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A page from Golādhyāya of the Siddhānta Śiromaṇī

Golayantra Nalakayantra Phalakayantra

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Contents

***About the Cover Page**: The top image on the front cover is of a page from Golādhyāya of the Siddhānta Śiromaṇī. The bottom images from left to right are astronomical instruments called Golayantra, Nalakayantra, Phalakayantra respectively.*

A Golayantra (Armillary sphere) is a model of objects in the sky, consisting of a spherical framework of rings, centered on the Earth or the Sun, that represent lines of celestial longitude and latitude and other astronomically important features. (https://www.scienceofgadgets.com/post/how-armillary-sphere-works)

Nalakayantra[1] or Nalikāyantra (Tube instrument) is an optical instrument, having a long tube or tubes, used to watch distant objects (https://www.wisdomlib.org/definition/nalikayantra)

The Phalakayantra[2] (Board Instrument) is an instrument devised by Bhāskara himself to measure the hour angle, described in the yantrādhyāya, the chapter on instruments in Golādhyāya (See Article 3 by Prof. Sriram- page 21 of this issue).

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1, 2. The images of these yantras have been reproduced here with the permission of the Hindustan book agency.

From the Editors' Desk

The beauty of mathematics often lies in mysterious and profound connections between seemingly distinct areas of Mathematics. One of the many such instances is the connection between Number Theory and Quantum Mechanics.

The Riemann Zeta function in number theory is deeply connected with the distribution of prime numbers. The non-trivial zeros of the function are conjectured to lie on the critical line with real part 1/2 (the Riemann Hypothesis). Surprisingly, the statistical distribution of these zeros appears to be the same as the energy levels of heavy nuclei in quantum mechanics (specifically, random matrix theory). This suggests a deep, yet not fully understood, link between the fundamental building blocks of numbers and the behavior of matter at the atomic level. The partition functions of certain string theories in Physics are related to modular forms in Number theory, hinting at a deep mathematical structure underlying the physical universe at its most fundamental level.

Brownian motion, the random movement of particles, can be described by stochastic differential equations (SDE). Remarkably, the probability distribution of a Brownian particle at a given time satisfies the heat equation, a fundamental partial differential equation in physics and engineering. This connection allows us to use the tools of analysis (like PDEs) to understand probabilistic phenomena and vice versa. For example, the Black–Scholes equation (a well-known SDE) estimates the value of the financial instrument called option, over time in terms of the price of the underlying asset.

One of the news items in this Issue describing current status of Hilbert's 10^{th} problem, reveals profound connections among Gödel's Impossibility theorem in Mathematical Logic, Halting problem of Turing machines in computer Science, and impossibility of determining whether a given Diophantine equation has integer solutions in number theory.

In the opening article, Prof. S. D. Adhikari discusses some early Ramsey-type results, demonstrating that when a large structure is partitioned into finitely many parts, at least one of these parts retains certain regularity properties of the original structure, in the context of r-colouring of sets.

In Article 2, Dr. Sagnik Chakraborty gives a self-contained proof of the simplicity of the alternating groups based on elementary considerations of orders of elements, avoiding intricate manipulations with the class equation.

There are various coordinate systems on a sphere associated with a positioning of celestial object. While the relations among them can be obtained using modern spherical trigonometry, one also finds a detailed explanation of these in the twelfth century Indian mathematician Bhāskarācārya's magnum opus, Siddhāntaśiromaṇi (1150 CE). In Article 3, Prof. M. S. Sriram provides an exposition on this theme.

In Article 4, Dr. D. V. Shah gives an account of significant developments in the Mathematical world during recent past, including updates on 'The Zero Height Conjecture', Hilbert's 10th problem, Descartes Circle Theorem, 'McKay conjecture' and 'Kakeya conjecture'. The article also includes a brief write-up on important contributions of awardees of Leibniz Prize, and the 2025 Leroy P. Steele Prizes in three different categories.

In Article 5, Prof. Dani reviews two recently published papers in History of Mathematics, one by Dipak Jadhav and the other by Adrian Rice.

In the Problem Corner, Dr. Udayan Prajapati presents a solution to one of the two problems posed in the January 2025 issue. Two new problems are also posed for our readers. Dr. Ramesh Kasilingam gives a calendar of academic events, planned during July, 2025 to November, 2025, in Article 7.

We are happy to bring out this forth issue of Volume 6 in April, 2025. We thank all the authors, all the editors, our designers Mrs. Prajakta Holkar and Dr. R. D. Holkar, and all those who have directly or indirectly helped us in bringing out this issue on time.

Chief Editor, TMC Bulletin

1. Degree of Regularity of a Linear Diophantine Equation

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1.1 Introduction

For a positive integer r, an *r-colouring* of a set S is a map $\chi : S \to \{1, \dots, r\}$. If s is an element of S, then $\chi(s)$ is called the colour of s. A set $T \subset S$ is called *monochromatic* with respect to a colouring χ if χ is constant on T.

One observes that writing $S = \chi^{-1}(1) \cup \chi^{-1}(2) \cup \cdots \cup \chi^{-1}(r)$, an r-colouring of a set S is nothing but a partition of S into r parts.

In what follows, we shall be using the notation $[n] = \{1, 2, \dots, n\}$ and for a set S, $\binom{S}{k}$ will denote the collection of k-subsets of S. The set of integers and the set of positive integers will respectively be denoted by $\mathbb{Z}$ and $\mathbb{Z}^+$.

In the next section, we will discuss some early Ramsey-type theorems. The existence of regular substructures within general combinatorial structures is a fundamental phenomenon that characterizes Ramsey theory. Most commonly, we encounter results demonstrating that when a large structure is partitioned into finitely many parts, at least one of these parts retains certain regularity properties of the original structure. Additionally, some Ramsey-theoretic results establish that sufficiently large substructures exhibit specific regularities.

In Section 1.3, we shall take up our main theme.

1.2 Some early Ramsey-type theorems

Schur [18] proved the following result in 1916, making it one of the earliest results in the field.

Theorem 1. (Schur's Theorem) *Given a positive integer r, there is a positive integer $S(r)$, such that for any r-colouring of $[S(r)]$, $\exists$ a monochromatic subset $\{x, y, z\}$ of $[S(r)]$ such that $x + y = z$. (The situation is described by saying that the equation $x + y = z$ has a monochromatic solution in $[S(r)]$.)*

By a Compactness argument (see [11], for instance), the above result is equivalent to the following.

Theorem 2. *Given a positive integer r, for any r-colouring of $\mathbb{Z}^+$, the equation $x + y = z$ has a monochromatic solution in $\mathbb{Z}^+$.*

The classical Ramsey theorem [15], which appeared in 1930 and was later rediscovered by Erdős and Szekeres [7] in 1935, can be viewed as a generalization of the pigeonhole principle. The theorem originally appeared as a lemma in the above mentioned foundational work [15] on Mathematical logic.

Theorem 3. (Ramsey's Theorem) *Given positive integers $k, r, l(\geq k)$, there exists a positive integer $n = n(k, r, l)$ such that for any r-colouring of $\binom{[n]}{k}$, there is an l-subset $L = \{c_1, \dots, c_l\} \subset [n]$ such that the elements of $\binom{L}{k}$ are of the same colour.*

After the result of Ramsey was rediscovered by Erdös and Szekeres, the branch of combinatorics known as Ramsey Theory developed, and with hindsight, we can now see the unifying features of the early Ramsey-type theorems, several of which appeared before Ramsey's theorem, which are seemingly unrelated.

We observe that Theorem [1] can be deduced from Theorem [3].

Take $S(r) = n(k, r, l)$, where $n(k, r, l)$ is as in Theorem [3]. Now, any colouring χ on $[S(r)]$ induces an r-colouring χ^* of $\binom{[S(r)]}{2}$:

$$\chi^*(\{i, j\}) = \chi(|i - j|), i \neq j \in [S(r)].$$

By the definition of $S(r)$, $\exists\ \{a,b,c\} \subset [S(r)]$, $a<b<c$ such that

$$\chi^*(\{a,b\}) = \chi^*(\{b,c\}) = \chi^*(\{c,a\}),$$

that is,

$$\chi(b-a) = \chi(c-b) = \chi(c-a).$$

However, $(b-a)+(c-b)=(c-a)$. Hence the colours must coincide.

□

Another Ramsey-type theorem which appeared before Ramsey's theorem is the theorem of van der Waerden [20] which led to many interesting developments in combinatorics and number theory.

Theorem 4. (van der Waerden's Theorem) *Given $k,r \in \mathbb{Z}^+$, there exists $W(k,r) \in \mathbb{Z}^+$ such that for any r-colouring of $[W(k,r)] = \{1,2,\cdots,W(k,r)\}$, there is a monochromatic arithmetic progression (A.P.) of k terms.*

The following statement can be established by applying induction on the number of colours in van der Waerden's theorem:

Theorem 5. *Given $k,r,s \in \mathbb{Z}^+$, there exists $N=N(k,r,s) \in \mathbb{Z}^+$ such that for any r-colouring of $[N]$, there are $a,d \in \mathbb{Z}^+$ such that the set*

$$\{a, a+d, \cdots, a+kd\} \cup \{sd\} \subset [N]$$

is monochromatic.

Remark 1. *Taking $s=1$ in the above, a monochromatic set $\{a, a+d,\} \cup \{d\}$ already implies Schur's theorem. We shall see later that a much stronger statement follows from the above theorem.*

1.3 A theorem of Rado and the notion of degree of regularity

From Schur's theorem and a special case of van der Waerden's theorem one sees that for any finite colouring of $\mathbb{Z}^+$, there are monochromatic solutions of the following equations:

$$\begin{aligned} x+y &= z, \\ x+z &= 2y. \end{aligned}$$

One is naturally led to the question that given an equation $c_1x_1+\cdots+c_nx_n=0, \quad c_i(\neq 0) \in \mathbb{Z}$, when does it have a monochromatic solution $(x_1, \cdots, x_n)$.

In fact, successful investigations of Rado ([12], [13], [14]) provided necessary and sufficient conditions for a system of homogeneous linear equations over $\mathbb{Z}$ to possess monochromatic solutions in any finite colouring of $\mathbb{Z}^+$. Here we state the following abridged version.

Theorem 6. (Rado) *Given an equation*

$$c_1x_1+\cdots+c_nx_n=0, \qquad c_i(\neq 0) \in \mathbb{Z},$$

it has a monochromatic solution $(x_1, \cdots, x_n)$, where x_i's may not be distinct, in $\mathbb{Z}^+$ with respect to any finite colouring if and only if some non-empty subset of $\{c_1, \ldots, c_n\}$ sums to zero.

Remark 2. *To prove that the condition in the above theorem is sufficient, one may use (see [11], for instance) Theorem [5]. It should be noted that from the general version of Rado's theorem (which we are not taking up here), van der Waerden's theorem follows (see [11]). For necessity, one uses the so called super modulo colour S_p (see [11]).*

Definition. If an equation $L : c_1x_1 + \cdots + c_nx_n = 0$ over $\mathbb{Z}$ has a monochromatic solution in $\mathbb{Z}^+$ with respect to any finite colouring of $\mathbb{Z}^+$, it is called *regular* over $\mathbb{Z}^+$.

One observes that if the equation (L) is not regular, then there is a super modulo colour S_p for which (L) is not regular. However, S_p is a $(p-1)$-colouring and depending on the coefficients, p has to be chosen large.

Rado made the conjecture that there is a function $r : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that given any equation $L : c_1x_1 + \cdots + c_nx_n = 0$ with integer coefficients which is not regular over $\mathbb{Z}^+$, there exists a partition of $\mathbb{Z}^+$ into at most $r(n)$ parts with no part containing a solution to the equation.

We state it for a single homogeneous equation as it has been proved by Rado [13] that if the conjecture is true for a single equation, then it is true for a system of finitely many linear equations, and Fox and Kleitman [8] have shown that if the conjecture is true for a linear homogeneous equation, then it is true for any linear equation. This conjecture is known as *Rado's Boundedness Conjecture*.

The first nontrivial case of the conjecture has been proved by Fox and Kleitman [8] where it was shown that $r(3) \leq 24$.

Definition. Given $n \in \mathbb{Z}^+$, the equation (L) is said to be *n-regular* over $\mathbb{Z}^+$ if, for every n-colouring of $\mathbb{Z}^+$, there exists a monochromatic solution $x \in (\mathbb{Z}^+)^{k+1}$ to (L). The *degree of regularity* of (L) is the largest integer $n \geq 0$, if any, such that (L) is n-regular. This (possibly infinite) number is denoted by $dor(L)$. If $dor(L) = \infty$, then (L) is regular.

In general one speaks about n-regularity over $A \subseteq \mathbb{Z}$ and defines $dor_A(L)$. For any $A \subseteq B \subseteq \mathbb{Z}$, clearly $1 \leq dor_A(L) \leq dor_B(L)$. It is not very difficult to observe that if (L_0) is regular, and the coordinate sum of α is nonzero, then $dor_{\mathbb{Z}^+}(L) = dor_{\mathbb{Z}}(L)$.

Now we state another conjecture of Rado [13].

Conjecture. For each positive integer r, there is a linear homogeneous equation that has degree of regularity r.

Alexeev and Tsimerman [5] proved the conjecture in 2010. Later, a proof of the following conjecture of Fox and Radoičić [9] by Golowich [10] supplied another proof of the above conjecture of Rado.

Conjecture (Fox and Radoičić). The degree of regularity of the following is $(n-1)$:

$$x_1 + 2x_2 + \cdots + 2^{n-2}x_{n-1} = 2^{n-1}x_n.$$

Recently, Adhikari and Goswami [4] have shown that for every $m, n \in \mathbb{Z}^+$, there exists an m-degree homogeneous equation that is n-regular but not $(n+1)$-regular.

Next we come to a conjecture due to Fox and Kleitman [8] for a very specific linear Diophantine equation.

Conjecture (Fox and Kleitman). Let $k \geq 1$. There exists an integer $b_k \geq 1$ such that the degree of regularity of the $2k$-variable equation $L_k(b_k)$,

$$x_1 + \cdots + x_k - y_1 - \cdots - y_k \;=\; b_k$$

is exactly $2k - 1$.

Fox and Kleitman [8] had shown that for any $b \in \mathbb{N}_+$, the equation $L_k(b)$ is not $2k$-regular. Indeed, if b is not a multiple of k, then considering the coloring given by the residue class modulo k, there is no monochromatic solution to the equation $L_k(b)$, and the equation is not even k-regular; we are through.

So, we assume that b is a multiple of k and consider the following $2k$-coloring of $\mathbb{N}_+$:

For $1 \leq i \leq 2k$, the set of integers colored i is defined to be

$$X_i = \bigcup_{j \geq 0} \left([(i-1)b/k + 1, ib/k] + 2bj \right).$$

Now, the set $X_i - X_i$ is independent of i. Since the set $k(X_1 - X_1) = \bigcup_{j \in \mathbb{Z}}([-b+k, b-k] + 2jb)$ is a union of translates of $[-b+k, b-k]$ by integer multiples of $2b$, it cannot contain b. Therefore, for any i, $1 \leq i \leq 2k$, $k(X_i - X_i)$ does not contain b. This shows that $L_k(b)$ is not $2k$-regular.

When $k = 2$, Adhikari and Eliahou [3] proved the Fox-Kleitman conjecture by establishing the following more general result:

For all positive integers b, we have

$$\operatorname{dor}(L_2(b)) = \begin{cases} 1 & \text{if } b \equiv 1 \bmod 2, \\ 2 & \text{if } b \equiv 2, 4 \bmod 6, \\ 3 & \text{if } b \equiv 0 \bmod 6. \end{cases}$$

From a result of Strauss [19], it follows that, for an appropriate b_k, the equation $L_k(b_k)$ is $\Omega(\log k)$-regular.

Adhikari, Balasubramanian, Eliahou and Grynkiewicz [1] gave a very short proof of the fact that, writing $c_{k-1} = \operatorname{lcm}\{i : i = 1, 2, \dots, k-1\}$, the equation $L_k(c_{k-1})$ is $(k-1)$-regular.

The full conjecture of Fox and Kleitman has been established by Schoen and Taczala [17] by generalizing a theorem of Eberhard, Green and Manners [6].

Adhikari, Boza, Eliahou, Revuelta, Sanz [2] considered the 4-variable Diophantine quadratic equation $(x_1 - y_1)(x_2 - y_2) = b$, denoted by $Q(b)$, where b is a given positive integer.

