The Mathematics Consortium

BULLETIN

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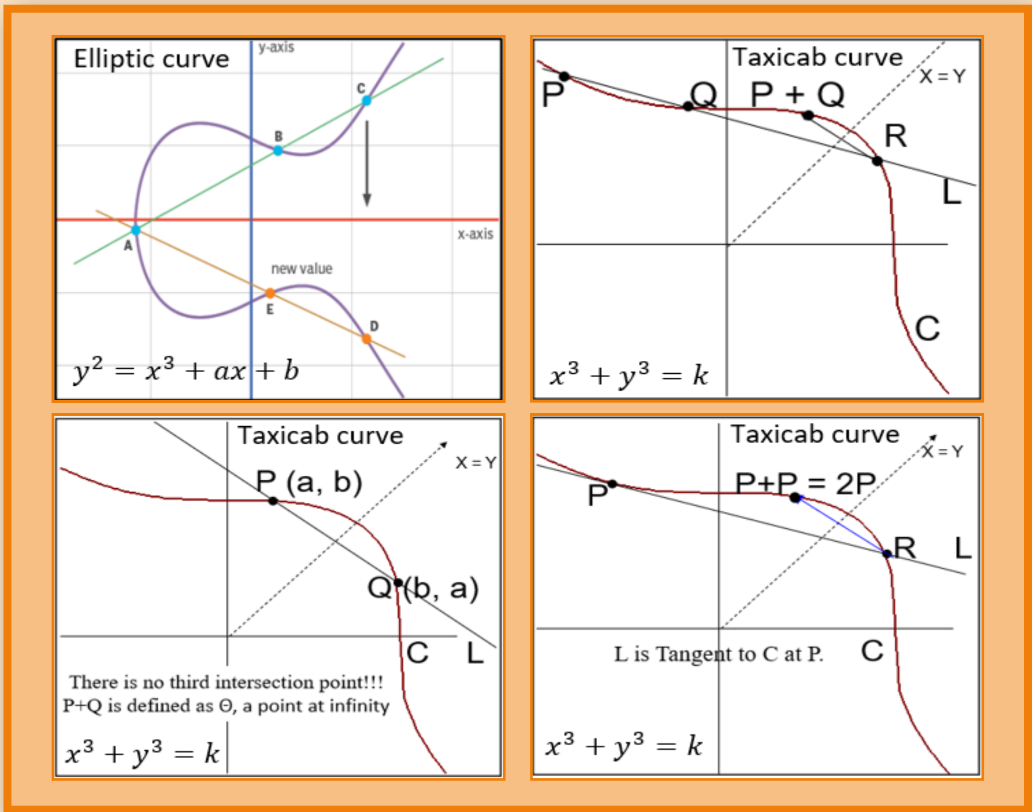
Generating Rational points on Elliptic curves



Binary operation '+' on the set of points on Taxicab/Elliptic curves

Chief Editor: Shrikrishna G. Dani Managing Editor: Vijay D. Pathak

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Contents

About the Cover Page: *The first curve in the top line, is an Elliptic curve represented by Weierstrass equation and other three images are of Taxicab curve which after rotating co-ordinate axes by 45° can be regarded as a non-Weierstrass Elliptic curve. The construction in the images indicate how binary operation on the set of points on the respective curves can be defined under different conditions. If one can find two rational points P and Q on the curve, then $P + Q$ gives the third rational point. Repeating the process, we can generate more and more rational points on the curve which may be helpful to visualize the pattern of rational points.*
https://en.wikipedia.org/wiki/Elliptic_curve;
https://www.math.brown.edu › jhs › Presenta... · PPT file · Web view

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From the Editors' Desk

Most of the development in Mathematics can be attributed to curiosity regarding various characteristics of abstract mathematical structures. Due to this curiosity mathematicians working in a specific area of their interest, raise challenging questions and seek answers to them. When they succeed, the mathematical world gets a new result, otherwise it is considered as an open problem. If someone involved has a firm belief regarding the answer and not able to give a formal proof for it, then it goes as a conjecture. The group of mathematicians vigorously attempt to find solutions to such conjectures and often new techniques, new methodologies, new subfields of mathematics emerge. There are many striking examples illustrating this phenomenon. For example, the pursuit of Fermat's Last Theorem emerging from Fermat's marginal note from the 17^{th} century that was proved in 1994, led to the development of Algebraic number theory, Modular forms, Elliptic curves. The Poincaré Conjecture proven in 2003, spurred the development of Geometric topology, Ricci flow. Work on many unproven conjectures has given indication of such new developments.

Thus, the skill of raising relevant questions / problems and a habit of working hard to find answers / solutions for them is key not only to new developments, but even to understanding deep concepts and methodologies of mathematics. The question is how to inculcate such skills among students at all levels. One way is to give challenging assignments to the students and making their submission an integral part of the assessment of the students. This certainly will help to develop problem-solving skills. Other options are organizing Study Group meetings or Problem-solving workshops where students can be encouraged to ask questions, evolve solutions to the problems by discussion in groups. We can also develop some team games based on specific topics in mathematics, a sort of quiz competition, except that the teams may raise their own questions instead of involving a quiz master. Some of these practices are adopted by leading academic institutions but not very common in colleges and universities in general.

On 22^{nd} December, 2024 we celebrated 137^{th} birth anniversary of the legendary Indian mathematician Srinivasa Ramanujan. The legacy from him includes a number of unpublished notebooks filled with theorems / conjectures that mathematicians have continued to study till today. Prof. Atul Dixit, in the opening expository article in this issue gives a survey of some of the developments in Partition theory arising from the Rogers-Ramanujan identities and also in the topic of modular relations satisfied by the Rogers-Ramanujan functions.

In Article 2, Prof. Raju K. George and Aleena Thomas explore how spectral analysis plays a significant role in control theory and discuss how spectral methods make it easier to come up with suitable control strategies. They illustrate this by applying the theory for the controllability study of artificial satellite and for developing control strategies to steer a satellite to a desired orientation.

Article 3 is our regular column on "A Peep into History of Mathematics" wherein Prof. S. G. Dani reviews four recent articles in History of Mathematics, on a variety of themes.

In Article 4, Dr. D. V. Shah gives an account of significant developments in the Mathematical world during recent past, including new results on prime numbers, evolution of new soft shapes and current status of Height zero Conjecture, and 'Moving Sofa Problem'. He also presents some highlights of the work of Prof. Neena Gupta who won the Infosys Prize 2024. We also pay tributes to mathematicians Walter David Neumann and R. Keith Dennis who passed away recently.

Article 5 is a review By Prof. Renu Jain on a book: I-Function and Its Applications by V. P. Saxena, P. Agrawal and Altaf A. Bhat. In the Problem Corner, Dr. Udayan Prajapati presents a solution to one of the two problems posed in the October 2024 issue. Two new problems are also posed for our readers. Dr. Ramesh Kasilingam gives a calendar of academic events, planned during April, 2025 to August, 2025, in Article 7.

We are happy to bring out this third issue of Volume 6 in January, 2025. We thank all the authors, all the editors, our designers Mrs. Prajakta Holkar and Dr. R. D. Holkar, and all those who have directly or indirectly helped us in bringing out this issue on time.

Chief Editor TMC Bulletin

1. The Multifaceted Rogers-Ramanujan Functions

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ABSTRACT. This expository article is a survey of some of the developments in partition theory arising from Rogers-Ramanujan identities and in the topic of modular relations satisfied by the Rogers-Ramanujan functions.

1.1 Introduction

Every college student has, at some point of time, encountered the number

$$\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\dots}}}. \tag{1.1}$$

The standard way with which a student proceeds, of course disregarding the convergence aspects, is to observe that if x denotes the above number, then $1/x = 1 + x$. This equation has two roots $\frac{1}{2}(-1 \pm \sqrt{5})$. Since x is positive, one concludes that it is equal to $\frac{1}{2}(-1 + \sqrt{5})$.

One of the triumphs of modern mathematics is the advancement made in the theory of functions. For example, what if we consider, instead of the number in (1.1), the function

$$F(q) := \cfrac{1}{1+\cfrac{q}{1+\cfrac{q^2}{1+\cfrac{q^3}{1+\dots}}}}? \tag{1.2}$$

One represents the above continued fraction in the following standard compact notation:

$$F(q) = \frac{1}{1}\;\underset{+}{}\;\frac{q}{1}\;\underset{+}{}\;\frac{q^2}{1}\;\underset{+}{}\;\frac{q^3}{1}\;\underset{+}{}\;\dots. \tag{1.3}$$

It is known that $F(q)$ converges for $|q| < 1$. Its behavior on the unit circle, however, is mysterious and still not understood completely. There are certain roots of unity where we know it converges (including, of course, $q = 1$) and certain others where it diverges; see [20, Theorem 7.2.1].

Consider a slight modification of $F(q)$, that is,

$$R(q) := q^{1/5}F(q). \tag{1.4}$$

The function $R(q)$ is the most celebrated continued fraction in Mathematics known as the *Rogers-Ramanujan continued fraction*. It was first studied by the English mathematician L. J. Rogers [39], and was rediscovered by the famous Indian mathematician S. Ramanujan before going to England. The latter obtained a plethora of results involving $R(q)$ in his Notebooks and the Lost Notebook [38].

The theory of $R(q)$ entails the study of two functions $G(q)$ and $H(q)$ called the *Rogers-Ramanujan functions*. These functions are defined by

$$G(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)} \quad \text{and} \quad H(q) := \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(1-q)(1-q^2)\cdots(1-q^n)}, \tag{1.5}$$

The following compact notation is generally adopted in q-series:

$$\begin{aligned}
(A)_0 &:= (A;q)_0 = 1,\\
(A)_n &:= (A;q)_n = (1-A)(1-Aq)\cdots(1-Aq^{n-1}), \qquad n \geq 1,\\
(A)_\infty &:= (A;q)_\infty = \lim_{n\to\infty}(A;q)_n, \qquad |q| < 1.
\end{aligned}$$

Using this notation, we can write the definitions of $G(q)$ and $H(q)$ in the form

$$G(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n} \quad \text{and} \quad H(q) := \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q)_n}. \tag{1.6}$$

These functions satisfy what are known as *Rogers-Ramanujan identities*. Many mathematicians consider them to be *'the most beautiful pair of formulas in all of mathematics'* [47]. These identities are given, for $|q| < 1$, by

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty}, \tag{1.7}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q)_n} = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}. \tag{1.8}$$

Rogers [39] derived these identities in 1894. Ramanujan rediscovered them in 1913 before going to England. An interesting account on Ramanujan's discovery of these identities in Rogers' work is given in [31].

Issai J. Schur, a German mathematician, who was cut off from England during the World War I, also rediscovered the identities, and gave two proofs [42], one of which was combinatorial in nature. Before we discuss this combinatorial interpretation[1] of the Rogers-Ramanujan identities (1.7) and (1.8) given by Schur, we need to discuss a fundamental construct in additive number theory: *partitions*.

1.2 Basics of partition theory

A partition of a non-negative integer n is a non-increasing sequence of positive integers which sum to n. For example, $3 + 2 + 1 + 1$ is a partition of 7.

The number of partitions of n is denoted by $p(n)$, *the partition function*. Thus, $p(4) = 5$ since there are five partitions of 4, namely, 4, $3 + 1$, $2 + 2$, $2 + 1 + 1$ and $1 + 1 + 1 + 1$. By convention, we take $p(0) = 1$.

While studying an arithmetic sequence $\{a(n)\}_{n=0}^{\infty}$, it is important to know its generating function. Different kinds of generating functions are suited for various purposes. The most common one is $\sum_{n=0}^{\infty} a(n)q^n$, where q is some real/complex variable. For example, the generating function for the sequence of whole numbers is $1 \cdot q + 2 \cdot q^2 + 3 \cdot q^3 + \cdots = q/(1-q)^2$, provided $|q| < 1$.

Euler was the first to obtain a representation for the generating function of the partition function $p(n)$, namely, he showed that

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(1-q)} \frac{1}{(1-q^2)} \frac{1}{(1-q^3)} \cdots = \frac{1}{(q;q)_\infty}. \tag{1.9}$$

The infinite product given above converges for $|q| < 1$. Henceforth, unless specified otherwise, we will always assume $|q| < 1$.

Euler's result is one of the first theorems proved in a standard course in the theory of partitions or in a book devoted to the subject, see [12] or [17], for example. The proof is not difficult and one only requires to know that the m in

$$\frac{1}{1-q^m} = \sum_{j=0}^{\infty} q^{jm}, \tag{1.10}$$

[1]This combinatorial interpretation was first found by MacMahon [35, Chapter 3]. However, MacMahon was unable to prove the identities themselves.

denotes a part of the concerned partition and j, the number of times m appears as a part. When $j = 0$, it, of course, means that m does not appear as a part in the partition.

Why are we interested in a generating function of a sequence? The first and the foremost reason is, it provides us an analytic object encapsulating the information of *every* element of the sequence, and which is also well-suited to algebraic and analytic manipulations. As rightly said by the famous combinatorialist Herbert Wilf [23, p. 145], '*a generating function of a sequence is a clothesline on which you hang all elements of the sequence*'.

1.3 Partition identities

In this section, we look at identities of the form $p_1(n) = p_2(n)$, where p_1 and p_2 are two restricted partition functions with different restrictions on their parts.

1.3.1 Euler's partition theorem

The generating function in (1.9) is extremely useful in finding important results in the subject. Consider, for example, the number of partitions of 6 into distinct parts. There are four such partitions, namely, $6, 5+1, 4+2$ and $3+2+1$. Now consider the number of partitions of 6 into odd parts. These are also four in number: $5+1, 3+3, 3+1+1+1$, and $1+1+1+1+1+1$. Is this a mere coincidence? One may check for a few more numbers that these two sets of partitions are of the same size. How do we show this for *every* positive integer n?

