 13-22, 2019 at the premises of CMS, Salt Lake City, Kolkata. Every day (excluding Sunday) there were 3 lecture sessions of one and half hour duration on diverse topics in Mathematics such as Complex System, Geometry (Geometric Calculus), Factorization of polynomials, Bezier curves and their application to font designing, Cryptography, Sum of Squares, PDE & Distribution Theory, Mathematical Modelling, Algebraic Geometry, Complex Analysis and Differential Geometry.

3. National Workshop on "Computational Mathematics" 2019 was organised at Devi Ahilya University, Indore (an Institutional Member of TMC) during 29-30 November 2019. The workshop was co-sponsored by TMC under Regional Programmes. There were 40 teacher participants and 8 one-hour talks were held on Representations of Finite Groups over Arbitrary Fields, Introduction to Bezier Curves and their applications to font designing, Singular value decomposition(SVD), Numerical Study on Air Pollution in Annular Regions with a Point Source like Chimney, Stochastic Modelling of High Frequency Intra-day Returns: Emergence of Cubic Power Law, Some models for Granular Flow and Crowd Dynamics, Accuracy and Stability Analysis of Gaussian Elimination and Introduction to Quantum Computing.

4. Gwalior Academy of Mathematical Sciences (a Society Member of TMC) organised 8th International Conference and 24th Annual conference of GAMS (ICGAMS-2K19) on "Mathematical and Computational Data Science with Applications" and a Symposium on "Advances

in Special Functions and Applications" during 13-15 Dec. 2019. A two days Pre-Conference workshop on "Mathematical Modelling & Simulation" was organised under TMC Regional Programmes during 11-12 Dec. 2019 at VIT, Bhopal University, M. P.-466114. 7 lectures of one or one and half hour duration (total 9 hours) were organised.

5. State Level Symposium on Number Theory and Computing, organised by Dept. of Mathematics, School of Sciences-II, Jain Deemed-to-be-University, Bangalore, was held during January 6-7, 2020. It was supported by TMC under Regional Programmes. In all 7 talks were organised on the topics: Image inpainting problems using difference equations and related numeric, the congruent number problem, Diophantine equations, Ring of polynomials over rationals, Applications of affine and projective algebraic curves, counting of rational points on a system of equations, Primality testing, Cryptosystems, and Machine Learning. Around 55 students participated in the symposium. There were also posters by students. (See photo on Inner back cover page.)

6. Department of Engineering Sciences at AISSMS Institute of Information Technology (IoIT), Pune, organised a Workshop on Applied Mathematics, during 9-10 Jan. 2020 in Association with TMC under Regional Programmes on the occasion of Silver Jubilee celebrations of IAIAM. Founder President of IAIAM, Prof. T. H. Date was felicitated at the Inaugural programme at the hands of Shri Prataprao Pawar, Chairman of Sakal Group. The workshop was attended by 25 teacher participants. In all 7 talks and 1 tutorial were organised on the topics: Eigen Values and Eigen Vectors, Cryptography, Machine Learning, Quantum Computing Fundamentals and applications, Applications of Linear Algebra (Google Page Rank Algorithm, font designing), Statistics and R-Programming, Linear Algebra in MR Image Reconstruction.

7. Workshop in Python Programming for B.Sc. (Computer Science) and B.Sc. (Mathematics) Teachers of S. P. Pune University was organised by Modern College of Arts, Science and Commerce (Autonomous), Pune-411005 (an Institutional Member of TMC), during 17-18 January 2020. It was arranged as a combined effort of TMC (Regional Programmes), IAIAM, Lokmanya Tilak Chair (SPPU), and Modern College. 5 hours of lectures with hands on sessions were organised each day. There were 65 teacher participants from various colleges.

8. Parvatibai Chougule College of Arts and Science, Margaon, Goa organised a 4-day National Level Seminar in Mathematics under D. B. Wagh Lecture Series held in honour of late first Principal of the college. The activity was held during 17-20 February 2020 under regional programs of TMC. In all 23 lectures were organised on the topics: Theory of Differential Forms and de Rham Cohomology (7 lectures), Fourier Analysis on Groups (7 lectures), Plane Geometry (7 lectures), Some Applications of Plane Geometry (2 lectures).

9. One Day Mathematics Workshop for Students at MIT World Peace University, Pune, was organised on 4th March 2020 in collaboration with IAIAM and TMC. Three lectures were organised on the topics: Industrial Applications of Mathematics and Statistics, Linear Algebra and Applications, Machine Learning which were followed by interactive session.

10. A workshop on "Algebra, Number Theory and their Applications" was organised on the occasion of International Mathematics Day during 13$^{\text{th}}$-14$^{\text{th}}$ March, 2020 by Department of Mathematics, Ramakrishna Mission Vivekananda Centenary College, Rahara, Kolkata-700118 (an Institutional Member of TMC) in collaboration with TMC. In all 8 lectures of 1 hour and 15 min. duration were organised on the topics: Linear Algebra and its Applications (4 lectures) and on Number Theory and its Applications (4 lectures). 100 students participated in the workshop.

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4.4: Prof. A. Raghuram & Prof. G. Harder at the Mathematisches Forschungsinstitut Oberwolfach, Germany in May, 2019

Jain University Bengaluru workshop
1st row - Prof. Ramana Raju, Dr Asha Rajiv, Prof. Dilip Patil, Prof. Venky Krishnan

Leonhard Euler (15 April 1707-18 September 1783)

A Swiss mathematician, Physicist, Astronomer, Logician, Geographer and Engineer.
Significant contributions to: Infinitesimal Calculus, Number theory, Mechanics, Fluid dynamics Optics and Music theory.

Rudolf Lipschitz (14 May 1832-7 October 1903)

A German mathematician.
Significant contribution to: Mathematical Analysis, Differential Geometry, Number theory, Algebra, Classical Mechanics.

Augustus De Morgan (27 June 1806–18 March 1871)

A British mathematician and Logician.
Significant contributions to: Logic, Complex numbers and Mathematical symbolism. Formulated De Morgan laws; Introduced the term Mathematical Induction; Gave Geometric interpretation of properties of complex numbers.

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