This equation is not regular. Indeed, it is not b-regular, and actually not even s-regular where $s = \lfloor \sqrt{b} \rfloor + 1$, as witnessed by the s-coloring given by the class mod s. For if x_1, y_1, x_2, y_2 are all congruent mod s, then $(x_1 - y_1)(x_2 - y_2)$ is divisible by s^2, and hence cannot equal b since $s^2 > b$. That is, we have $\operatorname{dor}(Q(b)) \leq \lfloor \sqrt{b} \rfloor$.

It was shown in [2] that, nevertheless, the numbers $\operatorname{dor}(Q(b))$ are unbounded as b varies:

Theorem 7. *Given a positive integer r, there is a positive integer $b = b(r)$ such that the equation $(x_1 - y_1)(x_2 - y_2) = b$ is r-regular.*

In the proof of the above, the following result was used in [2].

Theorem 8. (Szemerédi). *Given a desired length $k \in \mathbb{Z}^+$ and a specified density $0 < \delta \leq 1$, there exists a positive integer $N = N(k, \delta)$ such that every subset $A \subseteq [1, N]$ of density $|A|/N \geq \delta$ contains an arithmetic progression of length k.*

Theorem (7) was further generalized by Bidisha Roy and Subha Sarkar [16].

References

1. S. D. Adhikari, R. Balasubramanian, S. Eliahou and D. Grynkiewicz, *On the degree of regularity of a particular linear equation*, Acta Arith. **184**, no. 2, 187–191 (2018).

2. S. D. Adhikari, L. Boza, S. Eliahou, M. P. Revuelta and M. I. Sanz, *On the degree of regularity of a certain quadratic Diophantine equation*, European J. Combin. **70**, 50–60 (2018).

3. S. D. Adhikari and S. Eliahou, *On a conjecture of Fox and Kleitman on the degree of regularity of a certain linear equation*, Combinatorial and Additive Number Theory II, 1–8, Springer Proc. Math. Stat., 220, Springer, Cham, 2017.

4. S. D. Adhikari and S. Goswami, *Homogeneous Patterns in Ramsey Theory,* arXiv:2501.17203.

5. B. Alexeev, and J. Tsimerman, *Equations resolving a conjecture of Rado on partition regularity*, J. Combinatorial Theory Ser. A, **117**, 1008–1010 (2010).

6. S. Eberhard, B. Green and F. Manners, *Sets of integers with no large sum-free subset*, Annals of Math. **180**, 621–652 (2014).

7. P. Erdős and G. Szekeres, *A Combinatorial Problem in Geometry*, Compositio Math. **2**, 464–470 (1935).

8. J. Fox and D. J. Kleitman, *On Rado's Boundedness Conjecture*, J. Combin. Theory Ser. A **113**, 84–100 (2006).

9. J. Fox, and R. Radoičič, *The axiom of choice and the degree of regularity of equations over the reals*, Preprint, (2005).

10. N. Golowich, *Resolving a Conjecture on Degree of Regularity of Linear Homogeneous Equations*, Electronic Journal of Combinatorics, **21** (3), Paper 3.28, 8 pp. (2014)

11. R. L. Graham, B. L. Rothschild and J. H. Spencer, *Ramsey Theory*, John Wiley & Sons, 1980.

12. R. Rado, *Verallgemeinerung eines Satzes von van der Waerden mit Anwendungen auf ein problem der Zalentheorie, Sonderausgabe aus den Sitzungsberichten der Preuss*, Akad. der Wiss. Phys. -Math. klasse **17**, 1–10 (1933).

13. R. Rado, *Studien zur Kombinatorik*, Math. Z. **36**, 424 –480 (1933).

14. R. Rado, *Some recent results in combinatorial analysis*, Congrès International des Mathematiciens, Oslo, 1936.

15. F. P. Ramsey, *On a problem of formal logic*, Proc. London Math. Soc. (2) **30**, 264–285 (1930).

16. Bidisha Roy and Subha Sarkar, *Regularity of certain diophantine equation*, Proc. Indian Acad. Sci. Math. Sci. **129**, no. 2, Paper No. 19, 8 pp. (2019).

17. T. Schoen, and K. Taczala, *The degree of regularity of the equation* $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i + b$, Moscow J. Combin. and Number Th. **7**, 74–93 [162–181] (2017).

18. I. Schur, *Über die Kongruenz* $x^m + y^m \equiv z^m \pmod{p}$, Jber. Deutsch. Math. Verein. **25**, 114–117 (1916).

19. E. G. Straus, *A Combinatorial Theorem in Group Theory*, Math Comput. **29**, 303–309 (1975).

20. B. L. van der Waerden, *Beweis einer Baudetschen Vermutang*, Nieuw Arch. Wisk. **15**, 212–216 (1927).

□□□

How many subspaces V of $\mathbb{C}^n$ are there which satisfy the cyclic property:
"If $(a_1, \dots, a_n) \in V$, then $(a_n, a_1, \dots, a_{n-1}) \in V$"?

$V_1 = \{(a, a, \dots, a) | a \in \mathbb{C}\}$ and $V_2 = \{(a_1, \dots, a_n) \in \mathbb{C}^n | a_1 + \dots + a_n = 0\}$ are examples of such subspaces. Also $\{(0, \dots, 0)\}$ and $\mathbb{C}^n$ are trivial examples of such subspaces. Are there any more?

See the next issue of TMCB.

2. A(nother) Proof of the Simplicity of Alternating Groups

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ABSTRACT. The aim of this note is to give a self-contained elementary proof of the simplicity of alternating groups which avoids using clever manipulations of cycle decompositions of permutations as far as possible. We employ simple counting arguments to achieve this, instead of taking the more familiar route using class equations.

2.1 Introduction

Every graduate student in mathematics, at some point or other, comes across a somewhat loose statement like 'alternating groups are simple' (not entirely correct since the real story begins at $n = 5$), or at least its more precise but weaker version that 'A_5 is simple'. However, such a statement remains a folklore for the majority of students. The main reason behind this is that the proof of A_n's being simple for all $n \geq 5$, involves several steps. While none of these steps are particularly difficult to prove, the totality of the arguments often appears to be daunting, at least in its first appearance. The standard proofs of this fact either use clever manipulations using cycle decompositions of permutations, or a careful investigation of the class equations of A_5 and A_6, or a combination of both. The first approach involves a generous amount of trial and error unless one is able to memorize the clever tricks used in the cycle manipulations. On the other hand, the second approach of using class equations, though algorithmic, leaves no choice but to go through rather mundane computations.

In this note, we present a proof of the simplicity of the alternating groups A_n for $n \geq 5$, which does not involve too many computations using cycle decompositions. Also, it makes no use of class equations, something which plays a pivotal role in most standard proofs. The key idea behind our proof is a certain counting technique for finite groups, which, together with the use of embeddings of S_n (or, A_n) in S_{n+1} (or, A_{n+1}) allows us to avoid class equations.

Here is the basic layout of the paper. In §2, we recall some standard definitions as well as some basic facts about symmetric and alternating groups, which are used in this article. It also records an easy but useful observation about symmetric groups (*Proposition 2.2*), which practically allows us to cut down the size of symmetric and alternating groups. We note a well-known counting result for finite groups in *Lemma 2.2*, which is an essential ingredient in the proof of the simplicity of A_6.

In §3, we separately give a proof of the simplicity of A_5. The reason for this is twofold. Firstly, some readers may only be interested in the simplicity of A_5, without bothering too much about the simplicity of alternating groups of higher order. And secondly, the proof of the fact that A_5 is simple is so simple (pun intended), compared to the general case, that one can prove it without even having any substantial knowledge about the structure of alternating groups. A similar argument for the simplicity of A_5 can be found in [3].

In §4, we first prove that A_6 is simple. The crucial fact used in the proof is that A_6 *does not have any proper normal subgroup, which contains a 3-cycle (Lemma 4.2)*. We use the simplicity of A_5, together with *Lemma 2.2*, to show that every non-trivial normal subgroup of A_6, in fact, contains a 3-cycle, thereby proving that A_6 is simple. Finally, an application of *Proposition 2.2* reveals that the simplicity of A_5 and A_6 is sufficient to conclude the A_n is simple for all $n \geq 5$.

2.2 Preliminaries

Definitions and conventions. A nontrivial group G is called *simple* if it has no normal subgroup other than the obvious ones, namely the trivial subgroup $\{e\}$ and the whole group G.

If X is a set, the set of all bijections of X forms a group with respect to composition of functions, where the identity map plays the role of the identity element. It is called the *symmetric group over* X, and is denoted by S_X or $\mathrm{Sym}(X)$. The elements of S_X, i.e., bijections of X, are also called

permutations of X. A permutation is called *non-trivial* if it is not the identity map. The name symmetric group, which carries a distinct geometric flavor, is inspired by the fact that permutations of X are nothing but (set-theoretic) symmetries of X.
If X and Y are two sets and $f : X \to Y$ is a bijection, then f induces an isomorphism of groups $\tilde{f} : S_X \to S_Y$, given by $f(\sigma) := f \circ \sigma \circ f^{-1}$, where $\circ$ denotes the composition of functions.
If X is a non-empty finite set consisting of n elements, then S_X is called the symmetric group of degree n, and is usually denoted by S_n. It is customary to think of S_n as the group of all permutations of the set $\mathbb{N}_n := \{1, 2, ..., n\}$, and the identity element of S_n is denoted by e. Note that S_n contains $n!$ elements. We will be only interested in finite symmetric groups.
If n is a positive integer and σ, $\tau \in S_n$, we will denote the product of σ and τ by $\sigma \cdot \tau$, or simply by $\sigma\tau$ if there is no room for confusion. We follow the convention of applying functions from right to left, i.e., $(\sigma \cdot \tau)(i) := \sigma(\tau(i))$ for all $i \in \mathbb{N}_n$. Two permutations $\sigma, \tau \in S_n$ are said to be *conjugates* if there exists a permutation $\eta \in S_n$ such that $\tau = \eta \cdot \sigma \cdot \eta^{-1}$. Note that *conjugacy* is an equivalence relation on S_n.

If $\sigma \in S_n$, the support of σ, denoted by supp σ, is defined to be the set $\{i \in \mathbb{N}_n | \sigma(i) \neq i\}$. On the other hand, $i \in \mathbb{N}_n$ is called a *fixed point of* σ if $\sigma(i) = i$. Clearly, the set of all fixed points of σ is nothing but the complement of its support. A permutation σ acts trivially on the elements outside its support, as if it cannot 'see' them. We call two permutations $\sigma, \tau \in S_n$ disjoint if supp $\sigma \cap$supp $\tau = \emptyset$. If $\sigma, \tau \in S_n$ have disjoint supports, then $\sigma\tau = \tau\sigma$, simply because σ acts as the identity map on the support of τ, and vice versa. However, the converse is false since every non-trivial permutation $\tau \in S_n$ commutes with itself!

Let n be a positive integer and $\sigma \in S_n$. A set $Y \subseteq \mathbb{N}_n$ is called σ*-invariant* if $\sigma(i) \in Y$ for all $i \in Y$. The minimal non-empty σ-invariant subsets of $\mathbb{N}_n$ (with respect to set inclusion) are called the *orbits of* σ. An orbit of σ is called *trivial* if it contains only one element; otherwise, it is called *non-trivial*. Note that the trivial orbits of σ correspond to its fixed points. It is easy to see that the orbits of σ give rise to a partition of $\mathbb{N}_n$. An experienced reader will not fail to recognize that we are essentially speaking in the language of *group actions*. But we consciously avoid introducing group actions as we do not need them.

A permutation $\sigma \in S_n$ is called a *cycle* if it has exactly one non-trivial orbit. The *length of a cycle* σ is defined to be the number of elements in its unique non-trivial orbit. Note that the length of a cycle may take values between 2 to n. In particular, for us, the identity map is not a cycle. If σ is a cycle of length r, we call it an r-cycle. Other than the identity element, the simplest possible permutations are the 2-cycles, which are also known as *transpositions*.

If $\sigma \in S_n$, the *cycle type of* σ is defined to be the finite non-increasing sequence $(n_1, \dots, n_t)$, where $n_1, \dots, n_t$ are the sizes of the orbits of σ, arranged in a non-increasing order, so that $\sum_i n_i = n$. Clearly, the number of different possible cycle types of the elements of S_n is the same as the number of partitions of n.

A permutation $\sigma \in S_n$ is called an *even permutation* if σ can be written as a finite product of an even number of transpositions. As customary, the empty product is defined as the identity element of S_n. Since every permutation can be written as a finite product of transpositions (see *Proposition 2.1 below*), it immediately follows that the set of all even permutations of S_n is a normal subgroup of S_n (Do not forget that 0 is an even number!). We call it the *alternating group of degree n*, and denote it by A_n. Note that both A_1 and A_2 are trivial groups.

In the following *proposition*, we record some basic facts about symmetric and alternating groups, which we will freely use in the sequel. The proofs can be found in any standard graduate-level textbook on *abstract algebra* like [1], [2], [4] or [5] (for example, see *Section 4* of *Chapter IV* of [1]). By no means, this list is exhaustive, as there are hundreds of good algebra books available on the market.

Proposition 2.1. Let n be a positive integer.

(a) Every permutation in S_n can be written as a finite product of disjoint cycles.

(b) A permutation in S_n is either a product of an even number of transpositions or an odd number of transposition, but not both.

(c) Every r-cycle in S_n can be written as a product of $r-1$ transpositions. In particular, an r-cycle is an even permutation if and only if r is odd.

(d) The symmetric group S_n is generated by all transpositions of S_n.

(e) For all $n \geq 2$, there exist surjective group homomorphisms $\operatorname{sgn} : S_n \to \{\pm 1\}$, called the *sign map*, whose kernel is A_n.
In particular, A_n contains $\frac{n!}{2}$ elements for all $n \geq 2$.

(f) If $\sigma := (a_1\ a_2\ \dots\ a_r)$ is an r-cycle in S_n, then

$$\tau \cdot \sigma \cdot \tau^{-1} = (\tau(a_1)\ \tau(a_2)\ \dots\ \tau(a_r))$$

for all $\tau \in S_n$.

(g) Two permutations $\sigma, \tau \in S_n$ are conjugates (in S_n) if and only if they have the same cycle type.

We also need to analyze the centers of symmetric and alternating groups. Recall that the *center* of a group G, denoted by $Z(G)$, is defined as

$$Z(G) := \{a \in G \mid ag = ga \text{ for all } g \in G\}.$$

Lemma 2.1. The center of S_n (respectively, A_n) is trivial for all $n \geq 3$ (respectively, $n \geq 4$).

Proof. First we consider the case of symmetric groups. Let $n \geq 3$ and $\sigma \in S_n$ a non-trivial permutation. Then there exists $i \in \mathbb{N}_n$ such that $\sigma(i) = j \neq i$. As $n \geq 3$, we can find an element $k \in \mathbb{N}_n \backslash \{i, j\}$. Then it is clear that σ does not commute with the transposition $\tau := (j\ k)$, as $(\sigma \cdot \tau)(i) \neq (\tau \cdot \sigma)(i)$.
As for alternating groups, note that A_3 is the cyclic group of order 3. So let $n \geq 4$ and $\sigma \in A_n$ be a non-trivial even permutation. As before, there exists an element $i \in \mathbb{N}_n$ such that $\sigma(i) = j \neq i$. Then $(j\ k)$, as already noted, does not commute with σ for all $k \in \mathbb{N}_n \backslash \{i, j\}$. But $(j\ k)$ is not an even permutation. So we choose another element $l \in \mathbb{N}_n \backslash \{i, j, k\}$, and observe that σ does not commute with the 3-cycle $\tau := (j\ k\ l)$, as $(\sigma \cdot \tau)(i) \neq (\tau \cdot \sigma)(i)$.

Remark 2.1. In §4, we will prove that alternating groups are, in fact, generated by all 3-cycles. So, in hindsight, it is clear that one does not need to look beyond 3-cycles to determine the center of A_n.

The following counting argument will be used in the proof of the simplicity of A_6.

Lemma 2.2. Let H, K be subgroups of a finite group G, and

$$HK := \{hk \in G \mid h \in H, k \in K\}.$$

Then $|HK| = \frac{|H| \cdot |K|}{|H \cap K|}$. In particular, if $|H| \cdot |K| > |G|$, then $H \cap K$ is non-trivial.