This is where the generating functions can do magic. Let $p_d(n)$ denote the number of partitions of n into distinct parts. Since any part of a partition into distinct parts either appears only once or does not appear at all, to obtain $\sum_{n=0}^{\infty} p_d(n)q^n$, one needs to truncate the sum in (1.10) to $1+q^m$. Doing this for each of the factors in the middle expression in (1.9), we find that

$$\sum_{n=0}^{\infty} p_d(n)q^n = (1+q)(1+q^2)(1+q^3)\cdots. \tag{1.11}$$

whereas, if $p_o(n)$ denotes the number of partitions of n into odd parts, we must have

$$\sum_{n=0}^{\infty} p_o(n)q^n = \frac{1}{(1-q)(1-q^3)(1-q^5)\cdots}. \tag{1.12}$$

Using the elementary identity $(a-b)(a+b) = a^2-b^2$, one can formally[2] see that the right-hand sides of (1.11) and (1.12) are equal, which implies that $p_d(n) = p_o(n)$. This means the number of partitions of a positive integer into distinct parts is *always* equal to the number of partitions of that integer into odd parts.

Let us rephrase Euler's partition theorem as follows:

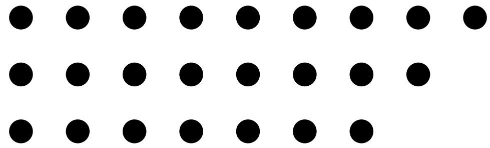
The number of partitions of a positive integer into parts which differ by at least 1 *is equal to the number of partitions of that integer into parts which are congruent to* ± 1 *modulo* 4.

1.3.2 MacMahon's and Schur's combinatorial interpretations of the Rogers-Ramanujan identities

In view of the paraphrasing of Euler's partition theorem given at the end of the above sub-section, one might wonder if there are more results of this type. For example, *what can be said about the number of partitions of a positive integer whose parts differ by at least* 2*?*

We now show that they are enumerated by the power series coefficients of the left-hand side of the first Rogers-Ramanujan identity, that is, by $\sum_{n=0}^{\infty} q^{n^2}/(q;q)_n$.

To prove this, however, we first need a way to diagrammatically represent a partition. This is done using the *Ferrers diagram* which uses dots to represent each part of a partition. For example, the Ferrers diagram of the partition $9+8+7$ of 24 is



Consider the black dots as lattice points in the xy-plane with the top leftmost dot being the origin. The line $y = -x$ restricted to the fourth quadrant is known as the *main diagonal*. If one reflects a partition in its main diagonal, what one gets is the *conjugate* of that partition. Thus, the conjugate of $9+8+7$ is $3+3+3+3+3+3+3+2+1$. It readily follows from conjugation that the number of partitions of an integer into parts whose size is less than or equal to n equals the number of partitions of that integer into less than or equal to n parts.

Now consider the summand $q^{n^2}/(q;q)_n$. The numerator q^{n^2} can be written as

$$q^{n^2} = q^1 \cdot q^3 \cdot q^5 \cdots q^{2n-1},$$

Moreover,

$$\frac{1}{(q;q)_n} = \frac{1}{(1-q)(1-q^2)\cdots(1-q^n)}$$

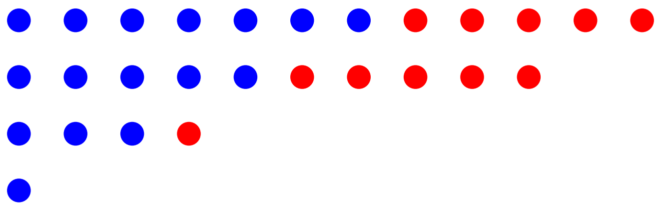
enumerates the number of partitions of an integer each of which has its number of parts less than or equal to n.

We now construct a partition whose Ferrers diagram is constructed as follows:

(1) The parts $2n-1, 2n-3, \cdots, 5, 3, 1$ (denoted in blue color in the diagram below) are placed below one another in the same order. They constitute n rows of the Ferrers diagram.

(2) Concatenate the largest part of a partition generated by $1/(q;q)_n$ (which is taken to be the sample partition $5+5+1$ in red color in the diagram below) with the $2n-1$ blue dots, the next largest with $2n-3$ blue dots, and so on. If there are fewer than n parts coming from $1/(q;q)_n$, the corresponding last few parts formed by blue color remain unchanged.

The figure below shows this construction for $n = 4$.



Since the blue sub-parts of the partition formed by concatenation differed by 2 to begin with, the parts of the complete partition differ by at least 2. To get all such partitions, we must sum $q^{n^2}/(q;q)_n$ over n from $n = 0$ to ∞ which proves the claim.

Now, clearly, $\dfrac{1}{(q;q^5)_\infty (q^4;q^5)_\infty}$ generates partitions having parts congruent to either 1 or 4 modulo 5. Therefore, the combinatorial equivalent of the first Rogers-Ramanujan identity (1.7) is

The number of partitions of a positive integer into parts differing by at least 2 equals the number of partitions of that integer into parts which are congruent to ± 1 modulo 5.

We have thus found the next Euler-type partition theorem! Similarly, the second Rogers-Ramanujan identity (1.8) is equivalent to the statement that *the number of partitions of a positive integer into parts differing by at least 2 and not having 1 as a part equals the number of partitions*

[2] This argument can be made rigorous. See [12, pp. 4-5].

of that integer into parts which are congruent to 2 *or* 3 *modulo* 5. This is owing to the fact that $n^2 + n = 2 + 4 + 6 + \cdots + 2n$.

It is to be noted, however, that each of the available proofs of (1.7) and (1.8), or their aforementioned combinatorial equivalents, is, in some or the other way, quite deep! Hardy [31] noted, "*...None of the proofs of these identities can be called both 'simple' and 'straightforward', since the simplest are essentially verifications; and no doubt it would be unreasonable to expect a really easy proof*". Hardy's statement remains true to this day, for, each proof (barring those which are mere verifications) involves an ingenious idea.

Andrews [14] surveyed all known proofs of the Rogers-Ramanujan identities until 1989. New proofs continue to emerge to this day, the most recent one being that given by Rosengren [41]. That being said, there is no "simple" bijective proof of fact that the partitions enumerated by the sum-sides of (1.7) and (1.8) are respectively equal in number to those enumerated by the product-sides! A proof along these lines by Garcia and Milne [28] falls short of being called a 'direct bijection' since it involves intermediate transformations, and is 51 pages long! On the other hand, the "short" bijective proof due to Bressoud and Zeilberger [25]is quite difficult and not really short (see the MathSciNet review of [25] and also [6, p. 3]).

Besides the Theory of Partitions and Modular Forms (more generally, Number Theory), Rogers-Ramanujan identities have played a significant role in several diverse areas of Mathematics and Science such as Commutative Algebra, Knot Theory, Statistical Mechanics, Representation Theory of Affine Lie Algebras and Algebraic Geometry, to name a few. See [1, 18, 19, 33, 34, 26]. An interested reader is encouraged to read the excellent book by Sills [44], entirely devoted to the Rogers-Ramanujan identities!

1.3.3 Schur's partition theorem, Alder's conjecture, and Andrews-Gordon identities

The topic of finding partition identities similar to the combinatorial interpretations of the Rogers-Ramanujan identities has blossomed ever since the identities in (1.7) and (1.8) were obtained. While a complete survey of these developments is beyond the scope of this review, we indicate only a few of them to give an idea of their vast expanse.

In the previous subsection, we saw that the combinatorial interpretation of the first Rogers-Ramanujan identity is, in some sense, the next level result of Euler's partition theorem.

What next? In 1926, Schur [43] made further progress in this direction by deriving the following beautiful result.

The number of partitions of an integer into parts that differ by at least 3*, and with no consecutive multiples of* 3 *as parts, equals the number of partitions of the integer into parts which are congruent to* ± 1 *modulo* 6*.*

One immediately observes the extra condition needed to make the two sets of partitions equinumerous, namely, that the partitions on which the difference conditions are imposed have to have a difference of at least six between any two of its parts which are multiples of 3. Such an additional condition was not needed for example in Euler's partition theorem or in the combinatorial interpretation of (1.7).

However, Lehmer [32] proved a general result whose special case is that if $q_d(n)$ denotes the number of partitions of a positive integer into parts differing by at least d, and $Q_d(n)$ denotes the number of parts congruent to ± 1 modulo $(d+3)$, then, for $d \geq 3$, then $q_d(n) \neq Q_d(n)$. However, based on the fact that $q_3(n) - Q_3(n) \geq 0$, as can be checked from the aforementioned result of Schur, Alder [3], [4] conjectured that $q_d(n) - Q_d(n) \geq 0$ for all $d, n \in \mathbb{N}$. This conjecture was proved in complete generality only fifty years later, through the combined efforts of Andrews [10] (for $d = 2^r - 1, r \geq 4$), of Yee [46] (for $d = 7, d \geq 32$), and of Alfes, Jameson and Lemke Oliver [5] (for all remaining values of d).

On the other hand, Alder [3, Theorem 3] showed that to have partition theorems similar to Schur's 1926 theorem or the combinatorial interpretation of (1.7) for the difference d between the parts greater than 3, one must have more complicated conditions. In this direction, Andrews [8],

[9] obtained two infinite families of results for $d \geq 3$ encompassing Schur's 1926 result. The two families intersect only for $d = 3$. Since then, there have been many exciting developments for which the reader is referred to a nice survey by Alladi [6].

Moreover, Gordon 30 obtained a different generalization of (1.7) and (1.8) wherein the modulus associated to certain congruence conditions is any odd number. His theorem reads [30, Theorem 1]:

For $d \geq 2$, the number of partitions of N into parts not congruent to $0, \pm t \pmod{2d+1}$, where $1 \leq t \leq d$, is equal to the number of partitions of the form $N = N_1 + \cdots + N_k$, where $N_i \geq N_{i+1}, N_i \geq N_{i+d-1} + 2$, and $N_{k-t+1} \geq 2$.

Andrews [11, Theorem 1] obtained the analytic counterpart of the above result. It is easy to conceive the generating function of the partitions not satisfying the congruence conditions. Indeed, it is $\displaystyle\prod_{\substack{n=1 \\ n \not\equiv 0, \pm t \pmod{2d+1}}}^{\infty} \frac{1}{1-q^n}$ for $1 \leq t \leq d$. However, the generating function of the partitions from the other set is a multi-dimensional sum. Andrews [7, Theorem 2] obtained an analogue of Gordon's result for the moduli of the form $4d + 2$. Finally, Bressoud [24] obtained a result for *all* moduli, thus encompassing the results of Andrews and Gordon.

1.4 Modular identities associated with the Rogers-Ramanujan functions

So far we have seen the combinatorics of partition functions arising from the Rogers-Ramanujan identities and their analogues. However, the Rogers-Ramanujan functions $G(q)$ and $H(q)$ defined in (1.6) are intimately connected to modular forms as well. Modular forms constitute an all-pervasive branch of number theory. A quote attributed to Martin Eichler reads, *'There are five fundamental operations in mathematics: addition, subtraction, multiplication, division, and modular forms.'*. This underscores the importance of modular forms in mathematics as well as other sciences. But what is a modular form?

A modular form f of weight k, where $k \in 2\mathbb{N}$, is a function defined [27, p. 4, Definition 1.1.2] on the upper half-plane $\mathbb{H} := \{x + iy \in \mathbb{C} : y > 0\}$ satisfying following three properties:

(i) f is holomorphic on the upper half-plane;

(ii) For any matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in $\mathrm{SL}_2(\mathbb{Z})$ and any $z \in \mathbb{H}$, we have $f\left(\dfrac{az+b}{cz+d}\right) = (cz+d)^k f(z)$;

(iii) f is holomorphic at ∞.

Observe that in (1.4) we have a $q^{1/5}$ in front of the continued fraction $F(q)$. The reader might have wondered whether this power of q has been put artificially. It turns out that for $q = e^{2\pi i z}$, $z \in \mathbb{H}$, the functions $\tilde{G}(q) := q^{-1/60}G(q)$ and $\tilde{H}(q) := q^{11/60}H(q)$ are modular forms with respect to z, and also that $R(q) = \tilde{H}(q)/\tilde{G}(q) = q^{1/5}H(q)/G(q)$. Thus, the $F(q)$ defined in (1.3) is simply the quotient of $H(q)/G(q)$.

The Rogers-Ramanujan functions $G(q)$ and $H(q)$ satisfy scores of modular relations, meaning there is a relation connecting them with $G(q^n)$ and $H(q^n)$ for some natural number $n > 1$. Ramanujan initiated the topic of finding modular relations for the Rogers-Ramanujan functions by obtaining the result [36] (see also [37, p. 231])

$$H(q)G(q^{11}) - q^2 G(q)H(q^{11}) = 1. \tag{1.13}$$

One of the manuscripts of Ramanujan transcribed by Watson (see [22] for details), contains 40 identities involving G and H which resemble (1.13)! Rogers [40] not only proved (1.13) but also nine other identities which were communicated to him by Ramanujan. The proofs of most of the forty identities were given by various mathematicians such as Darling, Watson, Bressoud, Biagioli, Yesilyurt in a series of papers culminating into the monograph [21] by Berndt, Choi, Choi, Hahn, Yeap, Yee, Yesilyurt, and Yi. Regarding these forty identities, Watson [45] says,

...the beauty of these formulae seems to me to be comparable with that of the Rogers-Ramanujan identities. So far as I know, nobody else has discovered any formulae which approach them even remotely ....