Proof. Let $\phi : H \times K \to G$ be the set-theoretic map, defined as $\phi(h, k) := hk^{-1}$. Since K is a subgroup of G, by definition, the image of ϕ is HK. As $|H \times K| = |H| \cdot |K|$, the assertion follows if we can show that the pre-image of every element of HK contains exactly $|H \cap K|$ elements. Let $g := hk^{-1} \in HK$. If $y \in H \cap K$, then $\phi(hy, ky) = (hy)(ky)^{-1} = hk^{-1}$. So, the pre-image of g contains at least $|H \cap K|$ elements. Conversely, let $h_1 \in H, k_1 \in K$ be such that $\phi(h_1, k_1) = g$. Then $h_1 k_1^{-1} = hk^{-1}$, implying that $h^{-1}h_1 = k^{-1}k_1 \in H \cap K$. Setting $y := h^{-1}h_1 = k^{-1}k_1$, it is clear that $h_1 = hy$ and $k_1 = ky$. Therefore, the pre-image of g contains exactly $|H \cap K|$ elements, and hence the assertion follows. The second claim is obvious since $|HK| \leq |G|$.

Let S_X be the symmetric group over a set X, and Y a subset of X. Then one may identify S_Y with the set of all permutations of X, which do not disturb any element outside Y. This simple observation about symmetric groups, which we formally record as the following *proposition*, will later allow us to reduce the sizes of symmetric and alternating groups.

Proposition 2.2. Let X be a non-empty subset of $\mathbb{N}_n$ containing r elements. Let $H_X \subseteq S_n$ be the set of all permutations, which fix every element outside X. Then H_X is a subgroup of S_n. Further, if $X := \{a_1, \dots, a_r\}$ and $f : \mathbb{N}_r \to X$ is the set-theoretic function which sends i to a_i for all i, then $\tilde{f} : S_r \to H_X$, defined as

$$(\tilde{f}(\sigma))(x) = \begin{cases} (f \circ \sigma \circ f^{-1})(x) & \text{if } x \in X, \\ x & \text{otherwise,} \end{cases}$$

for all $\sigma \in S_r$, is a group isomorphism, which preserves l-cycles for all $2 \le l \le r$. In particular, if K_X is the set of all even permutations of S_n, which fix every element outside X, then $K_X = H_X \cap A_n = \tilde{f}(A_r)$.

Proof. The routine verifications are left as an exercise for the reader. Note that if $\sigma := (b_1\ b_2\ \cdots\ b_l) \in S_r$ is an l-cycle, then $\tilde{f}(\sigma) = (f(b_1)\ f(b_2)\ \cdots\ f(b_l))$.

2.3 A_5 is simple

The simplicity of A_5 turns out to be an immediate consequence of the following observation.

Lemma 3.1. Let G be a group and N a normal subgroup of G with a finite index. Let $x \in G$ be an element of finite order. If the order of x is relatively prime to $[G : N]$, the index of N in G, then $x \in N$. Thus N contains all elements of G whose orders are coprime to $[G : N]$.

Proof. Let $\pi : G \to G/N$ be the natural projection, which takes an element $g \in G$ to the corresponding left coset gN. Then, by *Lagrange's theorem*, the order of $\pi(x)$ divides the order of x as well as $[G : N]$, implying that the order of $\pi(x) = xN$ is one, or equivalently, $x \in N$.

Remark 3.1. The above result may fail if N is not a normal subgroup of G. For example, if we take $G := S_3$ and H the cyclic subgroup of order 2 generated by $(1\ 2)$, then the 2-cycle $(1\ 3)$ is not contained in H.

Theorem 3.1. A_5 is a simple group.

Proof. Looking at the possible cycle types of various elements of A_5, it is easy to see that A_5 consists of the identity element, twenty-four 5-cycles, twenty 3-cycles and fifteen elements which are products of two disjoint 2-cycles. Consequently, A_5 contains 24 elements of order 5, 20 elements of order 3 and 15 elements of order 2.
Now, let N be a non-trivial normal subgroup of A_5. By *Lagrange's theorem*, the possible orders of N are $2, 3, 4, 5, 6, 10, 12, 15, 20$ and 30. If $|N| = 2$, then $N = \{e, \sigma\}$, for some $\sigma \in A_5$, satisfying $\sigma^2 = e$. Then for each $\tau \in A_5$, $\tau\sigma\tau^{-1} \in N$, implying that $\tau\sigma\tau^{-1} = \sigma$. But that means σ is contained in the center of A_5, which is a contradiction as the center of A_5 is trivial by *Lemma 2.1*. We now invite the reader to use *Lemma 3.1* in verifying that A_5 cannot contain a normal subgroup of the remaining possible orders. For example, if N is a normal subgroup of order 12, then $[G : N] = 5$, so by *Lemma 3.1*, N must contain all elements of A_5 of order 3, which is not possible as A_5 contains 20 elements of order 3. As for one more example, if N is of order 30 then N has index 2, so N contains all elements of order 3 and 5, implying that N contains at least $20 + 24 = 44$ elements, which is not possible.

2.4 A_n is simple for all $n \ge 6$

First, we prove that A_6 is simple. For that, we need a few preparatory results which are interesting in their own right. Recall that every permutation in S_n can be written as a finite product of transpositions, i.e., 2-cycles. However, 2-cycles are odd permutations; and therefore, are not elements of A_n. So we look at 3-cycles, which are even permutations, and it should not come as a surprise to the reader that alternating groups are generated by 3-cycles.

Lemma 4.1. Every alternating group is generated by its 3-cycles.

Proof. The alternating groups A_1 and A_2 are trivial; and A_3 is a cyclic group of order 3, which is generated by each of its two 3-cycles $(1\ 2\ 3)$ and $(1\ 3\ 2)$. Next, in A_4, there are eight 3-cycles. With the order of A_4 being 12, it is clear from *Lagrange's theorem* that no proper subgroup of A_4 can contain all 3-cycles. So A_4 is also generated by 3-cycles. Now, we consider A_n for some $n \geq 5$. Since every even permutation can be written as a product of the product of a pair of transpositions, it is sufficient to prove that the product of every pair of transpositions can be written as a finite product of 3-cycles. So, we consider two transpositions $(a\ b), (c\ d) \in S_n$, and let $\sigma := (c\ d)(a\ b) \in A_n$. We want to prove that σ can be written as a (finite) product of 3-cycles. We can find a set $X := \{a_1, a_2, a_3, a_4\} \subseteq \mathbb{N}_n$, consisting of four elements, such that $a, b, c, d \in X$ (we do this because the two transpositions $(a\ b)$ and $(c\ d)$ may not be disjoint).
Let $K_X \subseteq A_n$ be the subset of all even permutations which fix every element outside X. Then $K_X = H_X \cap A_n$ and by *Proposition 2.2*, there exists an isomorphism between K_X and A_4 which preserves the 3-cycles. As the 3-cycles of A_4 generate A_4 and $\sigma \in K_X$, we conclude that σ can be written as a finite product of 3-cycles, which finishes the proof.

Remark 4.1. If $a, b, c, d \in \mathbb{N}_n$ are distinct elements, then one may observe that $(a\ c)(a\ b) = (a\ b\ c)$ and $(c\ d)(a\ b) = (a\ c\ d)(a\ b\ d)$. This proof, although very neat, does not really tell us how to anticipate the second equality. As a result, it may require some trial and error, and that is why we chose to give a proof that is more conceptual. The preference will largely depend on the taste of the reader, and we leave it at that.

It follows from *Proposition 2.1(f)* that any two 3-cycles of S_n are conjugates of each other. Since A_3 is a cyclic group of order 3, its 3-cycles are not conjugates; and we leave it as an easy exercise for the reader to check that not all 3-cycles of A_4 are conjugates in A_4 (a little knowledge about *group actions*, and, in particular, the *orbit-stabilizer theorem* might help!). However, starting from $n = 5$, all 3-cycles of A_n are actually conjugates in A_n.

Lemma 4.2. Any two 3-cycles of A_n are conjugates (in A_n) for all $n \geq 5$.

In particular, if a normal subgroup $N \trianglelefteq A_n$ contains a 3-cycle, then $N = A_n$.

Proof. Let $(a\ b\ c)$ and $(d\ e\ f)$ be two 3-cycles in A_n. Since they are conjugates in S_n, there exists a permutation $\tau \in S_n$ such that $\tau(a\ b\ c)\tau^{-1} = (d\ e\ f)$. If τ is an even permutation, we are done. Otherwise, choose $g, h \in \mathbb{N}_n \backslash \{a, b, c\}$, and replace τ by $\tau \cdot (g\ h)$. This makes τ an even permutation satisfying $\tau(a\ b\ c)\tau^{-1} = (d\ e\ f)$.
The second assertion is trivial since all 3-cycles of A_n are conjugates, and by *lemma 4.1*, they generate A_n.

Therefore, if $N \trianglelefteq A_n$ is a non-trivial normal subgroup for some $n \geq 5$, then to prove that $N = A_n$, it is enough to show that N contains a 3-cycle. Sadly, we cannot prove it right now, but we can prove something very close.

Lemma 4.3. Let N be a non-trivial normal subgroup of A_n for some $n \geq 4$. Then N contains a product of two 3-cycles, which is not the identity element.

Proof. Since N is non-trivial, we can find a non-trivial permutation $\sigma \in N$. With the center of A_n being trivial and the set of all 3-cycles generating A_n, we can choose a 3-cycle $\tau \in A_n$ such that $\sigma \cdot \tau \neq \tau \cdot \sigma$, or equivalently, $\tilde{\sigma} := \sigma \cdot \tau \cdot \sigma^{-1} \cdot \tau^{-1}$ is not the identity element. Since N is normal, $\tau \cdot \sigma^{-1} \cdot \tau^{-1} \in N$, implying that $\tilde{\sigma} \in N$. As $\sigma \cdot \tau \cdot \sigma^{-1}$ and τ^{-1} are both 3-cycles, the assertion follows.

Remark 4.2. Can you see that the analogous assertion fails in A_3?

Since we are interested in proving the simplicity of A_6, and *Lemma 4.3* ensures that every non-trivial normal subgroup of A_6 contains a non-trivial element that is a product of two 3-cycles, let us try to find out what can be said about the number of elements in a normal subgroup of A_6, which contains a product of two disjoint 3-cycles. We are interested in the size of the normal subgroup as we want to apply *Lemma 2.2*.

Lemma 4.4. Let N be a normal subgroup of A_6, which contains a product of two disjoint 3-cycles, say $(a_1\ a_2\ a_3)$ and $(b_1\ b_2\ b_3)$. Then N contains at least 10 elements.

Proof. There are six permutations of the form $(a_2\ c)(a_3\ d)$, where $c, d \in \{b_1, b_2, b_3\}$ are distinct elements. If $\sigma := (a_2\ c)(a_3\ d) \in A_6$ is any such permutation, then $\sigma(a_1\ a_2\ a_3)\ (b_1\ b_2\ b_3)\sigma^{-1} = (a_1\ c\ d)(\sigma(b_1)\ \sigma(b_2)\ \sigma(b_3))$ is contained in N. Therefore, it is clear that N contains at least seven elements, including the identity element. So we can use *Lagrange's theorem* to conclude that $|N| \geq 10$.

Remark 4.3. Let $\sigma \in S_6$ be a product of two disjoint 3-cycles. Then it is easy to see that the conjugacy class of σ (in S_6) contains 40 elements. If one knows the basics of *group actions*, it is not difficult to check that the conjugacy class of σ in A_6 contains either 20 or 40 elements. But since we promised not to use group actions, we avoid this line of argument.

We are now in a position to prove that A_n is simple for all $n \geq 6$.

Proposition 4.1. A_6 is a simple group.

Proof. Let $N \trianglelefteq A_6$ be a non-trivial normal subgroup. By *Lemma 4.3*, N contains a non-identity element, which is a product of two 3-cycles, say σ and τ. First, suppose that σ and τ have a common fixed point, say $i \in \mathbb{N}_6$. Let K_i' denote the set of all even permutations which fix i. By *Proposition 2.2*, K_i' is isomorphic to A_5. Since A_5 is simple, and $K_i' \cap N$ is a nontrivial normal subgroup of K_i', it follows that $K_i' \subseteq N$. Clearly, K_i' contains a 3-cycle, implying that $N = A_6$. Next, let us assume that σ and τ are disjoint 3-cycles. Let K_1' denote the set of all even permutations which fix 1. Again, by *Proposition 2.2*, K_1' is isomorphic to A_5. By *Lemma 4.4*, N contains at least 10 elements. Since K_1' contains 60 elements, it follows form *Lemma 2.2* that $K_1' \cap N$ is non-trivial. As K_1' is simple and $K_1' \cap N$ a non-trivial normal subgroup of K_1', we conclude that $K_1' \subseteq N$. With K_1' containing a 3-cycle, it follows that $N = A_6$.

Theorem 4.1. A_n is simple for all $n \geq 6$.

Proof. Let $N \trianglelefteq A_n$ be a non-trivial normal subgroup. By *Lemma 4.3*, N contains a non-trivial product of two 3-cycles, say σ and τ. As the supports of both σ and τ contain three elements, we can find a set $X \subseteq \mathbb{N}_n$, consisting of 6 elements, such that $\operatorname{supp}\sigma \cup \operatorname{supp}\tau \subseteq X$. Let K_X denote the set of all elements of A_n whose supports are contained in X. By *Proposition 2.2*, K_X is isomorphic to A_6. So K_X is simple by *Proposition 4.1*, and $K_X \cap N$ is a non-trivial normal subgroup of K_X. Therefore, $K_X \subseteq N$. In particular, N contains a 3-cycle, implying that $N = A_n$.

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Bibliography

1. P. Aluffi, *Algebra : Chapter 0*; Graduate Studies in Mathematics (**104**), American Mathematical Society, 2009.

2. M. Artin, *Algebra*; Pearson Education India, 2nd edition, 2015.

3. J. A. Gallian, *Another proof that A_5 is simple*; The American Mathematical Monthly **91(2)**, 134-135, 1984.

4. I. N. Herstein, *Topics in Algebra*; John Wiley & Sons, 2006.

5. T. W. Hungerford, *Algebra*; Graduate Texts in Mathematics (**73**), Springer, 1974.

□□□

3. Spherical Trigonometry in Bhāskarācārya's Siddhāntaśiromaṇi (1150 CE)

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ABSTRACT. There are various coordinates on a sphere like the zenith distance, hour angle, azimuth, declination, right ascension, celestial longitude and latitude, etc. associated with a celestial object. There are not all independent, and relations among them can be obtained using modern spherical trigonometry. In ancient Indian astronomy texts, these were obtained using various geometrical constructions. The twelfth century Indian astronomer Bhāskarācārya has given detailed explanations of these relations in his magnum opus, Siddhāntaśiromaṇi (1150 CE). We give some representaive examples of these in this article.

3.1 Introduction

Bhāskarācārya is one of the greatest names in the history of ancient and medieval Indian mathematics and astronomy. He was born in 1114 CE, and probably hailed from the region around the present Patne or Patan in the western Indian province of Maharashtra. Bhāskara's *Līlāvatī* (The sportive one) on arithmetic and geometry, and *Bījagaṇita* (Algebra) are standard works on Indian mathematics [3]. The *Siddhāntaśiromaṇi* ('Crest jewel among the treatises on astronomy') composed in 1150 CE by Bhāskarcārya is one of the most comprehensive treatises on Indian astronomy [2, 8]. These were canonical textbooks for students of astronomy and mathematics in India for the next few centuries, and are taught in the Sanskrit institutes in India, even now[1].

The *Siddhāntaśiromaṇi* has two parts, namely, *Grahagaṇita* ('Planetary computations') and *Golādhyāya* ('Chapter on spherics'[2]). *Grahagaṇita* expounds on all the standard calculations and algorithms in astronomy of Bhāskara's times. It has 460 verses in 12 chapters. *Golādhyāya* which has more than 490 verses, has the definitions, more fundamental issues (like the nature of the earth, the placement of stars and planets around it and so on), and the principles and theoretical details of the calculations in *Grahagaṇita*. The verses in these two parts have been translated into English with notes [1, 7]. An important feature of the *Siddhāntaśiromaṇi* is that Bhāskara himself has written a commentary on it, known as the '*Vāsanābhāṣya*' or the '*mitākṣarā*' which explains all the algorithms contained in the verses, and also gives their derivations (*upapattis*). Recently, we have translated the verses and the *vāsnabhāṣya* of the *Grahagaṇita* part of the treatise, and prepared detailed explanatory notes based on the *bhāṣya* [12].

Spherical trigonometry in the sense of relations among the variables on the celestial sphere is very much needed for solving diurnal problems, eclipse calculations etc. It is natural that the Indian astronomy texts deal with the relations among the spherical variables. The geometrical insights of Bhāskara come into full play in handling such problems. We present some representative examples of Bhāskara's solutions of spherical problems in this article.

In Section 2, we introduce the celestial coordinates and spherical trigonometry relevant for this article. In Section 3, we take up the geometrical constructions of Bhāskara to obtain expressions for the declination and the right ascension (R.A.) of a celestial object on the ecliptic in terms of its longitude. In Section 4, we discuss the very important expression for the zenith distance (z) in terms of the latitude (ϕ), declination (δ), and the hour angle (H). This is used by Bhāskara to devise an instrument called "*phalakayantra*" (rectangular board instrument) to find the hour angle of the Sun at any instant. In Section 5, we consider the expression for (z) in terms of (ϕ), (δ), and the azimuth (A) which is more difficult, as it involves the solution of a quadratic equation for

[1]These include the Lal Bahadur Shastri Rashtriya Sanskrit Vidyapeetha in New Delhi, Benares Hindu University and Sampurnanand Sanskrit Vishwavidyalaya in Varanasi, Madras Sanskrit College in Chennai, and Rashtriya Sanskrit Vidyapeetha in Tirupati, to name a few.

[2]Spherical Geometry.

the cosine of the zenith distance (z). We summarise Bhāskara's detailed method for solving the problem. Spherical trigonometry is taken forward considerably by the astronomer-mathematicians of the Kerala school (main works during 14^{th}-17^{th} century C.E.), and we touch upon this in Section 6 on the concluding remarks.

3.2 Celestial coordinates and Spherical trigonometry

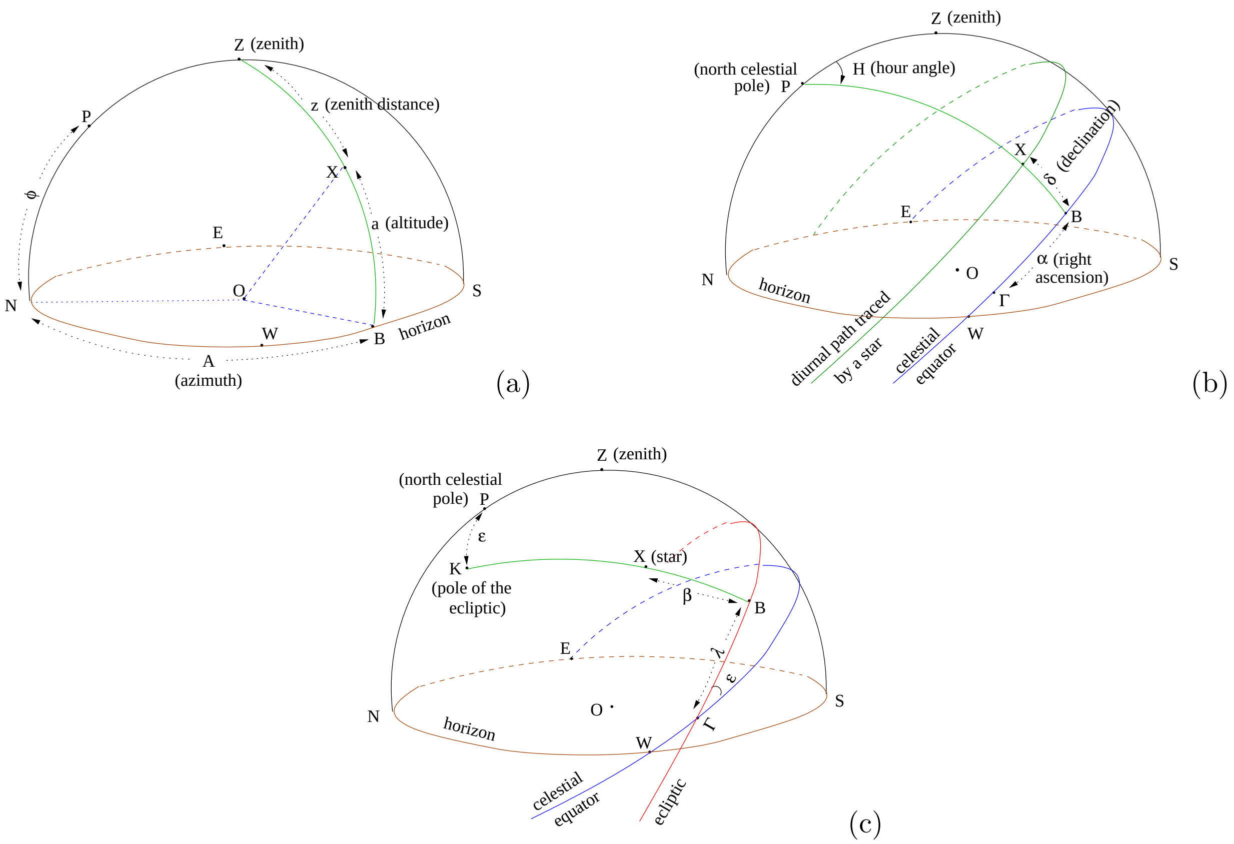

Fig. 1. Celestial coordinates. (a). z, a, A, (b). H, δ, α, (c). λ, β.

We depict the variables of interest on the sphere associated with a celestial object at X, like the zenith distance ($z = ZX$), altitude ($a = XB$), azimuth ($A = NB$) in Fig. 1(a), hour angle ($H = Z\hat{P}X$), declination ($\delta = XB$), right ascension ($\alpha = \Gamma B$) in Fig. 1(b), and celestial longitude ($\lambda = \Gamma B$), celestial latitude ($\beta = XB$) in Fig. 1(c). Here Γ is the vernal equinox (where the celestial equator and the ecliptic intersect). We can obtain relations among them using spherical trigonometry.

3.2.1 Basic relations in Modern Spherical Trigonometry

Now, spherical triangles are made of great circle arcs only. In the spherical triagle below, a, b, c are the sides (arc lengths; can be measured in angles also), and A, B and C are the spherical angles.

In modern spherical trigonometry, we have the cosine formula,

$$\cos a = \cos b \cos c + \sin b \sin c \cos A, \quad \text{and, similarly for } \cos b, \text{ and } \cos c, \tag{3.1}$$

and the sine formula,

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}. \tag{3.2}$$

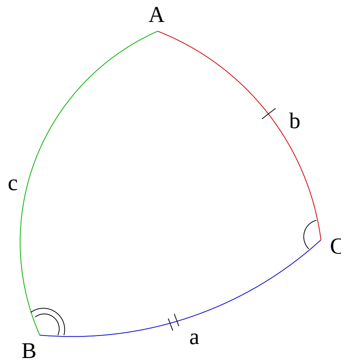

Fig. 2. Spherical triangle with sides a, b, c and angles A, B, C.

Many more relations can be derived using these basic formulae and related spherical triangles.

3.3 Spherical variables and the relations among them in Indian texts

How did Indian texts handle the spherical variables and the relations among them? In the earliest text on mathematical astronomy in India, namely, *Āryabhaṭīya* (499 CE) [9], one quarter is on *Gola* (sphere). The formula for the declination,

$$\sin\delta = \sin\epsilon \sin\lambda, \tag{3.3}$$

is used implicitly in this text, though not stated explicitly. In his commentary (*Vāsanābhāṣya*) on verses 47 and 48 in the second chapter (on "True longitudes of planets") in the *Grahagaṇita* part of *Siddhāntasiromaṇi* (c. 1150 CE) [3], Bhāskarācārya explains this relation using the "rule of three" which would amount to comparing similar triangles.[3]

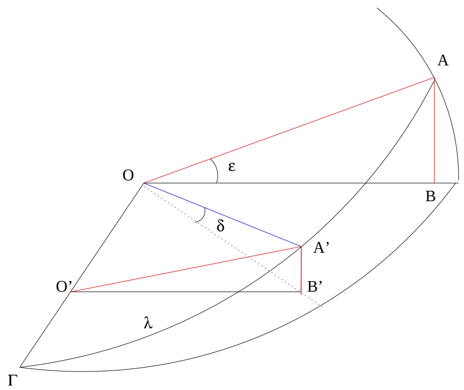

Fig. 3. Declination ($\delta = A'\hat{O}B'$) of an object at A'

From now onwards, R is the radius of the (celestial) sphere. In Fig. 3, the plane of the equator is that with points Γ, O, B, and the plane of the ecliptic is the one with points Γ, O, A. The angle between them is $\epsilon = A\hat{O}B$. Corresponding to a point A' on the ecliptic, arc $\Gamma A' = R\lambda$, where Γ is the intersection point of the two planes on the sphere, and $\lambda = \Gamma\hat{O}A'$ is the (celestial) longitude. Let $A'B'$ be perpendicular to the plane of the equator. Then $\delta = A'\hat{O}B'$ is the declination of A'. Let $A'O'$ be perpendicular to $O\Gamma$. Then,

$$A'B' = R\sin\delta, A'O' = R\sin\lambda, AB = R\sin\epsilon, AO = R.$$

Now, triangles $A'B'O'$ and ABO are similar. Hence,

$$\frac{A'B'}{A'O'} = \frac{AB}{AO}, \text{ or } \frac{R\sin\delta}{R\sin\lambda} = \frac{R\sin\epsilon}{R}, \text{ or,}$$
$$\sin\delta = \sin\epsilon\sin\lambda.$$

[3]He has not given the geometrical construction in Fig. 3, which is however implicit.

3.3.1 Relation between the longitude, λ and the R.A., α

A verse in the *Golapāda* (the quarter-part on spherics) *Āryabhaṭīya* [6] says:

> *iṣṭajyāguṇitamahorātravyāsārdhameva kāṣṭāntyaṃ|*
>
> *svāhorātrārdhahṛtaṁ phalamajāllaṅkodayaprāgjyā ||*
>
> Multiply the day radius corresponding to the greatest declination (on the ecliptic) by the desired Rsine, and divide by the corresponding day radius: the result is the Rsine of the right ascension measured from the first point of Aries along the equator. [Translation by K. S. Shukla and K. V. Sarma]

Here,

> the desired Rsine is $R\sin\lambda$,
> the day-radius corrsponding to the greatest declination is $R\cos\epsilon$,
> the day-radius is $R\cos\delta$, and
> the Rsine of the right ascension (R.A.) is $R\sin\alpha$.

Then,

$$R\sin\alpha = \frac{R\sin\lambda \times R\cos\epsilon}{R\cos\delta}. \tag{3.4}$$

Āryabhaṭīya has only algorithms and no explanations. This can be easily derived using the sine formula of modern spherical trigonometry.

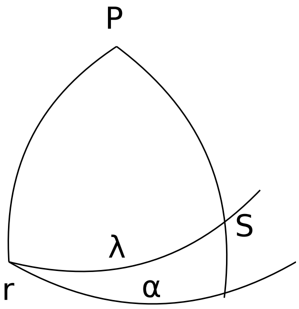

Fig. 4. The longitude, λ, and the R.A., α.

In Fig. 4, consider the spherical triangle ΓPS. $\Gamma S = \lambda$, is an arc along the ecliptic, $\Gamma\hat{P}S = \alpha$ is the R.A.. It is also an arc along the equator. $PS = 90 - \delta$, where δ is the declination, and $P\hat{\Gamma}S = 90 - \epsilon$, where ϵ is the obliquity of the ecliptic. Using the sine formula,

$$\frac{\sin\Gamma S}{\sin\Gamma\hat{P}S} = \frac{\sin PS}{\sin(90-\epsilon)}, \text{ or, } \frac{\sin\lambda}{\sin\alpha} = \frac{\cos\delta}{\cos\epsilon}.$$

$$\text{So, } \sin\alpha = \frac{sin\lambda\cos\epsilon}{\cos\delta}.$$

In his commentary (*Vāsanābhāṣya*) on verses 54 and 55 in the second chapter (on the "True longitudes of planets") in the *Grahagaṇita* part of *Siddhāntaśiromaṇi* [8], Bhāskara proves this with a geometrical construction as shown in Fig. 5. We present the essence of Bhāskara's proof in the following.

In Fig. 5, the celestial equator, ecliptic and the diurnal circle of radius $R\cos\delta$ of the Sun at S on the ecliptic, with longitude λ and R.A. α are shown. Consider the vertical right triangle with the hypotenuse marked "$R\sin\lambda$" on the ecliptic plane, and with the "opposite side" (*bhuja*) as a vertical dashed line marked "$R\sin\delta$" in the plane of the equatorial horizon (*Laṅkā*) which is perpendicular to the celestial equator. Then, the "adjacent side" (*koṭi*) is on the diurnal circle of S and is given by

$$\sqrt{R^2\sin^2\lambda - R^2\sin^2\delta} = \sqrt{R^2\sin^2\lambda - R^2\sin^2\epsilon\sin^2\lambda} = R\sin\lambda\cos\epsilon.$$

But, this *koṭi* is actually $R\cos\delta\sin\alpha$, as it is the Rsine of the R.A. α in a circle of radius $R\cos\delta$, and shown as such in the figure. Hence,

$$R\cos\delta\sin\alpha = R\sin\lambda\cos\epsilon, \text{ or,}$$

$$\sin\alpha = \frac{\sin\lambda\cos\epsilon}{\cos\delta}.$$

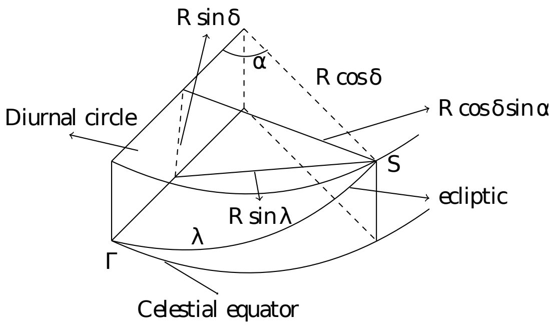

Fig. 5. Geometrical construction for the relation between $R\sin\alpha$ and $R\sin\lambda$

3.4 Zenith distance (z) in terms of the latitude (ϕ), declination (δ), and the hour angle (H)

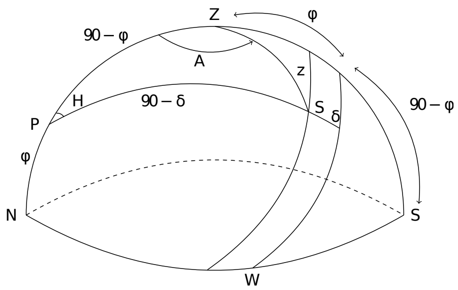

Fig. 6. Spherical triangle PZS with sides $PZ = 90^\circ - \phi, ZS = z, PS = 90^\circ - \delta$, and angle $Z\hat{P}S = H$.

Consider a celestial object S, typically, the Sun. In the spherical triangle PZS in Fig. 6, $ZS = z$, the zenith distance, $PZ = 90^\circ - \phi$, where ϕ is the latitude, $PS = 90 - \delta$, where δ is the declination, $Z\hat{P}S = H$, the hour angle, and $P\hat{Z}S = A$, the Azimuth.

Applying the cosine formula to ZS:

$$\cos ZS = \cos PZ\cos PS + \sin PZ\sin PS\cos Z\hat{P}S, \text{ or,} \tag{3.5}$$

$$\cos z = \sin\phi\sin\delta + \cos\phi\cos\delta\cos H. \tag{3.6}$$

What was the method of Bhāskara to obtain this relation? [4]

3.4.1 *Bhāskara's method for obtaining z in terms of ϕ, δ and H*

In Fig. 7, O is the centre of the celestial sphere of radius R. NPZS is the Meridian. At some instant, the Sun is at S'. Its declination is δ. $S_rS_2US'S_1S_t$ is the diurnal circle of the Sun with C as the centre. Its radius is $R\cos\delta$. Just as the plane of the equator, the plane of the diurnal circle is inclined to plane of horizon at an angle, $90^\circ - \phi$. S_rTS_t is the Rising-Setting line of the Sun. The straight line S_2CS_1 is parallel to this.

[4] See Reference [10], pp. 370-376 for details.

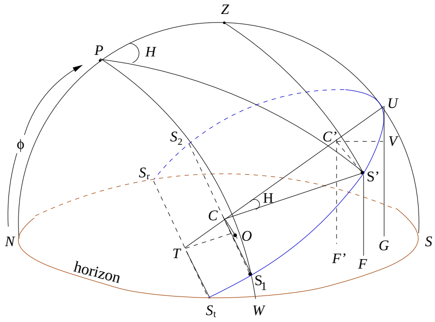

Fig. 7. Geometry for the zenith distance z for declination, δ, and hour angle, H.