But Ramanujan went beyond these formulae! To see how, let us first state another identity from among the set of forty:

$$G(q)G(q^4)+qH(q)H(q^4)=\frac{\varphi(q)}{(q^2;q^2)_\infty}=(-q;q^2)_\infty^2, \tag{1.14}$$

where $\varphi(q):=\sum_{n=-\infty}^{\infty} q^{n^2}$ is the Jacobi theta function. Regarding (1.14), Andrews [38, p. xxi] says,

This sort of identity has always appeared to me to lie totally within the realm of modular functions and to be completely resistant to q-series generalization. One of the greatest shocks I got from the Lost Notebook was the following assertion...

The assertion referred to in the above quote of Andrews is the special case $b=1$ and $q\to q^4$ of the following exquisitely beautiful identity occurring on page 27 of Ramanujan's Lost Notebook [38], and valid for $a\in\mathbb{C}\backslash\{0\}$, and $b\in\mathbb{C}$:

$$\sum_{m=0}^{\infty}\frac{a^{-2m}q^{m^2}}{(bq)_m}\sum_{n=0}^{\infty}\frac{a^nb^nq^{n^2/4}}{(q)_n}+\sum_{m=0}^{\infty}\frac{a^{-2m-1}q^{m^2+m}}{(bq)_m}\sum_{n=0}^{\infty}\frac{a^nb^nq^{(n+1)^2/4}}{(q)_n}$$
$$=\frac{1}{(bq)_\infty}\sum_{n=-\infty}^{\infty}a^nq^{n^2/4}-(1-b)\sum_{n=1}^{\infty}a^nq^{n^2/4}\sum_{\ell=0}^{n-1}\frac{b^\ell}{(q)_\ell}. \tag{1.15}$$

To see why (1.15) is important, let $a=b=1$ in it, replace q by q^4, and then use Rogers' identities [39 pp. 330-331]

$$G(q)=(-q^2;q^2)_\infty\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q^4;q^4)_n},\qquad H(q)=(-q^2;q^2)_\infty\sum_{n=0}^{\infty}\frac{q^{n^2+2n}}{(q^4;q^4)_n}. \tag{1.16}$$

This leads to the modular relation (1.14). Moreover, if we let $a=b=-1$ in (1.15), replace q by q^4, multiply the resulting two sides by $(-q^2;q^2)_\infty$ and use (1.16), we obtain an identity connecting $G(q)$ and $H(q)$ with certain fifth order mock theta functions of Ramanujan [13 p. 28][3]. This is the only result of its kind so far available in the literature.

All of this is well documented in Andrews [13] where (1.15) was proved by showing that the coefficients of $a^N, -\infty<N<\infty$, in the Laurent series expansions of both sides are equal. The present author and Kumar [29] recently generalized (1.15) and gave compelling evidence as to how Ramanujan may have arrived at this generalized modular relation. In Section 6 of the same paper, they showed that (1.15) and another identity on page 26 of the Lost Notebook, namely [29, Equation (6.1)], together give yet another among the forty identities of Ramanujan.

In [22, p. 73], Birch says,

'...They support the view that Ramanujan's insight into the arithmetic of modular forms was even greater than has been realized.'

The word *'They'* refers to the various manuscripts of Ramanujan transcribed by Watson which include the one containing the forty identities for $G(q)$ and $H(q)$. Also, after finding two algebraic relations between these two functions in [36], Ramanujan said, '*Each of these formulae is the simplest of a large class*'. It is believed that the *'large class'* referred to by Ramanujan is the set of forty identities. However, in light of the fact that (1.15) and the other multi-parameter identity on page 26 of the Lost Notebook lead to certain identities among the forty as corollaries, it was

[3]There are two minor corrections in the version stated in [13], namely, the right-hand side should be multiplied by $(-q^2;q^2)_\infty$ and the n in the $q-$product in the denominator of the double sum should be j.

speculated in [29] whether Ramanujan intended the term *'large class'* to mean identities of the form in (1.15). This definitely calls for serious research in this direction.

In any case, the identities such as (1.15) show Ramanujan's leaps of imagination that transcended the horizon of the modular landscape.

1.5 Concluding remarks

The aim of this article was to introduce an uninitiated reader to the magnificent world of partitions through the lens of Rogers-Ramanujan identities and to delineate two of the multitudinous aspects of the Rogers-Ramanujan functions. Being such a fundamental construct, partitions penetrate almost every branch of mathematics and mathematical sciences. Likewise, the Rogers-Ramanujan identities seem to be universal, with more and more seemingly distant areas of Mathematics discovering their presence and importance.

A beautiful and a comprehensive text on partition theory is Andrews' *Theory of Partitions* [12]. For an elementary introduction to partitions, the reader is encouraged to read the book by Andrews and Eriksson [17] which contains a nice exposition of several interesting topics in the theory, for example, *'Discovering the first Rogers-Ramanujan identity'*. Another book on the subject which may act as a supplement to [12] is the one by Agarwal, Padmavathamma and Subbarao [2]. The only text written on the Rogers-Ramanujan identities, and a must-read, is the book by Sills [44]. On the other hand, to see the proofs of the forty identities for $G(q)$ and $H(q)$ together in one place, and that too, in the spirit of Ramanujan, the reader is referred to the monograph [21]. An excellent book to know Ramanujan's contributions to the Rogers-Ramanujan as well as other continued fractions along with their proofs is Part I of the series of books on Ramanujan's Lost Notebook by Andrews and Berndt [15].

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□□□

We Congratulate Padma Shri awardee Professor M. D. Srinivas

We are very happy that Professor M. D. Srinivas, an eminent multi-faceted personality engaged in the promotion of awareness of Indian Knowledge Systems, is being awarded a Padma Shri award by the Government of India.

Born on October 2, 1950, Prof. Srinivas did his B.Sc. (Honours) and M.Sc. in Physics, from Bangalore University, in 1969 and 1971 respectively. He secured a fellowship from the University of Rochester to pursue graduate studies, and worked on Quantum Optics and the Foundations of Quantum Mechanics under the supervision of Professor Emil Wolf, and completed the Ph.D. in 1976. Returning to India he joined as a faculty member in the Department of Theoretical Physics, University of Madras, where he served till 1996, making significant research contributions along the way on various themes in the areas of Quantum optics, quantum probability and measurement theory.

In 1996 he took voluntary retirement from the University to devote full time to the Centre for Policy studies, Chennai, which he had founded, in 1990, together with J. K. Bajaj. Here he worked on the study of the functioning of the Indian society in the pre-colonial era, based on the data collected by British officials during late 18^{th} century - early 19^{th} century in northern Tamilnadu, prior to establishing their rule.

Prof. Srinivas has made notable contributions on medieval Indian sciences, especially Indian astronomy and mathematics; these include well-quoted articles on proofs in the Indian tradition, methodology of Indian sciences, the importance of commentaries, etc.. He also worked on the mathematics and astronomy of the Kerala school, and co-authored the books, *Yuktibhasha* of Jyesthadeva (jointly with K. V. Sarma, M. S. Sriram and K. Ramasubramanian), and *Karana paddhati* of Putumana Somayaji (jointly with Venketeswara Pai, M. S. Sriram, and K. Ramasubramanian). Apart from his role in scholarly papers and books he has also participated in producing study material in electronic form on ancient Indian mathematics and astronomy and also Indian knowledge systems in general.

He served on various academic bodies including Research Council for the History of Science of INSA (1998- 2000), Indian National Commission for History of Science (INSA, 1998-2004). He was Vice Chairman of the Indian Institute of Advanced Study, Shimla during 1999-2004: He has been a Member of the Central Sanskrit Board, since 2006 and Chairman, the MOP Vaishnav College for Women, Chennai, since 2013. He was elected Fellow of the Indian National Science Academy, in 2022. The honour is now topped with the Padma award of this year.

His soft-spoken and persuasive mannerism and scholarly demeanor have been very inspirational to the community. We heartily congratulate Professor Srinivas on the occasion the latest recognition, and wish him further successes in his endeavours.

Chief Editor TMC Bulletin

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2. Spectral Approach to Controllability Analysis and its Application to Satellite Control

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ABSTRACT. This article explores how spectral analysis plays a significant role in control theory. The eigenvalues and eigenvectors of the state matrix and the spectral properties of the controllability Gramian are used to characterize controllability. This approach also helps in developing a computational algorithm for the computation of steering control. Spectral methods make it easier to come up with suitable control strategies, and also give us a better understanding of how systems work, which leads to more effective and efficient system design. The theory is applied for the controllability study of artificial satellite and for developing control strategies to steer a satellite to a desired orientation.

2.1 Introduction

Consider a system, described by the first-order initial value problem

$$\frac{dx}{dt} = -x, x(0) = 1 \tag{2.1}$$

The solution is exponentially decaying as shown in Figure [2.1]. If we add a forcing term, $\cos t$ to the system [2.1], it becomes

$$\frac{dx}{dt} = -x + \cos t, x(0) = 1. \tag{2.2}$$

It can be shown that the solution is oscillatory as shown in Figure [2.2].

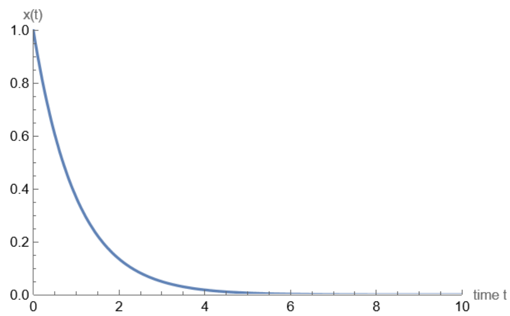

Figure 2.1: $x(t) = e^{-t}$ is the solution to System [2.1]

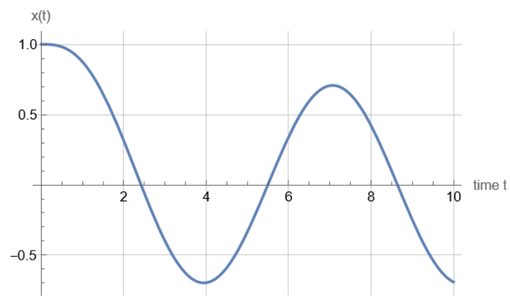

Figure 2.2: $x(t) = \frac{\cos t + \sin t + e^{-t}}{2}$ is the solution to System [2.2]

Thus, we see that on adding a forcing term (called the control term), there is a considerable change in the evolution of the state. The controllability problem is to check for the existence of a forcing function $u(t)$ such that the solution to a system that passes through a given initial state $x(t_0) = x_0$, also passes through a desired final state $x(t_1) = x_1$.

First order, n-dimensional linear time-varying control system is described by the following differential equation

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$

where, the state $x(t) \in \mathbb{R}^n$ and the control $u(t) \in \mathbb{R}^m$ for each fixed t and $A(t)$ and $B(t)$ are matrices of of order $n \times n$ and $n \times m$, respectively, having elements as functions of t. If $A(t) = A$ and $B(t) = B$ are constant matrices, the system is called Linear Time Invariant (LTI) system. In this article, we characterize controllability of the system and apply the theoretical results to a satellite control problem.

2.2 Fundamental Matrix Solution and Transition Matrix

Consider the linear homogeneous system

$$\dot{x}(t) = A(t)x(t) \tag{$\mathcal{LH}$}$$

where, $A(t) = [a_{ij}(t)]_{n\times n}$, $a_{ij}(t)$ are functions of t. If $a_{ij}(t)$'s are piecewise continuous functions of time t, then $\mathcal{LH}$ has infinitely many solutions. If an initial condition is given along with the system, then there exists a unique solution as we see in the following theorem.

Theorem 1. *If $a_{ij}(t)$ are piecewise continuous functions on an interval $I = [t_0, t_1]$, then there exists a unique solution to the initial value problem (IVP)*

$$\dot{x}(t) = A(t)x(t), \;\; x(t_0) = x_0$$

for any $t_0 \in I$ and any given vector $x_0 \in \mathbb{R}^n$.

Let $\{x_0^i : i = 1, 2, \ldots, n\}$ be a basis of $\mathbb{R}^n$. For each i, let $\phi_i(t)$ be the unique solution to the homogeneous system $\mathcal{LH}$ with initial condition $x(t_0) = x_0^i$. Now, $\{\phi_i(t) : i = 1, 2, \ldots, n\}$ is a basis of the solution space of $\mathcal{LH}$. Consider the $n \times n$ matrix

$$\Phi(t) = [\phi_1(t) \mid \phi_2(t) \mid \ldots \mid \phi_n(t)] \tag{2.3}$$

with n linearly independent solutions of $\mathcal{LH}$ as columns. $\Phi(t)$ is called a *fundamental matrix solution (fms)* and it satisfies $\dot{\Phi}(t) = A(t)\Phi(t)$.
Remark:

(i) If Ψ is a fundamental matrix solution, then ΨC is also a fundamental matrix solution for any $n \times n$ non-singular constant matrix C. (Interestingly, $C\Psi$ need not be a fundamental matrix solution of $\mathcal{LH}$, check!)

(ii) For any $s \in I$, $\Psi(s)$ is a non-singular, constant $n \times n$ matrix.

As $\Psi(t_0)$ is invertible, $C = \Psi^{-1}(t_0)$ is also invertible. Therefore, the matrix $\Phi(t, t_0) = \Psi(t)C = \Psi(t)\Psi^{-1}(t_0)$ is also a fundamental matrix solution to $\mathcal{LH}$. This special fms, $\Phi(t, t_0)$ is called the **State Transition Matrix**. The transition matrix of the system $\mathcal{LH}$ can be computed as the limit of the following infinite series of matrices called the *Peano-Baker series* (see Brockett [1]),

$$\Phi(t, t_0) = I + \int_{t_0}^{t} A(\tau)d\tau + \int_{t_0}^{t} A(\tau) \int_{t_0}^{\tau} A(\tau_1)d\tau_1 d\tau + \ldots \tag{2.4}$$

Note. If $A(t) = A$, an $n \times n$ constant matrix, then the Peano-Baker series reduces to the following matrix exponential:

$$\Phi(t, t_0) = I + A(t - t_0) + \frac{1}{2!}A^2(t - t_0)^2 + \frac{1}{3!}A^3(t - t_0)^3 + \ldots \triangleq e^{A(t-t_0)} \tag{2.5}$$

Theorem 2 (Solution to the linear non-homogeneous system)**.** *If $\Phi(t, t_0)$ is the transition matrix to the homogeneous system $\dot{x} = A(t)x(t)$, then the solution to the non-homogeneous system $\dot{x} = A(t)x(t) + B(t)u(t)$ with initial condition $x(t_0) = x_0$ is given by (using the variation of parameters method)*

$$x(t) = \Phi(t, t_0)x_0 + \int_{t_0}^{t} \Phi(t, \tau)B(\tau)u(\tau)d\tau. \tag{2.6}$$

2.3 Controllability of Linear Systems

In this section, we study the controllability of the linear non-homogeneous system

$$\dot{x} = A(t)x(t) + B(t)u(t) \tag{2.7}$$

Definition 1 (Controllability). *The system $\dot{x} = A(t)x(t) + B(t)u(t)$ is controllable on the time interval $[t_0, t_1]$, if for any given two states $x_0, x_1 \in \mathbb{R}^n$, there exists an admissible control function $u \in \mathcal{L}^2([t_0, t_1], \mathbb{R}^m)$, such that the corresponding solution of the system with the initial condition $x(t_0) = x_0$ also satisfies $x(t_1) = x_1$.*

For a concise introduction to controllability, refer Brockett [1], Sontag [7] and Terrell [8].

2.3.1 Characterization of Controllability

The linear control system [2.7] is controllable if and only if there exists $u(t) \in \mathcal{L}^2([t_0, t_1]$ such that

$$\begin{aligned} x(t_1) = x_1 = \Phi(t_1, t_0)x_0 + \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)u(\tau)d\tau \\ i.e., \; x_1 - \Phi(t_1, t_0)x_0 = \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)u(\tau)d\tau \end{aligned}$$

Denoting $x_1 - \Phi(t_1, t_0)x_0$ by w_1, the above equation becomes

$$w_1 = \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)u(\tau)d\tau$$

Define an operator $\mathcal{C} : \mathcal{L}^2([t_0, t_1]; \mathbb{R}^m) \to \mathbb{R}^n$ as

$$\mathcal{C}u = \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)u(\tau)d\tau \tag{2.8}$$

Obviously, $\mathcal{C}$ is a bounded linear operator and $\mathcal{C}$ defines its adjoint operator $\mathcal{C}^* : \mathbb{R}^n \to \mathcal{L}^2([t_0, t_1], \mathbb{R}^m)$ in the following way:

$$\begin{aligned} \langle \mathcal{C}^* v, u \rangle_{\mathcal{L}^2} &= \langle v, \mathcal{C}u \rangle_{\mathbb{R}^n}, \forall \; u \in \mathcal{L}^2([t_0, t_1], \mathbb{R}^m), v \in \mathbb{R}^n \\ &= \left\langle v, \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)u(\tau)d\tau \right\rangle_{\mathbb{R}^n} \\ &= \int_{t_0}^{t_1} \langle v, \Phi(t_1, \tau)B(\tau)u(\tau) \rangle_{\mathbb{R}^n} d\tau \\ &= \int_{t_0}^{t_1} \langle B^T(\tau)\Phi^T(t_1, \tau)v, u(\tau) \rangle_{\mathbb{R}^m} d\tau \\ &= \langle B^T(.)\Phi^T(t_1, .)v, u \rangle_{\mathcal{L}^2} \end{aligned}$$

Hence, the adjoint operator $\mathcal{C}^* : \mathbb{R}^n \to \mathcal{L}^2([t_0, t_1]; \mathbb{R}^m)$, is a linear operator given by

$$(\mathcal{C}^* v)(t) = B^T(t)\Phi^T(t_1, t)v \tag{2.9}$$

The composition of $\mathcal{C}$ and $\mathcal{C}^*$ defines a bounded linear operator given by $\mathcal{C}\mathcal{C}^* : \mathbb{R}^n \to \mathbb{R}^n$ by,

$$\mathcal{C}\mathcal{C}^* v = \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)B^T(\tau)\Phi^T(t_1, \tau)v d\tau \tag{2.10}$$

Note that the operator $\mathcal{C}\mathcal{C}^*$ can be realized by an $n \times n$ positive semi-definite matrix, called *Controllability Gramian* of the system and is denoted by $\mathcal{W}(t_0, t_1)$. The following theorem relates controllability of [2.7] and the properties of linear operators $\mathcal{C}, \mathcal{C}^*$ and the spectrum $\sigma(\mathcal{C}\mathcal{C}^*)$ of $\mathcal{C}\mathcal{C}^*$.

Theorem 3. *The following statements are equivalent:*

(i) The system [2.7] is controllable.

(ii) The operator $\mathcal{C}$ is onto.

(iii) The adjoint operator $\mathcal{C}^*$ is one-one.

(iv) $0 \notin \sigma(\mathcal{W}(t_0, t_1)) = \sigma(\mathcal{C}\mathcal{C}^*)$ *(i.e., $\mathcal{W}(t_0, t_1)$ is an invertible matrix).*

Proof. Clearly, $(i) \iff (ii)$ by definition of the operator $\mathcal{C}$ in [2.8].
Now, let us show $(ii) \Rightarrow (iii)$. Suppose that $\mathcal{C}$ is onto. We have to show that $\mathcal{C}^*$ is one-one. It is enough to show that $\mathcal{C}^* v = 0$ if and only if $v = 0$. Let $v \in \mathbb{R}^n$ such that $\mathcal{C}^* v = 0$. As $\mathcal{C}$ is onto, there exists $u \in \mathcal{L}^2([t_0, t_1]; \mathbb{R}^m)$ such that $\mathcal{C}u = v$. Then

$$\langle v, v\rangle = \langle \mathcal{C}u, v\rangle = \langle u, \mathcal{C}^* v\rangle = \langle u, 0\rangle = 0.$$

This implies that $v = 0$. Hence $\mathcal{C}^*$ is one-one.
To prove $(iii) \Rightarrow (iv)$, suppose λ is an eigenvalue of $\mathcal{C}\mathcal{C}^*$, i.e. there exists $v \neq 0$ such that $\mathcal{C}\mathcal{C}^* v = \lambda v$. Now,

$$\|\mathcal{C}^* v\|^2 = \langle \mathcal{C}^* v, \mathcal{C}^* v\rangle = \langle \mathcal{C}\mathcal{C}^* v, v\rangle = \langle \lambda v, v\rangle = \lambda \|v\|^2 \geq 0$$

Now, if $\lambda = 0$, then $\|\mathcal{C}^* v\|^2 = 0$, which implies that $\mathcal{C}^* v = 0$, which is a contradiction to the one-one-ness of $\mathcal{C}^*$. Hence $\lambda > 0$.
$(iv) \implies (i)$ Suppose that zero is not an eigenvalue of $\mathcal{C}\mathcal{C}^* = \mathcal{W}(t_0, t_1)$, this implies the invertibility of $\mathcal{C}\mathcal{C}^* = \mathcal{W}(t_0, t_1)$.
Define a control function

$$u(t) = B^T(t)\Phi^T(t_1, t)\mathcal{W}^{-1}(t_0, t_1)[x_1 - \Phi(t_1, t_0)x_0] \tag{2.11}$$

Using this control, the state of the system is given by

$$x(t) = \Phi(t, t_0)x_0 + \int_{t_0}^{t} \Phi(t, \tau)B(\tau)B^T(\tau)\Phi^T(t_1, \tau)\mathcal{W}^{-1}(t_0, t_1)[x_1 - \Phi(t_1, t_0)x_0]d\tau$$

Then

$$\begin{aligned} x(t_0) &= \Phi(t_0, t_0)x_0 + \int_{t_0}^{t_0} \Phi(t_0, \tau)B(\tau)B^T(\tau)\Phi^T(t_1, \tau)\mathcal{W}^{-1}(t_0, t_1)[x_1 - \Phi(t_1, t_0)x_0]d\tau \\ &= x_0 \end{aligned}$$

$$\begin{aligned} x(t_1) &= \Phi(t_1, t_0)x_0 + \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)B^T(\tau)\Phi^T(t_1, \tau)\mathcal{W}^{-1}(t_0, t_1)[x_1 - \Phi(t_1, t_0)x_0]d\tau \\ &= \Phi(t_1, t_0)x_0 + \mathcal{W}(t_0, t_1)\mathcal{W}^{-1}(t_0, t_1)[x_1 - \Phi(t_1, t_0)x_0] \\ &= \Phi(t_1, t_0)x_0 + x_1 - \Phi(t_1, t_0)x_0 = x_1 \end{aligned}$$

Since, x_0 and x_1 are arbitrary, the system is controllable. □

Spectral analysis of the state and input matrices is of importance in the study of controllability of LTI systems, i.e., $A(t) = A$ and $B(t) = B$. The *Popov-Belevitch-Hautus* eigenvector test is a controllability condition which provides the geometry between the left eigenspace of the state matrix A and the column space of the input matrix B.

Definition 2 (**Left eigenvector**). *Let A be an $n \times n$ matrix. A non-zero row vector v is said to be a left eigenvector of A if $vA = \lambda v$, for some scalar λ.*

Note that v is a left eigenvector of A if and only if v^T is a right eigenvector of A^T.

Theorem 4. *The LTI system $\dot{x} = Ax + Bu$ is controllable if and only if for every left eigenvector v of A, $v^T B \neq 0$.*

Proof. To prove this theorem, we need the notion of *Controllability Matrix*. The matrix $\mathcal{Q}(A,B) = [B|AB|\cdots|A^{n-1}B]$ is said to be the controllability matrix of the LTI system (A,B). (A,B) is controllable if and only if the controllability matrix has full rank. This is the celebrated *Kalman's Theorem* . Suppose that system (4) is not controllable. That is, $rank\,[\mathcal{Q}(A,B)] = r < n$. By Kalman controllability decomposition [8], there exists a non-singular matrix T such that

$$T^{-1}AT = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} = \hat{A} \text{ and } T^{-1}B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}.$$

Let $\tilde{v}$ be an eigenvector of A_{22}^T corresponding to the eigenvalue λ. That is $A_{22}^T\tilde{v} = \lambda\tilde{v}$. This implies that $\tilde{v}^T A_{22} = \overline{\lambda}\tilde{v}^T$. As A_{22} is a real matrix both λ and $\overline{\lambda}$ are eigenvalues of A_{22} and because of the similarity of A and $\hat{A}$ both λ and $\overline{\lambda}$ are eigenvalues of A also. Now, define $v^T = \begin{bmatrix} 0_{1\times r} & \tilde{v}^T \end{bmatrix} T^{-1}$. Then,

$$\begin{aligned} v^T A &= v^T T \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} T^{-1} \\ &= \begin{pmatrix} 0_{1\times r} & \tilde{v}^T \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} T^{-1} \\ &= \begin{pmatrix} 0 & \tilde{v}^T A_{22} \end{pmatrix} T^{-1} \\ &= \begin{pmatrix} 0 & \lambda\tilde{v}^T \end{pmatrix} T^{-1} = \lambda v^T \end{aligned}$$

Also, $v^T B = \begin{pmatrix} 0_{1\times r} & \tilde{v}^T \end{pmatrix} T^{-1} T \begin{pmatrix} B_1 \\ 0 \end{pmatrix} T^{-1} = 0$

Suppose that there exists $v \neq 0$ such that $v^T A = \lambda A$ and $v^T B = 0$ is satisfied. Then,

$$\begin{aligned} v^T \mathcal{Q}(A,B) &= v^T [B|AB|\cdots|A^{n-1}B] \\ &= [v^T B|v^T AB|\cdots|v^T A^{n-1}B] = [0|\lambda v^T B|\cdots|\lambda^{n-1}v^T B] = 0 \end{aligned}$$

which implies that $rank\,[\mathcal{Q}(A,B)] < n$. Hence, the system (4) is not controllable. □

Note. *The PBH condition states that an LTI system (A,B) is uncontrollable if and only if there exists a left eigenvector of A that is simultaneously orthogonal to all columns of B. This helps one to identify the possible choices of input matrices B so that the system (A,B) is controllable.*

The next theorem gives an explicit expression for steering control in terms of eigenvectors of the controllability Gramian matrix.

Theorem 5. *[6] For the control system [2.7], the control function defined by*

$$u(t) = B^T(t)\Phi^T(t_1,t)\sum_i^n \frac{c_i v_i}{\lambda_i} \tag{2.12}$$

steers the system from x_0 to x_1 during $[t_0,t_1]$, where λ_i is the $i-th$ eigenvalue of the Gramian matrix $W(t_0,t_1)$, $\{v_n\}$ is the orthonormal basis of $\mathbb{R}^n$ generated by eigenvectors corresponding to $\{\lambda_i\}_{i=1}^n$ and c_i's are the coordinates of the vector $x_1 - \Phi(t_1,t_0)x_0$ with respect to $\{v_n\}$.