$z = Z\hat{O}S'$ is the zenith distance of the Sun. It is also the angle corresponding to arc ZS'.

$H = Z\hat{P}S'$ is the hour angle of the Sun.

$S'F$ is perpendicular to the horizon, and $S'F = R\cos z$ is the '*Śaṅku*' or the 'Gnomon'.

$OC = R\sin\delta$ is perpendicular to UT.

$CU = R\cos\delta$ is the '*dyujyā*' or the 'Day-radius'.

OCT is a right-triangle with $OC = R\sin\delta$ and $C\hat{T}O = 90 - \phi$.

Hence, $CT = R\sin\delta\frac{\sin\phi}{\cos\phi}$. It is called the '*kṣitijyā*' or the 'Earth-sine'.

In verse 34 of the chapter on *Tripraśna* in the *Grahagaṇita* part of *Siddhāntaśiromaṇi* [8], *hṛti* is defined as the sum of *dyujyā* (day-radius) and *kṣitijyā* (Earth-sine). This is UT. So,

$$hṛti = UT = R\cos\delta + R\sin\delta\frac{\sin\phi}{\cos\phi}. \tag{3.7}$$

Draw UG perpendicular to the horizon. It is the '*Dinārdhaśaṅku*', or the 'Mid-day gnomon'. UGT is a right triangle with $U\hat{T}G = 90^\circ - \phi$. Then,

$$\begin{aligned} \text{Mid-day gnomon} = UG &= \cos\phi \times hṛti\ (= UT) \\ \text{So, Mid-day gnomon} &= R\cos\delta\cos\phi + R\sin\delta\sin\phi. \end{aligned} \tag{3.8}$$

Draw $S'C'$ perpendicular to UT. $S'\hat{C}C' = H$.

$CS' = CU = R\cos\delta,\ \ CC' = R\cos\delta\cos H$.

$C'U = CU - CC' = R\cos\delta(1-\cos H)$.

Draw $C'V$ perpendicular to UG. One can easily see that the plane with the points S', C', V is a horizontal plane. UV is called the *Ūrdhva* (Upwards), as it is the upper portion of the *Dinārdhaśaṅku* (Mid-day Gnomon), UG. $U\hat{C'}V = 90 - \phi$. Then,

$$\textit{Ūrdhva}\ \text{(Upwards)} = UV = C'V\sin(90^\circ - \phi) = R\cos\delta\cos\phi(1-\cos H).$$

This is how *Ūrdhva* is described in verses 58 and 59 of the mentioned chapter. Then in verse 60 it is stated that,

$$\begin{aligned} \text{Desired Gnomon}, R\cos z = S'F &= VG = UG - UV \\ &= \text{Mid-day Gnomon} - \text{Upwards}, \end{aligned}$$

$$\text{or,}\quad R\cos z = R\sin\phi\sin\delta + R\cos\phi\cos\delta\cos H, \tag{3.9}$$

$$\text{or,}\quad \cos z = \sin\phi\sin\delta + \cos\phi\cos\delta\cos H, \tag{3.10}$$

which is the desired result.

3.4.2 Hour angle, H in terms of z, ϕ, δ, and the Phalakayantra (Board instrument) of Bhāskara

The hour angle H can be determined in terms of z, δ, and ϕ, by rewriting equation (3.9) as,

$$R\cos H = \frac{R\cos z}{\cos\phi\cos\delta} - \frac{R\sin\phi\sin\delta}{\cos\phi\cos\delta}. \tag{3.11}$$

The *phalakyantra* (Board Instrument) is an instrument devised by Bhāskara himself to measure the hour angle based on the above relation, and described in the *yantrādhyāya*, the chapter on instruments in *Golādhyāya* [7, 10]. He is very proud of it, and introduces it thus:

> "As others have not stated the [determination of] correct time from [observations using] a vertical circle with ease, I have attempted [to devise] an instrument called *phalakayantra*, [which incorporates] the essence of calculations based on the true rationales pertaining to the sphere, which I will explain clearly."

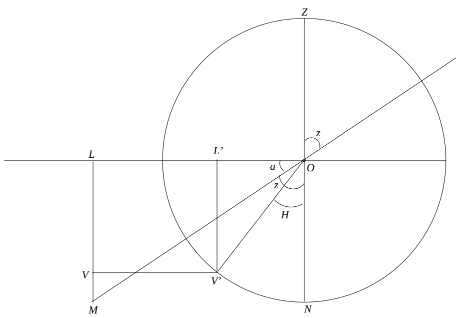

Fig. 8. *Phalakayantra* for measuring the hour angle, H.

In the upper part of the instrument, there is a circle with a radius of 30 units. There is a small hole at the centre with a pin placed in it. This pin is the axis of the instrument. The circumference of the circle is divided into "60 *ghaṭī*s"[5] and 360 degrees. Each degree is made of 10 "*pala*s". There is also a *Paṭṭikā* or an Index arm which is suspended vertically, with a hole at the centre. Also, a horizontal line is drawn through the central hole, and lines parallel to it drawn below it at equal intervals, for measuring distance along the vertical direction. The following is Bhāskara's instruction for using the instrument:

> "Now hold the instrument so that the rays of the Sun shall illuminate both of its sides equally [to secure its being in a vertical circle]; the place in the circumference marked out by the shadow of the axis should be assumed by an intelligent man to be the Sun's place."
>
> "Now place the index arm on the axis and putting it over the Sun's place, from the point at the end of the *yaṣṭi* set off above or below depending on the [hemi]sphere [above if the Sun is in the northern hemisphere, and below if it is in the southern hemisphere], the Rsine of the ascensional difference (*carajyā*). The distance from the point where the sine [which meeting the end of the *carajyā* thus set off,] cuts the circle, to the perpendicular line will represent the *nata* (hour angle) in *ghaṭikā*s."

[5]A *ghaṭī* corresponds to 24 minutes of time.

(Here *yaṣṭi* $= \frac{R}{\cos\phi\ \cos\delta}$, where R is 30. *carajyā* $= \frac{R\sin\phi\ \sin\delta}{\cos\phi\ \cos\delta}$.) [Translation by Wilkinson, 1861.]

Now, the zenith dist of the Sun is z, and the altitude $a = 90^\circ - z$.

$ML = OM\sin a = OM\cos z = \frac{R\cos z}{\cos\phi\ \cos\delta}$.

Mark a point V on ML such that

$MV = carajy\bar{a} = \frac{R\sin\phi\ \sin\delta}{\cos\phi\ \cos\delta}$.

Draw VV' parallel to OL, intersecting the circle at V'. Draw $V'L'$ parallel to VL. Then,

$$\begin{aligned} V'L' = VL = ML - MV &= \frac{R\cos z}{\cos\phi\ \cos\delta} - \frac{R\sin\phi\sin\delta}{\cos\phi\cos\delta} \\ &= R\cos H, \text{ from equation (11).} \end{aligned}$$

$$\text{But } V'L' = R\sin V'\hat{O}L' = R\cos V'\hat{O}N.$$

Hence, $V'\hat{O}N$ is the hour angle H. It is the number of divisions on the circle between N, and the point V' at which the horizontal line from V intersects the circle.

3.5 Zenith distance z in terms of the latitude ϕ, the declination δ and the azimuth A

Let z and A be the zenith distance and the azimuth of the Sun (S) when its declination is δ at a place with latitude ϕ, as shown in the following figure (Fig. 9).

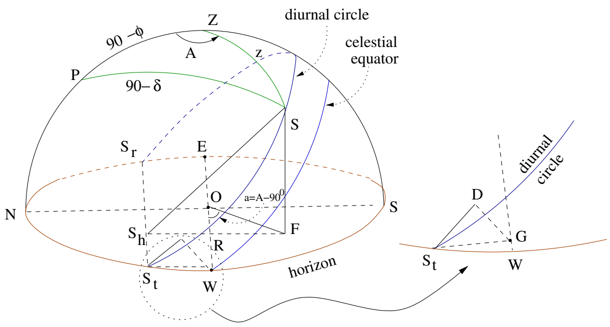

Fig. 9. Geometry for the zenith distance z, for declination δ and azimuth A.

Here A is the angle between the north-south or the meridian circle, and the vertical passing through S ($P\hat{Z}S$). In the Indian texts, the azimuthal angle, termed the *digaṃśa*, is the angle between the vertical passing through S and the prime vertical, which we denote by a. Clearly, $A = 90^\circ \pm a$, when $0^\circ \leq A \leq 180^\circ$. In the figure, $A = 90^\circ + a$.

Using the modern cosine formula for the side PS in the spherical triangle ZPS, where the sides are $PZ = 90 - \phi, ZS = z, PS = 90 - \delta$, and the angle $P\hat{Z}S = A$, we have

$$\begin{aligned} \cos(90-\delta) = \sin\delta &= \cos(90-\phi)\cos z + \sin(90-\phi)\sin z\cos A, \\ \text{or } \cos z\sin\phi &= \sin\delta \pm \sin z\sin a\cos\phi \ \text{ when } A = 90 \pm a. \end{aligned} \tag{3.12}$$

Now δ can be found directly from ϕ, z, and a from the above equation. However z cannot be found directly in terms of δ, ϕ, and a, as both $\cos z$ and $\sin z$ appear in the equation. One would have to solve a quadratic equation for $\sin z$, after squaring both sides and using $\cos^2 z = 1 - \sin^2 z$.

3.5.1 Bhāskara's method for obtaining z in terms of ϕ, δ and A: śaṅku, bhujā, agrā, and śaṅkutala

In Fig. 9, the Sun rises at S_r, moves along the diurnal circle and sets at S_t. If we assume that Sun's declination δ is constant through the day, the 'rising- setting' line, S_rS_t would be parallel to the east–west line. From S_t, draw S_tG perpendicular to the east–west line meeting it at G. S_tG is the "*arkāgrā*" or just "***agrā***". It is the distance between the 'rising-setting' line and the east-west line.

Now the plane of the diurnal circle is inclined at an angle $90 - \phi$ with the horizon. From G draw GD perpendicular to the plane of the diurnal circle meeting it at D. Join S_tD, which would be perpendicular to GD. Clearly, $D\hat{S_t}G = 90 - \phi$ and $D\hat{G}S_t = \phi$. S_tDG is a latitudinal triangle (a right-angled triangle with the latitude as one of the angles). Now $GD = |R\sin\delta|$. Hence, *agrā* $= S_tG = \left|R\frac{\sin\delta}{\cos\phi}\right|$.

From S, draw SF perpendicular to the plane of the horizon. In Indian astronomy texts, $SF = R\cos z$ is called the "***śaṅku***" or the gnomon and $OF = R\sin z$ is called "*dṛgjyā*". Draw RF perpendicular to the east-west line. $RF = R\sin z \sin a$ and it is called the "***bhujā***". It is the distance between the base of the *śaṅku* and the east-west line.

Extend FR to meet the rising-setting line perpendicularly, at S_h. S_hF is the distance between the base of the *śaṅku*, F and the rising-settting line, and is called the "***śaṅkutala***". SS_hF is a latitudinal triangle, with $S\hat{S_h}F = 90° - \phi$. Hence, the *śaṅkutala*, $S_hF = SF\frac{\sin\phi}{\cos\phi} = R\cos z\frac{\sin\phi}{\cos\phi}$.

To summarise,

$$\begin{aligned} SF : \textit{śaṅku} &= R\cos z, & (3.13)\\ S_tG : \textit{agrā} &= \left|R\frac{\sin\delta}{\cos\phi}\right|, & (3.14)\\ RF : \textit{bhujā} &= R\sin z \sin a, & (3.15)\\ S_hF : \textit{śaṅkutala} &= R\cos z\frac{\sin\phi}{\cos\phi}. & (3.16)\end{aligned}$$

Multiplying the equation for $\cos z \sin\phi$ by R, dividing by $\cos\phi$, and rearranging terms, we find:

$$\pm R\sin z \sin a = R\cos z\frac{\sin\phi}{\cos\phi} - \frac{R\sin\delta}{\cos\phi}. \tag{3.17}$$

Now Fig. 9 corresponds to the case of a northern declination, that is, $\delta = |\delta|$, and $A = 90° + a$, when we have to take the positive sign in the l.h.s. of the above equation. Hence,

$$\textit{bhujā} = \textit{śaṅkutala} - \textit{agrā}, \quad \delta \text{ north, and } A = 90° + a. \tag{3.18}$$

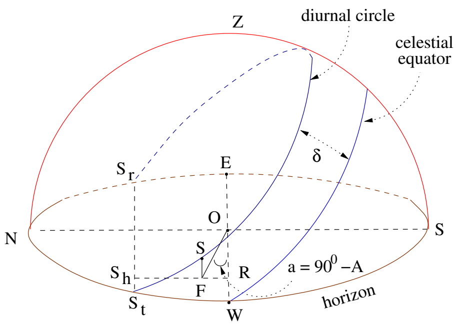

Fig. 10. Geometry for the zenith distance, z for a northern declination and $A < 90°$.

Fig. 10. depicts the situation when the declination is north, and $A = 90° - a$, in which case, we have to take the negative sign in the l.h.s. of the equation, and

$$\textit{bhujā} = \textit{agrā} - \textit{śaṅkutala}, \quad \delta \text{ north, and } A = 90° - a. \tag{3.19}$$

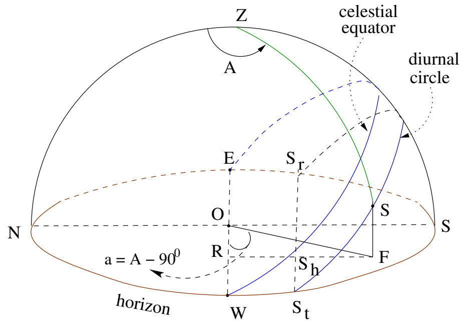

Fig. 11. Geometry for the zenith distance, z for a southern declination and $A > 90°$.

When the declination is south, $\delta = -|\delta|$, and *agrā* is $= -\frac{R\sin\delta}{\cos\phi}$. Here $A = 90° + a$, necessarily. In this case, shown in Fig. 11,

$$bhuj\bar{a} = \acute{s}a\dot{n}kutala + agr\bar{a}, \quad \delta \text{ south.} \tag{3.20}$$

Actually, these relations follow from the definitions of *bhujā*, *śaṅkutala*, and *agrā* and the geometry of the problem, as clear from the figures. They are equivalent to the cosine formula for the side PS.

In the explanation (*upapatti*) for verse 30 of the third chapter of *Grahagaṇita*[8], Bhāskara states these relations [6]:

> *svāgrāsvaśaṅkutalayoryāmyagoleyogaḥ soumyetvantaraṁ bhujo bhavati*|
>
> The sum of the *agrā* and the *śaṅkutala* in the southern hemisphere, and their difference in the northern hemisphere gives the *bhuja*.

3.5.2 Finding the zenith distance

The equation relating the *bhujā*, *agrā* and *śaṅkutala* is only the first step in solving for the *śaṅku*, $R\cos z$, and the zenith distance, z from that. In the explanation (*upapatti*) for the verses 49, 50 and 51 in the chapter on *Tripraśna* in the *Grahagaṇita* part of *Siddhāntaśiromaṇi* [8], Bhāskara casts the equation in terms of a '*chāyā- karṇa*' or the 'shadow hypotenuse', and then discusses the solution of the resulting quadratic equation. We first discuss the setting up of the equation, in the following.

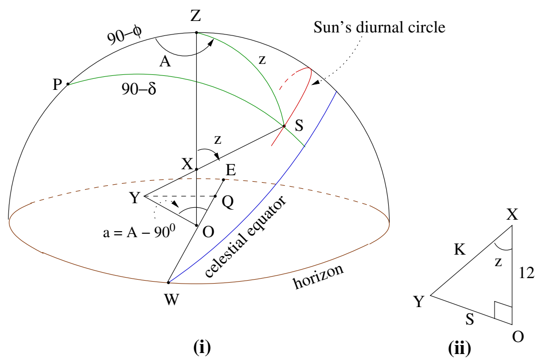

Fig. 12. The 12-digit gnomon (*dvādaśāṅgulaśaṅku*), the shadow (*chāyā*) S and the shadow-hypotenuse (*chāyākarṇa*) K.

In Fig. 12, we consider the same situation as in Fig. 9, when the Sun has a declination δ, zenith distance z, and azimuth A, for a location with latitude ϕ. OX is a 12-digit gnomon, or "*dvādaśāṅgula śaṅku*": $OX = 12$. Then $OY = S = 12\,\frac{\sin z}{\cos z}$ is the shadow of this gnomon, or the

[6] Reference [8], p. 57.

"*chāyā*", and $K = \frac{12}{\cos z}$ is the "*chāyā-karṇa*", or the shadow-hypotenuse. Draw YQ perpendicular to the east-west line. $YQ = B = S\sin a = 12\frac{\sin z}{\cos z}\sin a$, where $A = 90^\circ \pm a$. B is called the "*chāyābhujā*". In the figure, $A = 90^\circ + a$.

Note that $K^2 = S^2 + 12^2 = S^2 + 144$.

Multiplying equation (17) by $K = \frac{12}{\cos z}$, and dividing by R, we find

$$\pm B\ (chāyābhujā) = 12\frac{\sin\phi}{\cos\phi} - \frac{K}{R}\frac{R\sin\delta}{\cos\phi}. \tag{3.21}$$

On the equinoctial day, when $\delta = 0$, and the Sun moves on the equator, we note that the *chāyābhujā*, which is the distance between the tip of the shadow wnd the east-west line is a constant, $s = 12\frac{\sin\phi}{\cos\phi}$. This is called the "*palabhā*". Hence, on the equinoctial day, the tip of the shadow of the gnomon moves on a straight line parallel to the east-west line, at a distance equal to the *palabhā*. Note that the *chāyābhujā* is the shadow itself at noon, when the Sun is on the meridian, and $a = 90^\circ$. Hence, the *palabhā*, $s = 12\frac{\sin\phi}{\cos\phi}$, is the equinoctial mid-day shadow.

Now, denoting the *agrā* $|\frac{R\sin\delta}{\cos\phi}|$ by $\mathcal{A}$, and multiplying equation (3.21) by R, we find that

$$BR = sR \sim K\mathcal{A},\quad \delta \text{ north}, \tag{3.22}$$

$$BR = sR + K\mathcal{A},\quad \delta \text{ south}. \tag{3.23}$$

Now, $BR = s\sin a\ R = s\ D$, where $D = R\sin a$ is the *digjyā*. Squaring the equations (3.22, 3.23) and noting that $K^2 = s^2 + 144$, we obtain the following equation for K:

$$(K^2 - 144)D^2 = K^2\mathcal{A}^2 \pm 2K\mathcal{A}sR + s^2R^2,\quad \text{"}+\text{" for } \delta \text{ south, "}-\text{" for } \delta \text{ north.} \tag{3.24}$$

After rearranging the terms, and dividing by $D^2 - \mathcal{A}^2$ we have the following quadratic equation for K:

$$K^2 \mp 2K\frac{\mathcal{A}sR}{D^2 - \mathcal{A}^2} = \frac{s^2R^2 + 144D^2}{D^2 - \mathcal{A}^2},\quad \text{"}+\text{" for } \delta \text{ south, "}-\text{" for } \delta \text{ north.} \tag{3.25}$$

Here it can be recollected that

$$K\ (chāyākarṇa) = \frac{12}{\cos z}, \mathcal{A}\ (agrā) = |\frac{R\sin\delta}{\cos\phi}|,\ s\ (palabhā) = 12\frac{\sin\phi}{\cos\phi}, D\ (digjyā) = R\sin a. \tag{3.26}$$

The procedure for solving the above quadratic equation for K is which is spelt out clearly in the remaining part of the *upapatti* for the cited verses [7], and the explanation in modern notation are discussed in detail elsewhere [11]. We summarise it below.

Two variables "*Ādya*" denoted by x, and "*Anya*" denoted by y are defined through the relations:

$$x = \frac{s^2R^2 + 144D^2}{D^2 - \mathcal{A}^2},\quad \text{and}\quad y = \frac{\mathcal{A}sR}{D^2 - \mathcal{A}^2}. \tag{3.27}$$

Then, the formal solutions of the quadratic equation for K are given by:

$$K = y \pm \sqrt{x + y^2},\ \text{for } \delta \text{ south, and}\ \ K = -y \pm \sqrt{x + y^2},\ \text{for } \delta \text{ north.} \tag{3.28}$$

However, for the physical solutions, the zenith distance $z \leq 90^\circ$, and K should be positive. We consider the various cases now.

1. The declination δ is south.

[7] Reference [8], pp. 85-86.

In this case, from Fig. 9, it is clear that the *digjyā*, D is necessarily greater than the *agrā*, $\mathcal{A}$. Then x is positive, and $\sqrt{x+y^2} \geq y$. Hence, for positive K, only the "+" sign in front of the square root is permissible, and

$$K = y + \sqrt{x+y^2}, \text{ for } \delta \text{ south.} \tag{3.29}$$

2. The declination δ is north, and $D > \mathcal{A}$.
In this case also, only the "+" sign in front of the square root is permissible, and

$$K = -y + \sqrt{x+y^2}, \text{ for } \delta \text{ north, and } D > \mathcal{A}. \tag{3.30}$$

3. The declination δ is north, and $D < \mathcal{A}$.
In this case, both x and y are negative, $-y = |y|$, and $\sqrt{x+y^2} < |y|$. Then both the solutions result in positive K and

$$K = -y \pm \sqrt{x+y^2}, \text{ for } \delta \text{ north, and } D < \mathcal{A}. \tag{3.31}$$

The two solutions correspond to the location of the Sun south and north of the prime verical with the same value of a, and hence the same value of the *digjyā*, D, but different values of A, namely $90^\circ \pm a$. All these cases are discussed by Bhāskara.

3.6 Concluding remarks

Spherical trigonometry is very much needed for solving diurnal problems, eclipse calculations etc. In Indian texts correct solutions were indeed found for most of the problems using appropriate geometrical constructions. Bhāskara (born 1114 CE) was a master in this. He gives the algorithms and gives proofs of them in his *Siddhāntaśiromaṇi* (1150 CE) with his own commentary. In this article, we have given some examples of his treatment of problems involving spherical variables.

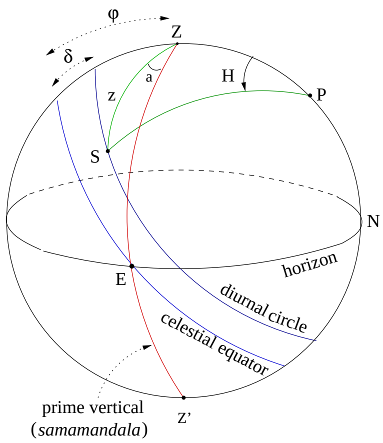

Fig. 13. The five variables: zenith distance (z), declination (δ), latitude (ϕ), azimuth (a), and the hour angle (H) of a celestial object.

There was a significant advance of spherical trigonometry by the Kerala school, especially in *Tantrasaṅgraha* of Nīlakaṇṭha Somayājī (1500 CE)[5], and the commentary on it, namely, *Yuktibhāṣā* of Jyeṣṭhadeva (1530 CE) [6]. In these texts, the problems involving spherical variables are systematised. Here, many derivations are based on manipulations **on** the spherical surface and **not necessarily in the interior** of the sphere as in Bhāskara's methods described in this article. This is true for the solutions of the "Ten problems" (*daśapraśnāḥ*). Consider the five variables: zenith distance (z), declination (δ), latitude (ϕ), (Indian) azimuth (a), and the hour angle (H), as shown in Fig. 13. The ten problems refer to the methods for obtaining any two of them, given the other three. The solutions are all exact. There are also new results like the exact expression for the declination of a planet with latitude, for instance, or an exact expression for the inclination of Moon's orbit with the equator at any instant, involving the longitude of the ascending node of the Moon. *Karaṇapaddhati* of Putumana Somayaji (around 1550 CE) [4] carries forward the tradition further.

References

1. Arkasomayaji, '*Siddhāntaśiromaṇi of Bhāskarācārya-II, Grahagaṇitādhyāya*', Translation of the text with explanations, Rashtriya Sanskrit Vidyapeetha, Tirupati, 2000.

2. Murali Dhara Chaturvedi (Ed.), '*Siddhāntaśiromaṇi* of *of Bhāskarācārya with Vāsanābhāṣya and Vārtika of Nṛsiṃha Daivajña*', Sampurnanand Sanskrit University, Varanasi, 1981.

3. R. C. Gupta, 'Bhāskara-II' in '*Encyclopaedia of the History of Science, Technology and Medicine in Non-Western Cultures*', ed. by Helaine Selin, pp. B17-B19, Springer, 2008.

4. R. Venketeswara Pai, K. Ramasubramanian, M. S. Sriram and M. D. Srinivas, '*Karaṇapaddhati* of Putumana Somayāji', Translation with mathematical notes Hindustan Book Agency, Delhi, 2017.

5. K. Ramasubramanian and M. S. Sriram, '*Tantrasaṅgraha of* Nīlakaṇṭha Somayājī', Translation with mathematical notes and jointly published by Hindustan Book Agency, Delhi and Springer, London, 2011.

6. K. Ramasubramanian, M. D. Srinivas and M. S. Sriram, '*Gaṇita-yukti-bhāṣā* of Jyeṣṭhadeva', Edition and Translation by K. V. Sarma, with Explanatory notes Hindustan Book Agency, Delhi, 2008; reprint by Springer, 2009.

7. Bāpū Deva Śāstri, '*Golādhyāya of Siddhāntasiromaṇi*', Translation by Wilkinson, revised, by Calcutta, 1861.

8. Gaṇapati Śāstri, '*Siddhāntaśiromaṇi of Bhāskarācārya with his Vāsanābhāṣya*', ed. Bāpū Deva Śāstri, revised Fourth edition, Chaukhambha Sanskrit Sansthan, Varanasi, 2005.

9. K. S. Shukla and K. V. Sarma, '*Āryabhaṭīya* of Āryabhaṭa', Edition with Translation, Indian National Science Academy, New Delhi, 1976.

10. M. S. Sriram, 'Bhāskarācārya's Astronomy' in '*History of Indian Astronomy - A Handbook*', ed. by K. Ramasubramanian, Aniket Sule and Mayank Vahia, SandHI, IIT Bombay and TIFR, Mumbai, pp. 349-378, 2016.

11. M. S. Sriram, '*Grahagaṇitādhyāya of Bhāskarācārya's Sidhhāntaśiromaṇi*', in 'Bhāskara Prabhā', Eds. K. Ramasubramanian, Takao Hayashi and Clemency Montelle, Hindusthan Book Agency, New Delhi, pp. 197 - 231, 2019.

12. M. S. Sriram, Sita Sundar Ram and R. Venketeswara Pai, '*Bhāskarācārya's Siddhāntaśiromaṇi: Grahagaṇitādhyāya*', Translation of the text and *Vāsanābhāṣya* with explanatory notes, Hindustan Book Agency, New Delhi, in Press.

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4. What is Happening in the Mathematical World?

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4.1 Two Long Standing Problems in Representation Theory have been Solved

Two mathematical breakthroughs have been achieved by Pham Tiep, a professor at Rutgers University, which could significantly advance the understanding of symmetries in nature and the behavior of various random processes. These findings could revolutionize our understanding of symmetries and random processes in fields such as physics, computer science, and even economics.

Tiep's first breakthroughs was solving 'The Zero Height Conjecture' proposed in 1955 by the renowned mathematician *Richard Brauer*.

The Conjecture is as follows: Let G be a finite group and p a prime. The set $\mathrm{Irr}(G)$ of irreducible complex characters can be partitioned into Brauer p-blocks. Each p-block B is canonically associated to a conjugacy class of p-subgroups, called the defect groups of B. The set of irreducible characters belonging to B is denoted by $\mathrm{Irr}(B)$.

Let ν be the discrete valuation defined on the integers by $\nu(mp^{\propto}) = \propto$ where m is coprime to p. Brauer proved that if B is a block with defect group D then $\nu(\chi(1) \geq \nu(\lceil G : D\rceil)$ *for each* $\chi \in \mathrm{Irr}(B)$. **Brauer's Height Zero Conjecture** asserts $\nu(\chi(1)) = \nu(\lceil G : D\rceil)$ that for all $\chi \in \mathrm{Irr}(B)$ if and only if D is abelian (See [2], [4]).

Tiep's solution unveils a hidden rule of symmetry that could transform how scientists model complex systems - from molecular structures to quantum mechanics. Understanding group structures plays a critical role in computer algorithms, data encryption, and even material science.

Tiep's work on the Height Zero Conjecture was a joint effort with several international colleagues, including Gunter Malle from Germany, Gabriel Navarro from Spain, and Amanda Schaeffer Fry, a former student now at the University of Denver. Proof of the conjecture was published in the September issue of the *Annals of Mathematics*.

Tiep's second major contribution is a solution of a difficult problem in what is known as the Deligne-Lusztig theory, part of the foundational machinery of representation theory. Tiep and coauthors have obtained bounds on traces, which confirm the long standing anticipations of experts in the field. The work is detailed in two papers, one was published in Inventiones mathematicae, vol. 235 (2024), the second in Annals of Mathematics, vol. 200 (2024). Tiep's discovery introduces a new method to solve matrix-related problems, potentially revolutionizing the way mathematicians and scientists analyze large-scale systems.

For the second breakthrough, Tiep worked with Robert Guralnick of the University of Southern California and Michael Larsen of Indiana University. On the first of two papers that tackle the mathematical problems on traces and solve them, Tiep worked with Guralnick and Larsen. Tiep and Larsen are co-authors of the second paper.

Tiep's solutions to long-standing problems in group theory and matrix analysis could influence everything from developing next-generation AI models to improving the efficiency of telecommunication networks.

Sources:

1. *https://glassalmanac.com/mathematician-cracks-two-decades-old-problems -a- historic -milestone/*
2. *https://www.thebr Defect group of a block - Encyclopedia of Mathematics ighterside.news/post /professor- solves-two-long-standing-math-problems-advances-our-understanding-of-nature/*
3. *https://www.rutgers.edu/news/double-breakthrough-mathematician-solves-two-long-standing - problems*
4. *https://en.wikipedia.org/wiki/Brauer%27s_height_zero_conjecture*

4.2 A Broader Version of Hilbert's Famous 10^{th} Problem has been Proved

In 1900, the eminent mathematician *David Hilbert* Introduced a list of 23 key problems which were to guide the next century of mathematical research. The problems aimed at building a firm foundation from which all mathematical truths could be derived. A key part of this vision was that mathematics should be "complete". That is, all its statements should be provably true or false.

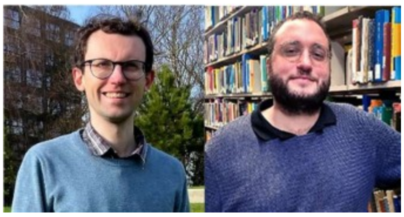
In the 1930s, *Kurt Gödel* demonstrated that this is impossible: In any mathematical system, there are statements that can be neither proved nor disproved. A few years later, *Alan Turing* and others built on his work, showing that mathematics is riddled with "undecidable" statements - statements whose validity cannot be confirmed or negated by any computer algorithm.

Hilbert's 10^{th} problem concerns Diophantine equations - polynomial equations saught to be solved in integers. For millennia, mathematicians searched for integer solutions to them.

Hilbert's 10^{th} problem asked whether it is always possible to tell if a given Diophantine equation has integer solutions. Does an algorithm exist to determine this for every equation?

In 1970, a Russian mathematician *Yuri Matiyasevich* showed that there is no general algorithm that can determine whether given Diophantine equation has integer solutions - that Hilbert's 10^{th} problem is an undecidable problem. You might be able to come up with an algorithm that can assess most equations, but it won't work for every single one.

Mathematicians wanted to test the reach of Matiyasevich's conclusion. The equations in question always have solutions over complex numbers, so if one of them is unsolvable for integers, then the question arises as to where the cut-off arises when you pass to larger systems of numbers. In the 50 years since Hilbert's 10^{th} problem was resolved, mathematicians have been searching for this cutoff.



Now, (from left to right) *Peter Koymans*, a mathematician at Utrecht University and his longtime collaborator, *Carlo Pagano* of Concordia University in Montreal - as well as another team of researchers working independently - have taken a major step toward that goal. Both groups have proved that for a vast and important collection of settings beyond integers, there is likewise no general algorithm to determine if any given Diophantine equation has a solution.