Proof. Since $W(t_0, t_1)$ is a symmetric matrix, it has n linearly independent eigenvectors $v_1, v_2, \dots, v_n$ forming an orthonormal basis of $\mathbb{R}^n$. Consider the vector $x_1 - \Phi(t_1, t_0)x_0$. As $\{v_i \mid i = 1, 2, \dots, n\}$ forms a basis, there exists scalars $c_i, i = 1, 2, \dots, n$ such that

$$x_1 - \Phi(t_1, t_0)x_0 = \sum_{i=1}^{n} c_i v_i \tag{2.13}$$

is the unique representation of $x_1 - \Phi(t_1, t_0)x_0$ with respect to the given basis. Now, we claim that the control defined by [2.12], steers the system [2.7] from x_0 to x_1 during the time $[t_0, t_1]$. We have $x(t_0) = \Phi(t_0, t_0)x_0 + \int_{t_0}^{t_0} \Phi(t_0, s)B(s)u(s)ds = x_0$ and at $t = t_1$, $x(t_1) = \Phi(t_1, t_0)x_0 + \int_{t_0}^{t_1} \Phi(t_1, s)B(s)u(s)ds$.

Using equation [2.12], we have

$$\begin{aligned}
x(t_1) &= \Phi(t_1, t_0)x_0 + \int_{t_0}^{t_1} \Phi(t_1, s)B(s)B^T(s)\Phi^T(t_1, s) \sum_{i=1}^{n} \frac{c_i v_i}{\lambda_i} ds \\
&= \Phi(t_1, t_0)x_0 + \sum_{i=1}^{n} \frac{c_i}{\lambda_i} \int_{t_0}^{t_1} \Phi(t_1, s)B(s)B^T(s)\Phi^T(t_1, s)v_i ds \\
&= \Phi(t_1, t_)x_0 + \sum_{i=1}^{n} \frac{c_i}{\lambda_i} W(t_0, t_1)v_i \\
&= \Phi(t_1, t_0)x_0 + \sum_{i=1}^{n} \frac{c_i}{\lambda_i} \lambda_i v_i \\
&= \Phi(t_1, t_0)x_0 + \sum_{i=1}^{n} c_i v_i = \Phi(t_1, t_0)x_0 + x_1 - \Phi(t_1, t_0)x_0 = x_1
\end{aligned}$$

Hence the system is controllable. □

2.4 The Satellite Control Problem

Consider a satellite orbiting the Earth. When a satellite is put into orbit, it may slightly deviate from the expected orbit or change its orientation because of different forces. To correct the deviation, the satellite have built-in control mechanism in the form of radial and tangential thrusters. We will perform the analysis in polar coordinate system. Let $r(t)$ be the radius from the origin to the mass and $\theta(t)$, the angle from the x-axis.

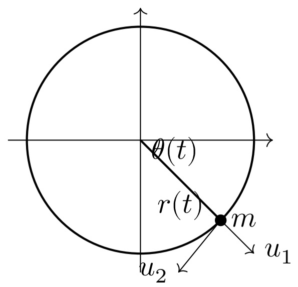

Figure 2.3: Satellite problem

The motion of the satellite is governed by the following nonlinear differential equations, which can be derived from Newton's laws of motions (see George & Ajayakumar [2]):

$$\frac{d^2 r}{dt^2} = r(t)\left(\frac{d\theta}{dt}\right)^2 - \frac{\beta}{r^2(t)} + u_1(t) \tag{2.14}$$

$$\frac{d^2\theta}{dt^2} = \frac{-2}{r(t)}\frac{d\theta}{dt}\frac{dr}{dt} + \frac{u_2(t)}{r(t)} \tag{2.15}$$

Let $u_1 = u_2 = 0$ and the initial conditions be $r(0) = \sigma, \dot{r}(0) = 0, \theta(0) = 0$ and $\dot{\theta}(0) = \omega$, where $\omega = (\frac{\beta}{\sigma^3})^{\frac{1}{2}}$, the solution is $r(t) = \sigma, \theta(t) = \omega t$ is an equilibrium solution. The above equations can be reduced into four first-order equations in terms of the variables $x_1 = r - \sigma, x_2 = \frac{dx_1}{dt}, x_3 = \sigma(\theta - \omega t), x_4 = \frac{dx_3}{dt}$. Let $x = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \end{pmatrix}^T$ and $u = \begin{pmatrix} u_1 & u_2 \end{pmatrix}^T$. Linearizing these equations about the above equilibrium solution, we get

$$\begin{pmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \frac{dx_3}{dt} \\ \frac{dx_4}{dt} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 3\omega^2 & 0 & 0 & 2\omega \\ 0 & 0 & 0 & 1 \\ 0 & -2\omega & 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} u \tag{2.16}$$

which is a linear system of the form

$$\dot{x} = Ax + Bu$$

where A and B are the state and control input matrices respectively, as given in equation (2.16).

2.4.1 Transition Matrix for the Satellite Problem

A being a constant matrix, transition matrix is the same as the matrix exponential. Note that A is a non-diagonalizable matrix with eigenvalues $0, 0, i\omega, -i\omega$. The corresponding generalized eigenvectors are $\tilde{v}_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}^T, \tilde{v}_2 = \begin{pmatrix} 1 & 0 & 0 & \frac{-3\omega}{2} \end{pmatrix}^T, \tilde{v}_3 = \begin{pmatrix} 1 & i\omega & 2i & -2\omega \end{pmatrix}^T$ and $\tilde{v}_4 = \begin{pmatrix} 1 & -i\omega & -2i & -2\omega \end{pmatrix}^T$. By the *Jordan-Chevalley Decomposition*[1], there exists a diagonalizable matrix S and a nilpotent matrix N such that $A = S + N$ and $SN = NS$.
Let $P = [\tilde{v}_1 | \tilde{v}_2 | \tilde{v}_3 | \tilde{v}_4]$,

$$i.e., P = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & i\omega & -i\omega \\ 1 & 0 & 2i & -2i \\ 0 & \frac{-3\omega}{2} & -2\omega & -2\omega \end{bmatrix}, P^{-1} = \begin{bmatrix} 0 & \frac{-2}{\omega} & 1 & 0 \\ 4 & 0 & 0 & \frac{2}{\omega} \\ \frac{-3}{2} & \frac{-i}{2\omega} & 0 & \frac{-1}{\omega} \\ \frac{-3}{2} & \frac{i}{2\omega} & 0 & \frac{-1}{\omega} \end{bmatrix}.$$

S is computed as $S = Pdiag\{0, 0, i\omega, -i\omega\}P^{-1}$.

$$S = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3\omega^2 & 0 & 0 & 2\omega \\ 6\omega & 0 & 0 & 4 \\ 0 & 2\omega & 0 & 0 \end{bmatrix} \text{ and } N = A - S = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -6\omega & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Note that N is nilpotent with index of nilpotency 2. Since S and N commute,

$$\begin{aligned} e^{A(t-t_0)} &= e^{S(t-t_0)} \cdot e^{N(t-t_0)} \\ &= \left[P \begin{pmatrix} e^0 & 0 & 0 & 0 \\ 0 & e^0 & 0 & 0 \\ 0 & 0 & e^{-i\omega(t-t_0)} & 0 \\ 0 & 0 & 0 & e^{i\omega(t-t_0)} \end{pmatrix} P^{-1} \right] [I + N(t-t_0)] \\ &= \begin{pmatrix} 4 - 3\cos\omega(t-t_0) & \frac{1}{\omega}\sin\omega(t-t_0) & 0 & \frac{-2}{\omega}(\cos\omega(t-t_0) - 1) \\ 3\omega\sin\omega(t-t_0) & cos\omega(t-t_0) & 0 & 2\sin\omega(t-t_0) \\ 6(\sin\omega(t-t_0) - \omega(t-t_0)) & \frac{2}{\omega}(\cos\omega(t-t_0) - 1) & 1 & \frac{4}{\omega}\sin\omega(t-t_0) - 3(t-t_0) \\ 6\omega(\cos\omega(t-t_0) - 1) & -2\sin\omega(t-t_0) & 0 & 4\cos\omega(t-t_0) - 3 \end{pmatrix} \end{aligned}$$

[1]This computation uses the *Jordan-Chevalley Decomposition*. Refer Saikia [5] and Moler [4] for details.

PBH Test for Satellite Problem

The left eigenvectors of A are $\begin{pmatrix}2\omega & 0 & 0 & 1\end{pmatrix}, \begin{pmatrix}\frac{3\omega}{2} & \frac{-i}{2} & 0 & 1\end{pmatrix}$ and $\begin{pmatrix}\frac{3\omega}{2} & \frac{i}{2} & 0 & 1\end{pmatrix}$. It is clear that none of these vectors are orthogonal to the columns of B. Hence, by the PBH eigenvector test, the linearized satellite problem is controllable.
The controlled trajectories and steering control of the system from $(1,2,3,4)^T$ to $(4,3,2,1)^T$ can be computed using equation [2.11] or [2.12] and plotted with the aid of computing softwares like MATLAB/ Mathematica.

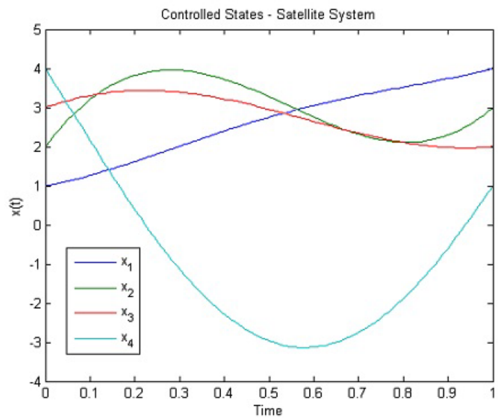

Figure 2.4: Controlled states of the linearized satellite problem

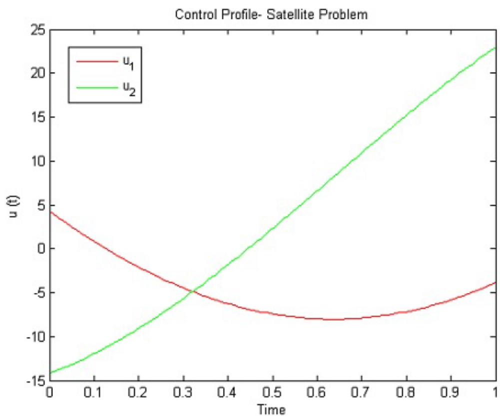

Figure 2.5: Steering control profile of the linearized satellite problem

2.5 Conclusion

Controllability and spectral characteristics of the system matrices are correlated in a deeper way. This provides easier tests for assessing the controllability of the system using spectral properties. This approach enables us to redesign the control matrix B, inorder to achieve controllability of the system. The idea of left eigenspaces and orthogonality play a key role in determining whether a system is controllable or not, this is the content of the PBH test. Theory is illustrated with the example of satellite control problem.

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A session in memory of Professor Radha Charan Gupta

Padma Shri
Prof. Radha Charan Gupta

A special session was held on 21 January 2025, in memory of Prof. Radha Charan Gupta, an eminent and much respected historian of mathematics who passed away on September 5, 2024; an obituary note on him, by Prof. M. S. Sriram, was carried by TMCB in the October 2024 issue.

The event was organized as a part of the Annual conference of the Indian Society for History of Mathematics, in the Ganita Sammelan, at IIT Gandhinagar, during January 19-22, 2025, organized under the HoMI Project, jointly with the Indian Society of History of Mathematics and the IKS Division of the Ministry of Education.

Prof. S. G. Dani presented an overview of the work of Prof. R. C. Gupta, highlighting its trend-setting aspects. Prof. M. S. Sriram spoke in detail, with illustrations, about two textbooks of Prof. Gupta in Hindi on History of mathematics. Speaking on the occasion Prof. Michel Danino mentioned that English translations of the two books are now being brought out under the HoMI project. Three video clips from a recent interview curated by the HoMI project were screened during the session. The session was moderated by Prof K. Ramasubramanian, who also narrated various anecdotes on Professor Gupta and his family, bringing out Prof. Gupta's deep commitment to the study of History of Indian Mathematics.

3. A Peep into History of Mathematics

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Here are my picks for a peep into history for this issue.

David Buckle, How the estimate of $\sqrt{2}$ on YBC 7289 may have been calculated, Historia Math. 62 (2023), 3-18.

The Babylonian cuneiform tablet[1] YBC 7289, dated to be from between 1800 and 1600 BCE, depicts a square and its diagonal, with number markings illustrating an approximation of $\sqrt{2}$, corresponding to the ratio of the diagonal to the side of the square. The numbers are represented in the *sexagesimal* system that they followed, and in effect $1 + \frac{24}{60} + \frac{51}{60^2} + \frac{10}{60^3}$ is described as an approximation for $\sqrt{2}$. In decimal expansion this corresponds to $1.41421\overline{296}$ (the bar stands for recurring decimals), which is remarkably accurate for their time. Now it is well-documented that the Babylonians had a method for approximating square roots, involving $a + \frac{b^2}{2a}$ as an approximation for $\sqrt{a^2 + b^2}$, when b is small compared to a. However, while they knew addition and multiplication well enough, they were much handicapped with regard to division, which they could carry out only with "regular" integers (those whose reciprocals have a finite sexagesimal fractional expansion). This also meant that they could not have used the formula iteratively to get finer approximations, as it would involve division by irregular denominators. The article discusses how they may have arrived at the close approximation as above, despite the handicaps. Along the way the article provides a good first introduction to the Babylonian arithmetic.

P. A. Jawalgekar, D. G. Sooryanarayan, and K. Ramasubramanian, Construction and application of third diagonal in cyclic quadrilaterals by Nārāyaṇa Paṇḍita, Indian J. Hist. Sci. 58 (2023), no. 4, 250-270.