The new proofs focused on a natural extension of Hilbert's 10^{th} problem. The extension deals with Diophantine equations whose solutions belong to number systems which can be obtained by starting with a finite set of numbers (like $\left\{1, -1, \sqrt{2}\right\}$) and adding those numbers in different combinations, called rings of integers. Mathematicians suspected that, for every single ring of integers the problem is still undecidable.

In general, undecidability proofs follow the same recipe: They show that the problem of interest is equivalent to a famous undecidable problem in computer science called the halting problem. The halting problem asks whether an idealized computational device called a Turing machine, when fed a given input, will run forever or eventually halt. It's known that there's no algorithm that can answer this for every Turing machine.

To settle Hilbert's original 10^{th} problem, mathematicians built on the work that began with Julia Robinson and others around 1950, and culminated in Matiyasevich's 1970 result, in which it was shown that for every Turing machine there is a corresponding Diophantine equation. The useful correspondence between Turing machines and Diophantine equations falls apart when the equations are allowed to have non-integer solutions.

A way to resolve the issue was found by Sasha Shlapentokh, and others. They decided to change the Diophantine equation by adding few terms so that solutions to original equation in a new ring of integers is equivalent to an integer solution to the revised equation and thereby re-establishing correspondence with Turing machines. They also figured out what terms they had to add to the Diophantine equations for various kinds of rings, could be determined using a special equation

representing an elliptic curve. However, building such an elliptic curve that worked for every ring of integers was an extremely subtle and difficult task.
In summer 2024, Koymans and Pagano could build an elliptic curve which gave them the recipe they needed to add terms to their Diophantine equations, which then enabled them to encode Turing machines - and the halting problem - in those equations, regardless of what number system they used. Thus, they proved that Hilbert's 10^{th} problem is undecidable for every ring of integers. The result was solidified further in February, 2025, less than two months after Koymans and Pagano posted their paper online, an independent team of four mathematicians announced a new proof of the same result. Instead of looking for a special elliptic curve, they had relied on a different kind of equation to do the same job.
Source: *https://www.wired.com/story/new-proofs-expand-the-limits-of-what-cannot-be-known/*

4.3 Mathematician Daniel Mathews Solve 380-Year-Old Problem Inspired by Descartes

A long-standing geometric mystery dating back to the 17^{th} century has finally been solved by Associate Professor *Daniel Mathews*, a mathematician at Monash University School of Mathematics, shedding new light on an equation first written by philosopher and mathematician *René Descartes*.
The discovery extends the famous Descartes Circle Theorem stated in 1643, which describes the relationship between four mutually tangent circles. The theorem describes a quadratic equation in the radii of the circles such that when it is satisfied, one can construct a fourth circle tangent to three given, mutually tangent circles. Despite centuries of mathematical progress, a general equation for larger configurations of circles remained elusive - until now. Daniel Mathews found the equation that governs these larger patterns of tangent circles, known as "n-flowers".

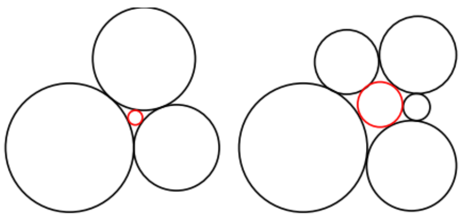


Figure 1: Left, a 3-flower. Right, a 5-flower.

For $n \geq 3$, an n-flower consists of a central circle C_∞, and n petal circles C_j , over integers $j \mod n$, so that the C_j are externally tangent to C_∞ in order around C_∞, and each C_j is externally tangent to C_{j-1} and C_{j+1}. (See Figure 1). The curvature of a circle C_* is denoted by k_*. Descartes' theorem gives an equation satisfied by the curvatures in a 3-flower:

$$(k_\infty + k_1 + k_2 + k_3)^2 = 2(k_\infty^2 + k_1^2 + k_2^2 + k_3^2).$$

Generalised Descartes Theorem, proved by Daniel Mathews is as follows: Let a n-flower consist of a central circle C_∞, and n petal circles $C_j (j \in \mathbb{Z}/n\mathbb{Z})$.
Define m_0, and m_j for $1 \leq j \leq n-1$ as $m_0 = \sqrt{\frac{k_0}{k_\infty} + 1}, m_j = \sqrt{\left(\frac{k_j}{k_\infty} + 1\right)\left(\frac{k_{j-1}}{k_\infty} + 1\right)}$.
Then for odd n, the following holds:

$$\frac{m_0^2 i}{2}\left(\Pi_{j=1}^{n-1}\left(m_j - i\right) - \Pi_{j=1}^{n-1}\left(m_j + i\right)\right) - \Pi_{j=1}^{(n-1)/2}\left(m_{2j-1}^2 + 1\right) = 0.$$

For even n, the following holds: $\frac{i}{2}\left(\Pi_{j=1}^{n-1}\left(m_j - 1\right) - \Pi_{j=1}^{n-1}\left(m_j + i\right)\right) - \Pi_{j=1}^{(n-1)/2}\left(m_{2j}^2 + 1\right) = 0.$
The proof, which draws on modern mathematical techniques involving spinors - objects that also play a role in quantum mechanics and relativity – solves a problem that has remained open for more than 380 years [3]. Mathews used a version of spinors developed by Nobel prize-winner Roger Penrose and Wolfgang Rindler, which they applied to the theory of relativity.
Others have generalised the result in other ways, but this is the first extension of the result to give an explicit equation relating the radii of an arbitrary number of circles in the plane.
This discovery is an exciting example of how classical problems can inspire new mathematics centuries later.

Sources:

1. *https://www.monash.edu/science/news-events/news/2025/mathematicians-solve-380-year-old- problem-inspired-by-descartes*
2. *https://en.wikipedia.org/wiki/Descartes%27_theorem*
3. *https://www.danielmathews.info/wp-content/uploads/2023/10/spinors_and_descartes _theorem.pdf*

4.4 A Major Group Theory Problem, 'Mckay Conjecture' has been Settled

In 2003 *Britta Späth*, a German graduate student, encountered a McKay conjecture, one of the big open problems in group theory and decided to dedicate all her time to working on it. She, now a Professor at the University of Wuppertal in Germany, has finally succeeded, together with her partner, Marc Cabanes, a mathematician now at the Institute of Mathematics of Jussieu in Paris.

The problem that absorbed them takes a key theme in mathematics and turns it into a concrete tool for group theorists. The McKay conjecture, named after the Canadian mathematician John McKay, who originally stated a limited version of it as a conjecture in 1971, for the special case of prime $p = 2$ and simple groups. The conjecture was later generalized by other mathematicians to a more general conjecture for any prime p and more general groups.

The conjecture states that: Suppose p is a prime number, G is a finite group, and P is a Sylow p-subgroup of G. Then the irreducible complex characters of G are in one-one correspondence with the irreducible complex characters of the normalizer of P.

After the conjecture was posed, several mathematicians tried their hand at proving it. They made partial progress - and in the process they learned a great deal about groups. But a full proof seemed out of reach.

The McKay conjecture for the prime 2 was proven by Gunter Malle and Britta Späth in 2016 [2]. A proof of the McKay conjecture for all primes and all finite groups was announced by Britta Späth and Marc Cabanes in October 2023 in various conferences, a manuscript on it was put out later in 2024 [3].

Sources:

1. *https://www.quantamagazine.org/after-20-years-math-couple-solves-major-group-theory-problem- 20250219/*
2. *Malle, Gunter; Späth, Britta (2016). "Characters of odd degree". Annals of Mathematics. 184: 869- 908. doi:10.4007/annals.2016.184.3.6*
3. *Marc Cabanes; Britta Späth (2024). "The McKay Conjecture on character degrees". arXiv:24-10.20392 [RT].*

4.5 The Kakeya Conjecture has been Solved for Dimension Three

Chinese mathematician *Wang Hong* has solved a geometry problem called the Kakeya conjecture in three dimensions. It is considered a breakthrough that could have implications for imaging, data processing, cryptography and wireless communication.

The conjecture goes back to 1917, when Japanese mathematician Sōichi Kakeya posed a problem: If you place an infinitely thin needle onto a surface and rotate it to point in every single direction, what is the smallest area the needle can cover?

In 1928, Besicovitch proved that one could in fact rotate a needle in arbitrary small amounts of area.This led to the definition of a Kakeya set in $\mathfrak{R}^2$ to be a set which contained a unit line segment in every direction. Besicovitch's construction showed that Kakeya sets in $\mathfrak{R}^2$ could have arbitrarily small measure; in fact, one can construct Kakeya sets which have Lebesgue measure zero. The same question of how small these Kakeya sets could be was then posed in higher dimensions, giving rise to Kakeya set conjecture: A Kakeya set in $\mathfrak{R}^2$ has Hausdorff and Minkowski dimension n. The Kakeya conjecture was solved for $n = 2$ by Davies in 1971, but remained open for $n \geq 3$. The updates on the conjecture were discussed in section 3.2 of Issue 2 of TMCB Vol. 5 in October 2023. In 1999, Nets Katz and Terence Tao showed that any counterexample to the conjecture must be "plany", which means that whenever line segments intersect at a point, those segments also lie nearly in the same plane. It must also be "grainy", which requires that the planes of nearby points of intersection be similarly oriented. However, they couldn't prove that all counterexamples must be sticky, which would complete the proof of the conjecture. In a "sticky" set, line segments that point in nearly the same direction also have to be located close to each other in space which in turn force a lot of overlap among the line segments, thereby making the set as small as possible-precisely what you need to create a counterexample.

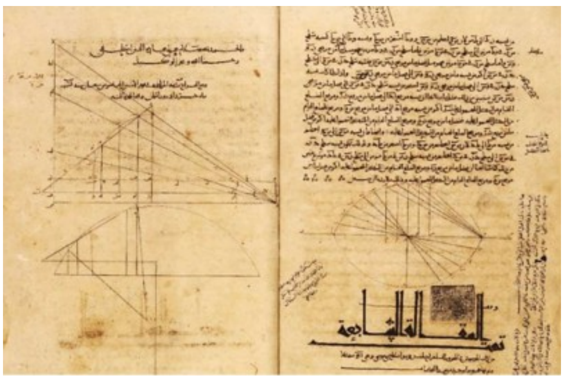
In October 2022, Wang Hong, an associate professor at the New York University Courant Institute of Mathematical Sciences (left) and her collaborator *Joshua Zahl*, from The University of British Columbia (right), proved that there are no sticky counter examples, with a Minkowski dimension of less than 3, to the Kakeya conjecture [2].

Now, Wang and Zahl have claimed to have proved that, in three dimensions, a Kakeya set does indeed have Hausdorff dimension 3 and Minkowski dimension 3. They presented their milestone proof in a 127-page preprint paper on the open-access repository arXiv on Feb. 24, 2025 [3] which is considered to be a significant progress in geometric measure theory.

Sources:

1. *https://www.scmp.com/news/china/science/article/3300958/chinese-maths-star-wang-hong-solves-infamous-geometry-problem*
2. *"Sticky Kakeya sets and the sticky Kakeya conjecture" (Submitted on 18 Oct 2022), arXiv: 2210.09581 [math.CA].*
3. *Hong Wang; Joshua Zahl (2025-02-24). "Volume estimates for unions of convex sets, and the Kakeya set conjecture in three dimensions". arXiv:2502.17655 [math.CA].*

4.6 Scientists Rediscover Lost Works of Apollonius in a 17^{th}-Century Manuscript



Researchers have found two lost books by *Apollonius*, the ancient Greek mathematician known as the Great Geometer. The works were found in an Arabic manuscript preserved at the Libraries of Leiden University in the Netherlands. The manuscript had been forgotten in the library, part of a collection acquired by Dutch mathematician*Jacob Golius* in the 17^{th} century.

The rediscovered manuscript contains the lost fifth and seventh books of Apollonius's renowned work, *the Conics*. It introduces fundamental geometric concepts such as hyperbolas, ellipses, and parabolas. These concepts had an impact on the scientific world and are noted for their influence on the ancient mathematics.

Only four of the original eight books of the Conics were available to European scholars during the Renaissance, as the fifth to eighth books were considered lost for centuries. The rediscovery of the lost books reflects the Islamic Golden Age's contributions to preserving knowledge. Islamic scholars preserved and expanded ancient knowledge, which later contributed to the European Renaissance.

The manuscript is a translation of books five to seven by *Thabit ibn Qurra*, edited by the *Banū Mūsā* brothers. It is accompanied by illustrations and Arabic calligraphy, illuminating the history of mathematics. These manuscripts not only contain mathematical knowledge but also attract attention with calligraphy and geometric illustrations.

Apollonius of Perga, born around 260 BCE in the ancient Greek city of Perga, is known for his pioneering work in geometry. He studied and taught in Alexandria and was one of the greatest mathematicians and geometers of antiquity. Of the 21 works on mathematics, geometry, astronomy, and mechanics that Apollonius wrote, only four have survived.

Source: *https://www.jpost.com/archaeology/archaeology-around-the-world/article-841280*

4.7 Awards

4.7.1 Angkana Rüland receives Leibniz Prize for her Outstanding Research

University of Bonn mathematician *Angkana Rüland* receives the Gottfried Wilhelm Leibniz Prize from the German Research Foundation (DFG) which is endowed with 2.5 million euros, in recognition of her excellent research work. The researcher from the Hausdorff Center for Mathematics (HCM) at the University of Bonn is honored with the award for her outstanding work in mathematical analysis, particularly on models for microstructures in phase transitions in solids and inverse problems with non-local operators.

In her research on microstructures, she is particularly interested in a class of alloys that have shape- memory properties.

Her work on inverse problems, is about reconstructing information from indirect measurements - such as is done with X-ray tomography or ultrasound scans, for instance. “This indirect information lets you infer information on someone’s body without having to take any tissue samples”, Angkana Rüland explains.

Rüland was born in 1987 in Chiang Mai, and was a mathematics student at the University of Bonn. She completed her doctorate in 2014 with the dissertation *On Some Rigidity Properties in PDEs* supervised by Herbert Koch. In 2014, she was awarded the “Hausdorff Memorial Prize” for the best doctoral thesis in mathematics.

After postdoctoral research at the University of Oxford, working there with John M. Ball, she became a researcher at the Max Planck Institute for Mathematics in the Sciences in 2017. She took a professorship at Heidelberg University in 2020 before returning to the University of Bonn in 2023.

The highly endowed Leibniz prize allows great freedom in research. Rüland, who is also a member of the Transdisciplinary Research Area “Modeling” at the University of Bonn, would like to use the prize money to further expand her research group at the HCM.

Sources:

1. *https://www.uni-bonn.de/en/news/240-2024*
2. *https://www.mathematics.uni-bonn.de/en/news/leibniz-prize-awarded-to-angkana-ruland*
3. *https://en.wikipedia.org/wiki/Angkana_R%C3%BCland*

4.7.2 Three Distinguished Mathematicians Receive The 2025 Leroy P. Steele Prizes

The *Leroy P. Steele Prizes* are awarded every year by the American Mathematical Society, for distinguished research work and writing in the field of mathematics. Since 1993, there has been a formal division into three categories. The Steele Prize for Lifetime Achievement is awarded for the cumulative influence of the total mathematical work of the recipient, high level of research over a period of time, particular influence on the development of a field, and influence on mathematics through Ph.D. students. The Leroy P. Steele Prize for Mathematical Exposition is awarded annually for a book or substantial survey or expository research paper. The Steele Prize for Seminal

Contribution to Research is awarded for a paper, whether recent or not, that has proved to be of fundamental or lasting importance in its field, or a model of important research.

2025 Leroy P. Steele Prize for Lifetime Achievement:

English mathematician *Dusa McDuff* received the 2025 Leroy P. Steele Prize for Lifetime Achievement from the American Mathematical Society for her outstanding contributions in C*-algebras, symplectic geometry and topology, as well as her leadership and mentoring in mathematics.

The prize is awarded for foundational and far-reaching contributions and long-continued leadership and mentoring in mathematics. Specializing in the structures and properties of space, she analyzes the interactions between pairs of quantities by measuring two-dimensional areas.

She was the first recipient of the Ruth Lyttle Satter Prize in Mathematics, was a Noether Lecturer, and is a Fellow of the Royal Society. She is currently the Helen Lyttle Kimmel '42 Professor of Mathematics at Barnard College, New York.

2025 Leroy P. Steele Prize for Mathematical Exposition:

New Zealand mathematician working in arithmetic geometry - *James S. Milne* has been awarded 2025 Leroy P. Steele Prize for Mathematical Exposition for his "extensive corpus of excellent expository works" provided on his website.