A detailed study of general cyclic quadrilaterals (those with vertices lying on a circle) can be traced back to Brahmagupta (b. 598), who in particular gave formulae for their area and the diagonals. The essential significance of Brahmagupta's work in this respect seems to have been missed, however, in the subsequent Indian works dealing with geometry of quadrilaterals, during several subsequent centuries, including by the renowned Bhaskaracharya. The study on the topic was taken to new heights by Nārāyaṇa Paṇḍita, a 14th century mathematician, in his voluminous work *Gaṇitakaumudī*, composed in 1356 CE. One of the key features attributable to Nārāyaṇa is the introduction of what is called the "third diagonal" and its application in various formulae, including especially for the circumradius of the quadrilateral; an earlier formula in this direction was valid only for a restricted class of cyclic quadrilaterals.

The present article begins with the history of the topic, and includes an exposition of the - rather limited - information available about the author and the content and structure of *Gaṇitakaumudī*. In the penultimate section, which forms the bulk of the article, the authors present Nārāyaṇa's treatment of cyclic quadrilaterals, from the fourth chapter of *Gaṇitakaumudī*, titled *kṣetravyavahāra*. The "third diagonal" is arrived at by interchanging adjacent sides; one may think of this geometrically as relocating the lens formed by a diagonal and the segment of the circle cut off by it, changing its orientation. Nārāyaṇa notes, in particular, that there can be only three diagonal lengths possible following the modfications as above, (though the general description as above may suggest otherwise), and that the number is attained when the sides of the given quadrilateral are pairwise unequal. He also notes that the circumdiameter of a triangle equals the product of the two sides (other than the base) divided by the altitude and deduces an elegant formula for the area of the cyclic quadrilateral as the product of the three diagonals divided by twice the circumdiameter. Formulas are also described for altitudes of the cyclic quadrilateral and areas of the triangles formed when the quadrilateral is cut along the two diagonals.

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[1] It may be recalled here that the "cuneiform tablets" have been our principal source of knowledge about the Babylonian civilization. They consist of clay tablets on which writing, involved in various contexts, was done when still wet, and then dried and used for various purposes.

Chuanming Zong, The journey of Euclid's *Elements* to China, Notices Amer. Math. Soc. 70 (2023), no. 6, 953-961.

Euclid's *Elements*, composed about 23 centuries ago, has been one of the most influential works, impacting the development of mathematics around the world, starting with Europe. The work has now been translated in almost all languages of scientific communication, running into over a thousand editions. In this context its spread, in the form of accessibility through translated and edited works, in different parts, makes an important component of study of the history of mathematics. The present article describes, as indicated by its title, the part of the story, indeed a dramatic one as it turns out, with regard to China.

The first copy of *Elements* arrived in China in 1582, brought with him by Matteo Ricci, an Italian priest. Together with the Chinese scholar Xu Guangqi he translated the first six books of the work during the period from 1606 until 1610, the year when Ricci passed away, in Beijing. The remaining seven books were translated only in 1857, after a gap of almost two and half centuries, by a British missionary Alexander Wylie in collaboration with the Chinese mathematician Li Shan. The article traces the Chinese background, the events in Europe leading to transmission of mathematical knowledge through the missionaries, various aspects of the impact of introduction of *Elements*, and other works that followed it, with numerous delightful illustrations.

William Dunham, Cauchy and his modern rivals, Math. Gaz. 107 (2023), no. 568, 103-113.

The title is a take on the 1879 paper *Euclid and his modern rivals* by Charles Dodgson (1832 - 1898), also known by his pen-name Lewis Carrol. While for Dodgson the ancients outshone the moderns, and the article meant to celebrate Euclid, in contrast the present article begins with a "spoiler alert" that all the mathematics considered is "worthy of celebration".

Augustin-Louis Cauchy (1789-1857) is of course remembered for a whole variety of contributions in mathematics. The content here however primarily concerns two inequalities introduced by him which have had a profound impact on subsequent developments in Analysis: One is the "Cauchy inequality" established in 1821, asserting that $|\sum_{i=1}^n a_i b_i| \leq \sqrt{\sum_{i=1}^n a_i^2}\sqrt{\sum_{i=1}^n b_i^2}$, for any real numbers $a_1, \dots, a_n$ and $b_1, \dots, b_n$ (with equality only when the n-tuples are proportional) and the second relating to the comparison of geometric mean of a set of numbers with their arithmetic mean, dubbed the AM-GM inequality. The paper recalls Cauchy's proofs, which incidentally are quite unlike what one gets to see in our textbooks, and also recalls various analogous inequalities noted by Cauchy, which are not commonly found in recent literature.

The "modern rival" referred to with regard to the first of the inequalities is Hermann Amandus Schwarz (1843-1921), who in his quest to extend Cauchy's result to the realm of integrals came up with a new line of attack; as such his name now features, hyphenated with that of Cauchy, in "Cauchy-Schwarz" inequality. The "modern rival" introduced in the case of the AM-GM inequality is the 20th century mathematician George Polya (1887-1985), for his alternative proof of the inequality, via the exponential function e^x. The spirit of the presentation in the article is best conveyed by the author's concluding observation "...mathematics is enriched when the same result generates radically different proofs. The variants we have seen here, both for Cauchy's inequality and for the AM-GM inequality, are but a sampling of the multiple ways these can be derived.", along with kudos to and photographs of the trio Cauchy, Schwarz and Polya.

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4. What is Happening in the Mathematical World?

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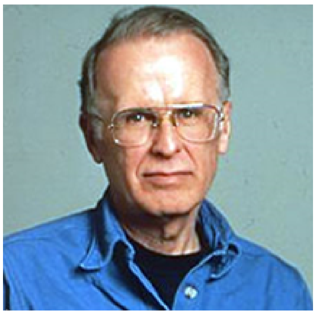
4.1 Celebrating 100th Birth Anniversary of John Warner Backus, Inventor of Fortran Language



John Warner Backus (Dec. 3, 1924 - Mar 17, 2007) was an American mathematician and computer scientist best known for the invention of FORTRAN, the first widely used high-level programming language, and was the inventor of the Backus-Naur form (BNF), a widely used notation to define syntaxes of formal languages.

Backus earned a B.S. (1949) and an M.A. (1950) from Columbia University in New York, both in mathematics, and joined International Business Machines (IBM) in 1950. During his first three years, he worked on the Selective Sequence Electronic Calculator (SSEC); his first major project was to write a program to calculate positions of the Moon. In 1953, Backus developed the language Speedcoding, the first high-level language created for an IBM computer, to aid in software development for the IBM 701 computer.

In 1954 he assembled a team to define and develop program for the IBM 704 computer. His small group at IBM developed the computer language FORTRAN for numerical computations. It was the first high-level programming language to be put to broad use. This widely used language made computers practical and accessible machines for scientists and others without requiring them to have deep knowledge of the machinery.

Backus served on the international committees that developed ALGOL 58 and the very influential ALGOL 60, which quickly became the *de facto* worldwide standard for publishingalgorithms.In 1959, he introduced the Backus Normal Form (BNF) notation for describing the syntax of any context-free programming language, which was published in the UNESCO report on ALGOL 58. This contribution helped Backus win the Turing Award of $1 million, in 1977. Backus later worked on a function-level programming language known as FP, which was described in his Turing Award lecture.

He was named an IBM Fellow in 1963. The IEEE awarded Backus the W. W. McDowell Award in 1967. He received the National Medal of Science in 1975 and Computer History Museum Fellow Award for his development of FORTRAN, contributions to computer systems theory and software project management in 1997. In 2007, Asteroid *6830 Johnbackus* was named in his honor. John Backus retired in 1991. He died at his home in Ashland, Oregon on March 17, 2007.

Sources:

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2. *https://mathshistory.st-andrews.ac.uk/Biographies/Backus/*

4.2 Amateur Investigator finds Largest known Prime Number with 41 Million Digits

A 36-year-old researcher, amateur mathematics investigator *Luke Durant* from San Jose, California used several graphics processing units (GPUs) to search for the so far largest prime number.

If the newly found prime number is written in binary, it will be a vast string of ones: 1111111...111 specifically, 13,62,79,841 ones in a row. If we stacked up that many sheets of paper, the resulting tower would stretch into the stratosphere. If we write this number in decimal, it starts out 8,816,943,275... and ends ...076,706,219,486,871,551. This would have 4,10,24,320 decimal digits and it would fill 20,000 pages in a book form! It is now 16 million digits longer than the previous record found in 2018. Another way to write this number is $2^{13,62,79,841} - 1$.

This prime number is the 52nd known Mersenne prime (A prime number that is one less than some power of two (or $2^p - 1$)) ever discovered (known as M136279841 for short).

Durant is one of thousands of people working as part of a long-running volunteer prime-hunting effort called the Great Internet Mersenne Prime Search, or GIMPS. Durant made his discovery through a combination of mathematical algorithms, practical engineering, and massive computational power. Where large primes have previously been found using traditional computer processors (CPUs), this discovery is the first to use a different kind of processor called a GPU.

GPUs were originally designed to speed up the rendering of graphics and video, and more recently have been repurposed to mine cryptocurrency and to power AI. Durant, a former employee of leading GPU maker NVIDIA, used network of thousands of GPUs housed in 24 data centers across 17 countries to create a kind of "cloud supercomputer". The lucky GPU was an NVIDIA A100 processor located in Dublin, Ireland.

Sources:

1. *https://www.newscientist.com/article/2452686-amateur-sleuth-finds-largest-known-prime-number -with-41-million-digits/*
2. *https://theconversation.com/a-41-million-digit-prime-number-is-the-biggest-ever-found-but -mathematicians-search-for-perfection-will-continue-242291*

4.3 New Elliptic Curve with the Most Complicated Pattern of Rational Points to Date, has been Found

Elliptic curves, which date back to at least ancient Greece, are central to many areas of study. They have a rich underlying structure that mathematicians have used to develop powerful techniques and theories. Mathematicians often try to characterize elliptic curves by studying the special "rational points" that lie on them. On a given curve, these points form clear and meaningful patterns. But it is not yet known whether there is a limit to how diverse and complicated these patterns can get.

Now, two mathematicians - *Noam Elkies* of Harvard University (left) and *Zev Klagsbru*n of the Center for Communications Research, California (right) - have found an elliptic curve with the most complicated pattern of rational points to date, breaking an 18-year-old record. The discovery opens new debate over what mathematicians think they know about elliptic curves.

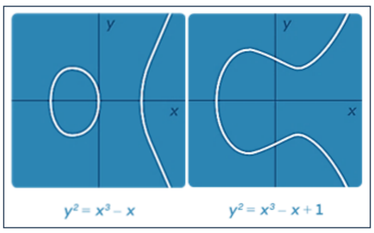


Figure1

Elliptic curves are equations of the form $y^2 = x^3 + Ax + B$, where A and B are rational numbers. When you graph the solutions to these equations, they look like as shown in Figure-1.

Mathematicians are particularly interested in a given elliptic curve's rational solutions. It is literally one of the oldest problems in the history of mathematics. While it is relatively straightforward to find rational solutions to simpler types of equations, elliptic curves are the first class of equations where there are really a lot of open questions.

To get a grip on the rational solutions of an elliptic curve, mathematicians often turn to the curve's *rank*, a number that measures how closely packed the rational points are along the curve. A rank 0 elliptic curve has only a finite number of rational points. A rank 1 elliptic curve has infinitely many rational points, but all of them line up in a simple pattern, so that if you know one, you can follow a well-known procedure to find the rest. The rank of an elliptic curve tells mathematicians how many "independent" points - points from different families - they need in order to define its set of rational solutions. The higher the rank, the richer in rational points the curve will be. A rank 2 and a rank 3 curve both have infinitely many rational solutions, but the rank 3 curve packs in rational points from an additional family, meaning that on average, a given stretch of it will contain more of them. Almost all elliptic curves are known to be either rank 0 or rank 1. But there are still infinitely many with higher rank - and they are exceedingly difficult to find. As a result, mathematicians are not sure if there is a limit to how high the rank can get.

Imagine a simple surface, a flat plane. You can slice it into infinitely many straight lines, laid side by side. Depending on how you make your slices, the lines you end up with will be defined by different equations. Similarly, there are more complicated, curvy surfaces that, when sliced up, yield infinitely many elliptic curves. Mathematicians have been using these surfaces to find higher-rank elliptic curves since the 1950s.

In 2006, Elkies sliced a particular K3 surface (for definition see source 2) and looked at the pieces. He found among the slices an elliptic curve that he could show had a rank of at least 28 - beating the previous record of 24. Elkies' record stood for nearly two decades.

Recently, Elkies along with Zev Klagsbrun sliced the K3 surface in a different way, getting a new pile of curves to work with. But there were hundreds of ways they could slice it, and most of those slicing methods seemed unlikely to produce the curve they sought. Then, entirely by accident, they found a slicing method that gave them a pile of curves, all guaranteed to have a rank of at least 17.

Using more powerful computational technique, they found within that pile an elliptic curve with a rank of at least 29.

The curve's equation, when written as $y^2 = x^3 + Ax + B$, has values of A and B that are each over 60 digits long. The 29 independent rational solutions that Elkies and Klagsbrun pinpointed involve numbers that are similarly huge.

It could still be a long way to go towards setting the debate about whether the rank of elliptic curves has an upper limit. But the discovery of a rank 29 curve expands the frontier of this unexplored realm.

Sources:

1. *https://www.quantamagazine.org/new-elliptic-curve-breaks-18-year-old-record-20241111/*
2. *https://www.ucl.ac.uk › archived › notes › K3*

4.4 Brauer's Height Zero Conjecture has been Settled

A Vietnamese American mathematician Pham Huu Tiep from Rutgers University, USA and his colleagues Robert M. Guralnick, Michael Larsen, Gunter Malle from Kaiserslautern, Germany, Gabriel Navarro from Universitat de València, València, Spain, and A. A. Schaeffer Fry from University of Denver, USA, have settled the longstanding open Brauer's Height Zero Conjecture [2].

Brauer's Height Zero Conjecture (BHZ), formulated in 1955, has been one of the most fundamental and challenging problems in modular representation theory of finite groups.

Deeply influencing the research in the field, it is also a source of many developments in the theory. One can state the BHZ conjecture in known terminology of representation theory as follows: Let G be a group and p be a prime. The set Irr(G) be the set of irreducible complex characters which can be partitioned into Brauer p- blocks. To each p-block there is canonically associated a conjugacy class of p-subgroups called the defect groups of B. Let χ be an irreducible character in a block B of a group G with defect group D. Let v be the discrete valuation defined on the integers with $v(np^\alpha) = \alpha$ whenever n is prime to p. By a theorem of Brauer, $v(\chi(1)) \geq v(|G : D|)$. The height of χ is defined to be $v(\chi(1)) - v(|G : D|)$.

Brauer's height-zero conjecture is the assertion that every irreducible character in B has height zero if and only if D is Abelian.

The "if" implication of the Conjecture was proven in 2013 by R. Kessar, G. Malle, using the classification of finite simple groups, after decades of contributions by many authors. In 1984, the "only if" implication was proven for p-solvable groups and for few other special cases including $p = 2$, later. In 2014, G. Navarro, B. Späth have shown that the Conjecture is implied by the inductive Alperin-McKay condition on simple groups. However, the verification of the inductive Alperin-McKay condition on simple groups for odd primes remains an enormous challenge. In their paper Tiep-et-al have taken a different approach and proved the open direction of Brauer's Height Zero Conjecture for odd primes p.

Tiep, Guralnick, and Larsen also studied a particular aspect of finite classical groups, producing two papers [3] and [4] on the subject.

In the first, Larsen and Tiep helped to prove existing beliefs about how an abstract algebra group can be instead represented as a matrix group, and that the representational matrix itself can be described using its diagonal elements, or its trace. Each element in the group can have the trace attached, creating the "character" of the group representation. Larsen and Tiep refined a mode of thinking about these characters by adjusting how they can be bound using the character and multiple other features.

In their second character paper, they deepen this understanding of the character and trace bounds by analyzing a ratio of components of the character.

Sources:

1. *https://www.aol.com/mathematicians-solved-notorious-old-problem-203400574.html*
2. Pham Huu Tiep, G. Malle, G. Navarro, and A. A. Schaeffer Fry, Brauer's height zero conjecture, Annals of Math. 200 (2024), 557 - 608.
3. Pham Huu Tiep, Michael Larsen, Uniform character bounds for finite classical groups, Annals of Math. 200 (2024), 1 - 70. Character levels and character bounds for finite classical groups, Invent. Math. 235 (2024), 151 - 210 (joint work with R. M. Guralnick and M. Larsen).
4. Pham Huu Tiep, M. Guralnick, Michael Larsen, Character levels and character bounds for finite classical groups, Invent. Math. 235 (2024), 151 - 210.

4.5 There are Infinitely many Primes of the form $p^2 + 4q^2$

In 1640, Pierre de Fermat conjectured that there are infinitely many primes that can be formulated by squaring two whole numbers and adding them together. (The prime number 13, for instance, can be written as $2^2 + 3^2$.) Leonhard Euler later proved it. But tweaking the question just a little bit - by insisting that one of the numbers you're squaring be a prime, perhaps, or a perfect square - makes the problem much harder.

In 2018, Friedlander and Henryk Iwaniec of Rutgers University asked if there are infinitely many primes of the form $p^2 + 4q^2$, where both p and q must also be prime. (For example, $41 = 5^2 + 4 \times 2^2$.) The constraint turned out to be particularly challenging to deal with. Now, two mathematicians *Ben Green* of Oxford University (left) and *Mehtaab Sawhney* of Columbia University (right) have proved that it is indeed true.

The breakthrough came when the duo undertook the challenge from a new angle. Rather than using traditional counting techniques, they used the method of Type I/II sums in the number field $\mathbb{Q}(\sqrt{-n})$. The main innovation is in the treatment of the Type II sums, where they make heavy use of two recent developments in the theory of Gowers norms in additive combinatorics: quantitative versions of socalled concatenation theorems, due to Kuca and to Kuca–Kravitz-Leng, and the quasi polynomial inverse theorem of Leng, Sah and the Sawhney.

Their success depends on a clever technique: instead of working directly with prime numbers, they first proved their result for "rough primes" - numbers that are not divisible by handful of small prime numbers (say, 2, 3, 5, 7). This intermediate step made the problem more manageable while still leading to the final solution.

They have also proved a more general result that there are infinitely many primes of the form $p^2 + nq^2$ with both p and$q$ prime, and $n \equiv 0$ or $n \equiv 4$ mod 6.

The discovery's significance extends beyond just solving this single problem. By successfully applying the Gowers norm to prime number theory, Green and Sawhney have potentially opened up an entirely new toolkit for mathematicians studying prime numbers.

Sources:

1. *https://boingboing.net/2024/12/13/two-mathematicians-just-solved-a-centuries-old-prime-number- puzzle-that-experts-thought-impossible.html*
2. Ben Green and Mehtaab Sawhney, Primes of the Form $p^2 + nq^2$, arXiv2410.04189v2[math.NT] 12 Oct 2024.

4.6 A Whole New type of Soft Shapes Found to Tile Space

A team of mathematicians from the Universities of Oxford and Budapest have discovered a new class of shapes that tile space without using sharp corners. These 'soft shapes' are found abundantly in nature - from sea shells to muscle cells. The findings not only explain the geometry of biological tissues, but could also unlock new building designs, without corners.

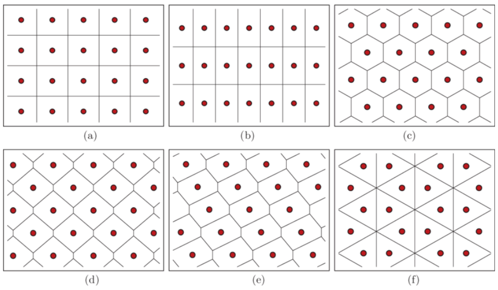


Figure 1: Two-dimensional Bravais lattices and their Voronoi tessellations

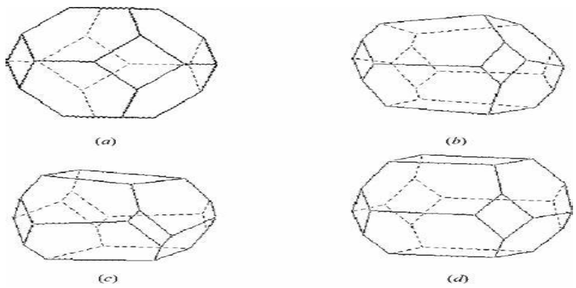


Figure 2: Lattice Voronoi-Dirichlet polyhedra

Fitting shapes together to cover surfaces without gaps has been studied since a long time. However, their typical approach – using shapes with sharp corners and flat faces - is rarely seen in the natural world. Instead, living organisms use an amazing array of patterns to form and grow. These patterns are described by shapes with curved edges, non-flat faces, and few sharp corners. Up to now, how nature achieves geometrical complexity using these 'soft shapes' was not explained mathematically.

Now, Alain Goriely (Univ. of Oxford), Gábor Domokos, Krisztina Regős and Ákos Horváth (Budapest Univ.) (from left to right) discovered a new class of mathematical shapes - soft cells, which are shapes with minimal sharp corners that cover space without gaps. They have proved that an infinite class of polyhedral tiling can be smoothly deformed into soft tiling and constructed the soft versions of all Dirichlet-Voronoi cells associated with point lattices in two and three dimensions (See figures 1 and 2). Remarkably, these ideal soft shapes, born out of geometry, are found abundantly in nature, from cells to shells.

In 2D, these soft cells (see Fig.3) have curved boundaries with only two corners. Such tiling patterns are found in muscle cells, zebra islands, in the layers of onion, in architectural design etc. (see Fig. 5). In 3D, these soft cells become more complex and interesting. The team established that, in 3D, soft cells have no corners at all (see Fig. 4). Then, starting with 3D tiling systems, team showed that they can be softened by bending the edges while minimizing the number of sharp corners in this process.

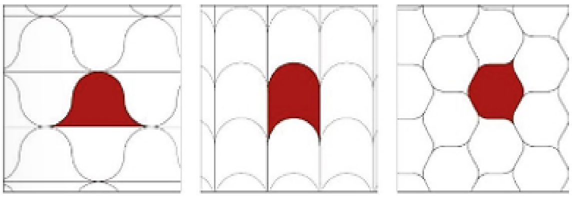


Figure 3: Soft tilings in the plane

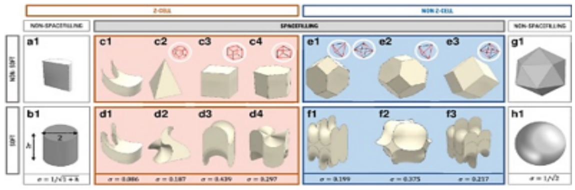


Figure 4: Genesis of soft 3-D cells

Figure 5: Soft 2-D cells in nature

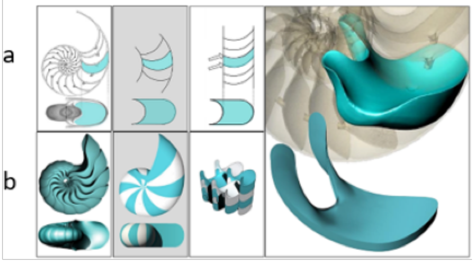


Figure 6: The geometry of the nautilus chamber Row (a): Sections. Row (b): 3D geometry.

Through doing this, they found entire new classes of soft cells with different tiling properties. Architects have constructed these kinds of shapes intuitively whenever they wanted to avoid corners. A central part of the study shows how the inner chambers of the nautilus are natural examples of 3D soft cells without corners (see Fig. 6, Row b). Surprisingly, the planar sections of the chambers are 2D soft cells (see Fig. 6, Row a).

Soft cells appear to be geometric building blocks of biological tissue and their existence open up questions in geometry and biology. Conditions for generating soft tilings could shed new light on why certain patterns are chosen by nature.

Sources:

1. *https://www.gamerstones.com/techalert/mathematicians-discovered-a-new-class-of-mathematical -shapes- soft-cells/?utm_source=Blubs&utm_medium=BlubButton&utm_campaign=Blubs&utm_id=Blubs*
2. *https://doi.org/10.1093/pnasnexus/pgae311*

4.7 60 Years-old 'Moving Sofa Problem' is now Solved

In mathematics, the moving sofa problem is a two-dimensional idealization of real-life furniture-moving problems and asks for the rigid two-dimensional shape of the largest area that can be maneuvered through a unit-width L-shaped corridor. Now, in November, 2024 the combinatorics and geometry enthusiast Jineon Baek of Yonsei University in Korea has posted an arXiv preprint claiming to have solved (giving a100-page proof [2]) this very problem.

The story begins in 1966 when Austrian-Canadian mathematician Leo Moser formalized the "moving sofa problem". While it might seem trivial, the puzzle knocks into deep mathematical principles of combinatorics and geometry. It asks an illusorily simple question: What is the largest two- dimensional shape that can navigate an L-shaped corner in a corridor of unit width?

Imagine trying to move a perfectly rectangular piece of furniture through a narrow hallway. A square unit fits easily, but as the object's dimensions grow, the movement becomes impossible. Moser's challenge captured the frustration of movers everywhere.

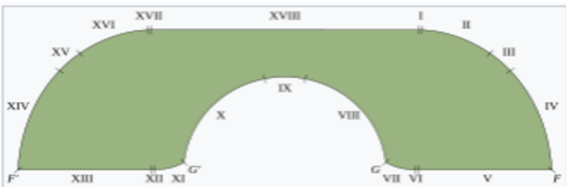


Figure 1: Gerver's sofa with 18 curve sections

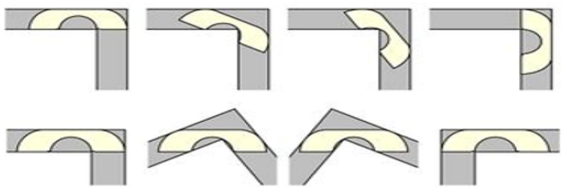


Figure 2: Movement through L-shaped corner

British mathematician John Hammersley took an early crack at the problem in 1968. He designed a sofa that combined a semicircle with a square featuring a semicircular piece. His creation was mathematically efficient and could move around an L-shaped corner with an area of 2.2074 units. He also established an upper limit of 2.8284 units, suggesting no sofa larger than that could fit.
Nearly 25 years later, Joseph Gerver of Rutgers University refined Hammersley's design. By rounding edges and incorporating additional arcs, Gerver proposed a new sofa shape (see Figures 1 and 2) with an area just over 2.2195 units. His solution was "locally optimal," meaning it was the best possible shape under the constraints he defined.
In 2018, Yoav Kallus of the Santa Fe Institute and Dan Romik from the University of California, added a modern twist. Using computer simulations, the team theorized that a sofa shape with an area as large as 2.37 units might be feasible. Their work reignited interest in the problem and pushed the mathematical boundaries further.
Jineon Baek's recent work builds on this foundation. By employing an advanced mathematical technique called an injective function, Baek mapped the key properties of Gerver's sofa design. Baek analyzed how these properties could be extended to larger dimensions, ensuring that the sofa's shape would remain feasible for navigating tight spaces. Through this rigorous approach, Baek conclusively demonstrated that 2.2195 units is the absolute maximum area for a sofa that can successfully pass through an L-shaped corner in a 1-unit-wide corridor.
Baek's work not only contributes to geometry and combinatorics but also showcases how abstract mathematics can have unexpected real-world applications.