The website, which Milne has been developing since 1996, now contains over 2,000 pages of notes, as well as other expository articles, covering a wide range of topics within algebra and number theory, from basic group theory to class field theory to abelian varieties to Shimura varieties to Tannakian categories and much more.

Many of the documents began as course notes but have been expanded and polished over decades to become some of the most thorough and well-written accounts available of the topics they cover. The inclusion of ample historical remarks and guides to the literature adds value for both newcomers and experts. The expository works have educated a generation of arithmetic geometers and will continue to do so for as long as they are available.

Milne is Professor Emeritus of Mathematics at the University of Michigan.

2025 Leroy P. Steele Prize for Seminal Contribution to Research:

Professor *Kenneth Alan Ribet* will receive the 2025 AMS Leroy P. Steele Prize for Seminal Contribution to Research for his groundbreaking 1976 paper "A modular construction of unramified p-extensions of $Q(\mu_p)$".

Kenneth Ribet is an American mathematician working in algebraic number theory and algebraic geometry. He is known for the Herbrand-Ribet theorem and Ribet's theorem, which were key ingredients in the proof of Fermat's Last Theorem, as well as for his service as President of the American Mathematical Society from 2017 to 2019.

Ribet is credited with paving the way towards Andrew Wiles's proof of Fermat's Last Theorem. In 1986, Ribet proved that the epsilon conjecture formulated by Jean-Pierre Serre was true, and thereby proved that Fermat's Last Theorem would follow from the Taniyama-Shimura conjecture. He is currently a professor of mathematics at the University of California, Berkeley.

Sources:

1. *https://www.ams.org/news?news_id=7402*
2. *https://en.wikipedia.org/wiki/Dusa_McDuff*
3. *https://en.wikipedia.org/wiki/Ken_Ribet*

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5. A Peep into History of Mathematics

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Here are my picks for a peep into history for this issue.

5.1 Dipak Jadhav Object-numerals as listed in Nijaguṇa Śivayogī's *Viveka-Cintāmaṇī*, Indian Journal of History of Science 58 (2023), no. 1, 13-19.

Though the decimal place-value system, together with zero, for representing natural numbers has been around at least since mid-first millennium CE, numbers were seldom written, until the recent centuries, using symbols for the digits as we do today, whether in literary or in scientific works. The reason was that the works were composed in poetic forms, and it would be highly cumbersome, perhaps also incongruous, to fit expressions for numbers into metrical patterns. Instead, digits, or segments of them, in the desired number were substituted by words that would convey them; thus, for example, *vedarandhrarasākṣi* would stand for 2694, with *veda* for 4, *randhra* for 9, *rasa* for 6 and *akṣi* for 2, (in reverse order). The system is known as *bhūtasaṅkhyā*, object numerals or word numerals in English. Several different words could be adopted for the same digit, thus giving considerable flexibility for usage in a poetic format.

There have been some notable compilations of the object-numerals (the words used to substitute for the digits), including by H. R. Kapadia and more recently by K. S. Shukla, containing 449 and 945 entries respectively. The present paper discusses a list of 59 object numerals from the work *Viveka-Cintāmaṇī* of Nijaguṇa Śivayogī. The latter was a philosopher who very likely flourished in the 15^{th} century. The work describes in particular, in Kannada but with many Sanskrit terms, philosophies from many Vedic, Buddhist, Jain and materialist works - the book was translated into Marathi in 1604 and into Sanskrit in 1652. The focus of the paper is on the object numerals listed in the work. Comparisons of entries are made with those from the earlier compilations mentioned above. The paper also provides a good introduction to the topic of object numerals, including some of the history.

5.2 Adrian Rice, An enchantress of number? Reassessing the mathematical reputation of Ada Lovelace, Notices of the American Mathematical Society, Vol. 71 (2024), no. 3, pp. 374-385.

Ada Lovelace (1815-1852), one of the earliest celebrated women mathematicians from the modern times, is renowned for her 1843-paper containing a theoretical account of the *analytical engine* designed by Charles Babbage, which is an important milestone in the development of computers; incidentally, she was the daughter of the renowned poet Lord Byron, but was raised by her mother, as the parents got divorced. The last appendix in the 66-page long paper, mentioned as her "chief claim to fame" contains some thoughts on the possibility of artificial intelligence, and an outline of an iterative process by which Babbage's machine could compute the Bernoulli numbers.

In the subsequent period however, in parallel with the appreciation of her work, there have been misgivings in some quarters, casting doubts on her ability to make such a technical contribution. In this context the present article takes up a reassessment of her mathematical ability, analyzing what mathematical topics she would have been exposed to in the ten year period until the 1843 paper, the changing perspectives on her over a period, and the arguments involved in the negative assessments concerned. The author notes that research on this, based on study of archival material, in collaboration with Christopher Hollings and Ursula Martin, has led to two papers and a book, and that it provides "strong evidence that she did indeed have the mathematical competence to write and understand the mathematics contained in her famous paper of 1843".

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6. Problem Corner

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In the January 2025 issue of TMC Bulletin, we posed two problems, one from Geometry and one from Number Theory. So far, we have received a solution of problem from Geometry from Anmol Mishra form Valsad, Gujarat. However, we have not received any solution for the other problem.

The solution provided by Anmol is correct but lengthy and hence we present here, solution provided by problem proposer, Priyamvad Srivastav.

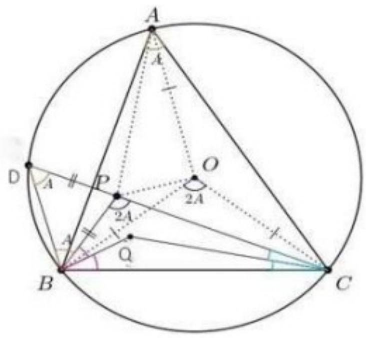
We would like to emphasize here that problem solving is an important activity in the process of learning mathematics. Hence, we appeal to all the teachers to encourage their students to attempt solving problems posed in this section.

In this issue, we pose two problems one from Number Theory by Dr. Vinaykumar Acharya and one form Combinatorics by Dr. Udayan Prajapati for our readers. Readers are invited to email their solutions to Dr. Udayan Prajapati (ganit.spardha@gmail.com), Coordinator, Problem Corner before 10^{th} June, 2025. Most innovative solution will be published in the subsequent issue of the Bulletin.

The first Problem posed in the last issue:

Let ABC be an actute angled triangle inscribed in a circle of radius R. Suppose P lies in the interior of the triangle such that $PB \neq PC, \angle BPC = 2\angle BAC$ and $PA^2 + PB \times PC = 2R^2$. If Q is the incenter of the triangle PBC, show that $\angle QBA = \angle QCA$.

Solution: (by the Problem proposer Priyamvad Srivastav)



Let O be the circumcenter of triangle ABC and denote the circumcircle by Γ. Then $OA = OB = OC = R$.

Moreover, since $\angle BPC = 2\angle BAC = 2\angle A$, it follows that B, P, O, C are concyclic.

Now, let CP meets Γ again in D. Then $\angle BDC = \angle A$ and $\angle PBD = \angle BPC - \angle BDC = 2\angle A - \angle A = \angle PDB$, from which it follows that $PB = PD$.

Therefore, $PB \times PC = PD \times PC = |OP^2 - R^2| = R^2 - OP^2$ (the power of the point P with respect to Γ).

Now, we have $2R^2 = PA^2 + PB \times PC = PA^2 + R^2 - OP^2$, and therefore $PA^2 = R^2 + OP^2 = OA^2 + OP^2$, which implies that $\angle AOP = 90^\circ$.

Now, in cyclic quadrilateral BPOC, we have, $\angle BCP = \angle BOP$, and therefore, $2\angle C = \angle AOB = \angle AOP + \angle BOP = 90^\circ + \angle BCP$.

And hence $\angle BCP = 2\angle C - 90\circ$. This immediately implies that $\angle CBP = 2\angle B - 90\circ$.

So, $\angle PBA = \angle ABC - \angle CBP = \angle B - (2\angle B - 90\circ) = 90 \circ - \angle B$.

Similarly, $\angle PCA = 90^\circ - \angle C$.

Now, since Q is the incenter of $\triangle PBC$, we have

$$\angle QBA = \angle QBP + \angle PBA = \frac{2\angle B - 90^\circ}{2} + 90^\circ - \angle B = 45^\circ. \text{ Similarly,}$$

$$\angle QCA = \angle QCP + \angle PCA = \frac{2\angle C - 90^\circ}{2} + 90^\circ - \angle C = 45^\circ.$$

Problems for this issue

Problem 1 (proposed by Vinay Acharya): Let a and b be distinct positive integers such that $3^a + 2$ is divisible by $3^b + 2$. Prove that $a > b^2$.

Problem 2 (proposed by Udayan Prajapati): Consider $n \times n$ grids having n^2 vertices (i, j) for all $i, j = 1, 2, \ldots, n$. Find the number of pairs of unit squares having vertices from the n^2 vertices such that the square regions have empty intersection (having no common edges or corners).

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7. International Calendar of Mathematics Events

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Note: Majority of events in July and August were included in the January Issue of TMCB. Here we include only those events of July and August which were announced later.

July 2025

- July 21-25, 2025, Formalizing Class Field Theory, Mathematical Institute, University of Oxford, UK. *www.claymath.org/events/formalizing-class-field-theory/*
- July 28-30, 2025, SIAM Conference on Computational Geometric Design (GD25), Montreal Convention Center, Montreal, Quebec, Canada. *www.siam.org/conferences/cm/conference/gd25*

August 2025

- August 10-12, 2025, The Mathematics of Various Entertaining Subjects Conference (MOVES 2025), NYU Courant, New York City, NY. *momath.org/moves-conference/*
- August 23-24, 2025, 2025 Fall Western Sectional Meeting, University of Denver, Denver, CO. *www.ams.org/meetings/sectional/2326_program.html*
- August 29-31, 2025, 25th International Pure Mathematics Conference 2025 (Silver Jubilee IPMC 2025), Islamabad, Pakistan. *www.pmc.org.pk*

September 2025

- September 1-4, 2025, Twelfth Conference on New Trends in the Applications of Differential Equations In Sciences (NTADES 2025), St. Constantin And Elena, Varna, Bulgaria. *www.ntades.eu*
- September 1-6, 2025, XV Annual International Conference of the Georgian Mathematical Union Batumi Shota Rustaveli State University, Batumi, Georgia. *gmu.gtu.ge/conferences/*
- September 2-4, 2025, 12^{th} International Congress on Fundamental and Applied Sciences 2025 (ICFAS2025), Fatih Sultan Mehmet Vakif University, Istanbul, Türkiye. *icfas2025.intsa.org/index.html*
- September 3-6, 2025, XII International Scientific Conference "Modern Problems of Mathematics and Mechanics" The Institute of Mathematics and Mechanics of The Ministry of Science and Education of The Republic of Azerbaijan, Baku/Azerbaijan. *mpmm.imm.az/*
- September 3-7, 2025, 9^{th} International Conference of Mathematical Sciences (ICMS 2025) Maltepe University Maltepe, Istanbul, Turkey. *www.maltepe.edu.tr/icms*

- September 4-6, 2025, International Conference on Mathematics and Mathematics Education (ICMME2025), Istanbul Medeniyet University, Ãœskudar, Istanbul, Turkey. *theicmme.org/*
- September 8-10, 2025, Advancing The Frontiers - International Conference on Algebra, Analysis, and Applications, Kutaisi International University, Georgia. *www.kiu.edu.ge/?m=530*
- September 8-12, 2025, The 12^{th} International Conference on Stochastic Analysis and Its Applications, University POLITEHNICA Bucharest, Bucharest, Romania. *sites.google.com/view/icsaa2025/acoel*
- September 29 - October 1, 2025, Conference on New Innovations in Material Science Frankfurt, Germany. *momentera.org/conferences/material-science/*
- September 29 - October 3, 2025, AIM Workshop: Multiscale Modeling of Ocular and Cardiovascular Systems, American Institute of Mathematics, Pasadena, California. *aimath.org/workshops/upcoming/ocularcardio/*

October 2025

- October 3-5, 2025, 2025 Fall Southeastern Sectional Meeting Tulane University, New Orleans, LA. *www.ams.org/meetings/sectional/2328_program.html*
- October 13-17, 2025, AIM Workshop: Flag Algebras and Extremal Combinatorics, American Institute of Mathematics, Pasadena, California. *aimath.org/workshops/upcoming/flagextremal/*
- October 14-17, 2025, SIAM Conference on Mathematical and Computational Issues in The Geosciences (GS25), Louisiana State University, Baton Rouge, Louisiana, U.S. *www.siam.org/conferences-events/siam-conferences/gs25/*
- October 15-16, 2025, The 1^{st} International Electronic Conference on Games (IECGA 2025) Online With Live Sessions. *sciforum.net/event/IECGA2025?utm_source=AMS&utm_medium=AMScal*
- October 18-19, 2025, 2025 Fall Central Sectional Meeting St. Louis University, St. Louis, MO. *www.ams.org/meetings/sectional/2322_program.html*
- October 19-23, 2025, 7^{th} School on Belief Functions and Their Applications Granada, Spain. *www.bfasociety.org/BFTA2025/*
- October 20-24, 2025, New Trends of Stochastic Nonlinear Systems: Well-Posedeness, Dynamics and Numerics, CIRM, 163 Avenue De Luminy, Case 916 13288 Marseille Cedex 9, France. *conferences.cirm-math.fr/3374.html*
- October 27-31, 2025, AIM Workshop: Computations in Stable Homotopy Theory American Institute of Mathematics, Pasadena, California. *aimath.org/workshops/upcoming/compstabhom/*

November 2025

- November 17-21, 2025, Recent Trends in Stochastic Partial Differential Equations, SL Math, 17 Gauss Way, Berkeley CA. *www.slmath.org/workshops/1148*
- November 17-20, 2025, SIAM Conference on Analysis of Partial Differential Equations (PD25), Sheraton Pittsburgh Hotel at Station Square Pittsburgh, Pennsylvania, U.S. *www.siam.org/conferences-events/siam-conferences/pd25/*

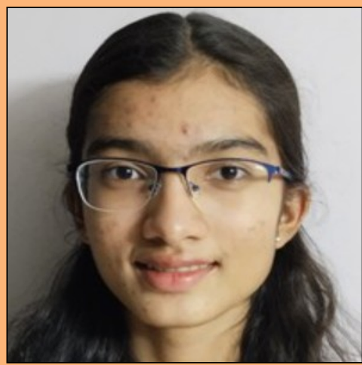
□ □ □

India at
European Girl's Mathematical Olympiad (EGMO) 2025
In Pristina, Kosovo (11-17, April, 2025)

Indian Contestants

Silver Medalists

Sanjana Chacko



Shreya Mundhada

Bronze Medalists

Saee Patil

Shreya Gupta Ray

Rank	Name	P1	P2	P3	P4	P5	P6	Total	Medal
28	Sanjana Chacko	7	4	1	7	7	0	26	Silver
42	Shreya Shantanu Mundhada	7	7	0	7	2	0	23	Silver
74	Saee Patil	4	2	1	7	4	0	18	Bronze
74	Shreya Gupta Ray	7	0	0	7	4	0	18	Bronze
National Result									
12	India (4 contestants)	25	13	2	28	17	0	85	2S + 2B
We congratulate all the contestants for their excellent performance.									

Kurt Gödel (28 April 1906 - 14 Jan. 1978)

Austrian-American logician, mathematician & philosopher. Had an immense effect upon scientific and philosophical thinking in the 20th century. Known for two Gödel incompleteness theorems. Developed a technique now known as Gödel numbering, which codes formal expressions as natural numbers. Also made important contributions to proof theory.

Edsger Wybe Dijkstra (11 May 1930 - 06 May 2002)

A Dutch computer scientist, physicist, mathematician. Contributed to diverse areas of computing science, including compiler construction, operating systems, distributed systems, sequential and concurrent programming, software engineering principles, graph algorithms. Coined the phrase "structured programming". Known for Dijkstra's shortest path Algorithm.

Alonzo Church (14 June 1903 - 11 Aug. 1995)

An American mathematician and logician who made major contributions to mathematical logic and the foundations of theoretical computer science. Best known for the lambda calculus, Church–Turing thesis, proving the undecidability of the Entscheidungs problem, Frege–Church ontology, and the Church–Rosser theorem. Also worked on philosophy of language.

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