Sources:

1. *https://www.earth.com/news/moving-sofa-problem-mathematician-solves-a-century-old- puzzle/*
2. *https://arxiv.org/abs/2411.19826*
3. *https://en.wikipedia.org/wiki/Moving_sofa_problem*

4.8 Babylonian Tablet Preserves A Student's 4,000-year-old Geometry Mistake

A small clay tablet from the site of Kish in Iraq reveals a student calculated the area of a triangle incorrectly 4,000 years ago.
This round clay tablet, which is in the collection of the Ashmolean Museum at the University of Oxford, is one of two dozen examples of ancient Babylonian mathematics homework found at the archaeological site of Kish in 1931.
However, the student who used this tablet to calculate the area of a triangle made a key mistake, and this error has been preserved for nearly 4,000 years.
The tiny tablet is just 3.2 inches (8.2 cms) in diameter and depicts a right-angled triangle with three sets of calligraphy style numbers – one set along each of the two sides representing the length and height of the triangle, and one in the middle for its area. Along the top line (height) of the triangle, the student has written 3.75, while the vertical line (base) is indicated as 1.875. These values mean the area of the triangle should be 3.5156. The student, though, has incorrectly calculated it as 3.1468.
Babylonian people understood the Pythagorean theorem more than a millennium before the ancient Greek philosopher Pythagoras became famous for establishing that the sum of the squares of two sides of a right triangle equals the square of the hypotenuse.
So, this student's messed-up maths actually shows an important cultural development: The way people collected and passed on knowledge was switching from memorization to written information.
Source: *https://www.livescience.com/archaeology/babylonian-tablet-preserves-students-4-000-year-old- geometry-mistake*

4.9 Awards

4.9.1 Prof. Neena Gupta of ISI, Kolkata Wins the Infosys Prize 2024 in Mathematical Sciences

Prof. Neena Gupta of Theoretical Statistics and Mathematics Unit, Indian Statistical Institute, Kolkata is awarded the Infosys Prize 2024 in Mathematical Sciences for her work on the Zariski Cancellation Problem, a fundamental problem in algebraic geometry, a version of which was first posed in 1949 by Oscar Zariski, one of the founders of modern algebraic geometry. The prize comprises a gold medal, a citation, and a prize of \$1,00,000. Infosys Prize remains the largest award in India that acknowledges excellence in science and research.

Prof. Neena Gupta's solution of the Zariski Cancellation Problem is a landmark achievement in affine algebraic geometry. Affine algebraic geometry explores what at first sight might look like very simple objects, namely collections of polynomials in some variables: $3x^2 + 5yz + w^3$ is an example of a polynomial in the variable x, y, z, w. Zariski's problem revolves around the cancellation property for polynomials, essentially asking whether an algebraic object (a quotient of a polynomial ring), that becomes isomorphic to the polynomial ring in variables $x_1 \ldots, x_n$ after adding a variable, is itself such a polynomial ring in some variables. In more geometric terms the problem asks: if two geometric objects have the same structure after adding a dimension to them (which is like considering cylinders over them), then do the objects themselves have the same structure?

In her breakthrough work she showed that, in positive characteristic, contrary to what one might expect, the answer is in the negative. Thus, the innocuous looking move of adding a dimension can destroy information. Prof. Gupta's intricate proof shows that a specific 3- dimensional affine variety constructed earlier by Asanuma yields a counterexample to Zariski's problem in positive characteristic. In a follow-up work, she revealed surprising connections of this problem with other fundamental problems and concepts on affine spaces. In subsequent work with her collaborators, she has established further striking results in commutative algebra and algebraic geometry.

Her solution also had a striking elegance and made a great impact in the mathematical community. Her work has had a significant and lasting impact on algebraic geometry and commutative algebra, areas that are central to modern mathematics.

Prof. Neena Gupta earned her Ph.D. from the Indian Statistical Institute in Kolkata in 2012, where she is now a professor. Her contributions have been widely recognized: she received the prestigious Shanti Swarup Bhatnagar Prize in Mathematical Sciences in 2019, making her one of its youngest recipients and one of the few women to receive this honor. She received the Nari Shakti Puraskar from the President of India for the year 2021, and the DST-ICTP-IMU Ramanujan Prize in 2021. She was an invited speaker at the International Congress of Mathematicians in 2022 and received the 2023 TWAS-CAS Young Scientist Award for Frontier Science. Most recently she has been selected to deliver the AWS-AMS Noether Lecture in January 2025 at the annual meeting of the American Mathematical Society.

Source: *https://www.infosysprize.org/laureates/2024/neena-gupta.html*

4.10 Obituary

4.10.1 Walter David Neumann, Renowned for Major Breakthroughs Passes away at the Age of 78

After a long struggle with Alzheimer, mathematician *Walter David Neumann* of Princeton University died on Sept. 24, 2024 at the age of 78.

Walter Neumann was a British-American mathematician, internationally renowned for major breakthroughs in fields of mathematics that included low-dimensional topology, hyperbolic geometry, geometric group theory, and singularity theory.

Unusually, Bernhard and Hanna Neumann (Walter's parents), older brother Peter and a cousin Mike Newman were all group theorists. But Walter earned his doctorate in topology from Bonn University, Germany. He felt that mathematicians do their best work not only when they are young but also when they are new to a field. His own career proved the truth of his theory.

Walter was also a wonderful human being - deeply modest, warm, humble, and remarkably generous, both as a person and as a mathematician. Walter was especially generous with young mathematicians, always offering his time and guidance. He had a rare ability to listen during mathematical discussions with his many collaborators and friends, making everyone feel heard and valued. His enthusiasm for mathematics was contagious. Walter retired in 2021 from Columbia University.

Sources:

1. *https://en.wikipedia.org/wiki/Walter_Neumann*
2. *https://planetprinceton.com/2024/09/27/mathematician-walter-david-neumann-dies-at-78/*

4.10.2 A renowned mathematician R. Keith Dennis passes away at the age of 80

A renowned mathematician *R. Keith Dennis*, professor emeritus of mathematics at Cornell died on Dec. 12, 2024 at the age of 80 following a long battle with metastatic prostate cancer.

Dennis made significant contributions to algebraic K-theory and group theory, publishing 25 papers. His graduate textbook, "Noncommutative Algebra," co-written with Benson Farb, remains a standard reference in the field. He discovered a remarkable mapping in K-theory that was subsequently developed into a very general tool that came to be known as the 'Dennis Trace'.

Keith was born on March 10, 1944, in Texas. He demonstrated his mathematical aptitude early in life and graduated from high school as valedictorian. He received a B.S. in 1966 and a Ph.D. in 1970, both in mathematics from Rice University. During 1970–71, he worked with Fields medal winner John Milnor at Princeton University's Institute for Advanced Study, then came to Cornell, where he spent the rest of his career.

In addition to his scholarly research, Dennis was deeply committed to expanding the availability of mathematics books. He personally donated thousands of books, journals, papers and letters from his own collection to AIM's library.

In addition to his efforts at Cornell, Dennis worked with the American Mathematical Society from 1995 to 2001, serving as the executive editor of Math Reviews for three years, and then as a consulting editor. During his tenure at MR, Dennis oversaw the shift from print to digital platforms.

Over the years, he supervised many undergraduate research projects and his students have gone on to become leaders in group theory and related fields.

In 1987, Dennis received a Humboldt Award, which honors leading researchers for their academic achievements. His other honors include being named an inaugural Fellow of the American Mathematical Society in 2013.

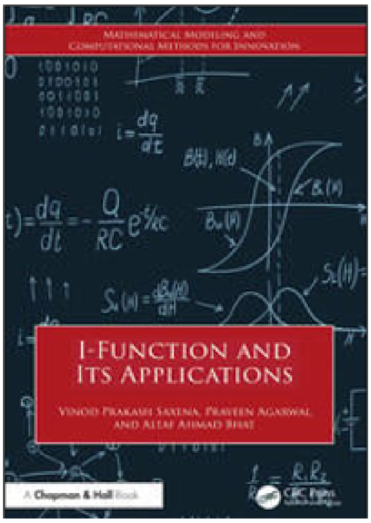
Source: *https://news.cornell.edu/stories/2024/12/mathematician-r-keith-dennis-dies-80*

□□□

5. Book Review

I-Function and Its Applications,
V. P. Saxena, P. Agrawal and Altaf A. Bhat
(C. R. C. Press (Taylor and Francis group))

Review by: Prof. Renu Jain
Ex Vice Chancellor, Devi Ahilya Vishv Vidyalaya, Indore.
E-mail: renujain3@rediffmail.com



The present book entitled I-*Function and Its Applications* provides an exhaustive coverage of the I function and its applications in specific areas. This function is the latest generalization of hyper geometric functions in the sequence of Mac Robert's E-Function, Meijer's G-Function and Fox's H-Function. The I-Function evolved while solving generalized dual integral equations, involving sums of H functions in their kernels. The definition and existence of the I-Function was established around early seventies by Professor V. P. Saxena, employing the generalized Hardy-Titchmarsh theorems for symmetric and unsymmetrical kernels. This book is the second publication on the I-Function, introduced and authored by Professor Saxena along with two co-authors. First one, published by Anamaya, New Delhi ((2008) contained basic properties, analysis and some selected properties of the function.

As a researcher in the field of Special functions, I found this function very interesting and hence started exploring different aspects of I-Function. I am still continuing with possible applications of the I-Function, particularly those emerging from its algebraic structure. The book has eight chapters including an introductory chapter which presents a survey of various special functions, including generalizations of hypergeometric functions, leading to the latest development in terms of I function.

In some other chapters, several aspects of the I function, like existence conditions, transformation and expansion formulas, algebraic properties and applications in diverse fields, have been covered. Another chapter shows the importance of various fundamental results and techniques derived from the theory of complex analysis, laying the groundwork for further exploration.

The special functions having emerged as solutions of certain well-known differential equations and boundary value problems arising in the field of mathematical physics, their use in solving problems becomes acceptable. In this way special functions "constitute a common currency" of mathematics. Underlining this fact, yet another chapter has been devoted to the solution of the dual integral equations. This chapter expands the area of special functions that have been developed and applied to a variety of fields such as combinatorics, astronomy, potential theory and population dynamics and bio-engineering.

In general, the book I-*Function and Its Applications* serves as a platform for exploring the recent theories and applications, and offers students, researchers and scholars of Mathematics a deep insight into advanced mathematical techniques and their practical applications across various fields.

□□□

6. Problem Corner

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In the October 2024 issue of TMC Bulletin, we posed two problems, However, so far, we have not received solution of any of the problems. We would like to emphasize here that problem solving is an important activity in the process of learning mathematics. Hence, we appeal to all the teachers to encourage their students to attempt solving problems posed in this section.
Here we present an outline of a solution to the first problem posed in the previous issue. This solution is provided by the problem proposer.
Also, in this issue we pose two problems, one from Geometry by Priyamvad Srivastav, and one form Number Theory by Amarnath Murthy, for our readers. Readers are invited to email their solutions to Dr. Udayan Prajapati (ganit.spardha@gmail.com), Coordinator, Problem Corner, before 15th March, 2025. Most innovative solution will be published in the subsequent issue of the bulletin.

First problem posed in the previous issue:
Let $f(n) = n^3$ for every natural number n.
Let $x = 0.f(1)f(2)f(3)\cdots$. In other words, to obtain the decimal representation of x write the numbers $f(1), f(2), f(3)\cdots$ in base 10 in a row (without leading zeroes). Is x a ratoinal?

Solution: In the decimal representation of x, the j^{th} group of digits is as in $f(j) = j^3$. As $j > 0$, the leftmost digit of j^3 is nonzero.
Hence, the decimal representation of x is infinite. (6.1)
Suppose that x is rational.
We know that every rational number has unique decimal representation which is either terminating or recurring (Here we follow a convention that if a real number r is of the form
$r = 0.a_1a_2\ldots.a_n999\ldots$ with $a_n < 9$, then we write $r = 0.a_1a_2\ldots.a_{n-1}e_n$, where $e_n = a_n + 1$).
Then by [6.1], x must have recurring decimal representation.
So, $x = 0.a_1a_2\ldots.a_nb_1b_2\ldots.b_k$, for some fixed integers n and k.
Then $10^{(n+k)}th$ group of digits in the decimal expansion of x, is as in $f(10^{(n+k)}) = 1000^{(n+k)}$, which contains $3(n + k) > n + k$, consecutive trailing zeroes. This is feasible only if $b_i = 0$, for each $i = 1, 2, \ldots, k$.
But this means that x has terminating decimal representation.
This contradicts (1). And hence, x is irrational.

Problem for this issue

Problem 1: Let ABC be an acute angled triangle inscribed by a circle of radius R. Suppose P lies in the interior of the triangle such that $PB \neq PC, \angle BPC = 2\angle BAC$ and $PA^2 + PB \times PC = 2R^2$. If Q is the incenter of the triangle PBC, show that $\angle QBA = \angle QCA$.

Problem 2: Prove that there are infinitely many triplets (a, b, c) of positive integers such that $(a \times b \times c)$ is a tetrahedral number and $(a + b + c)$ is a triangular number.

Note: The nth triangular number is $T_n = \frac{n(n+1)}{2}$ and the nth tetrahedral number is $Te_n = \frac{n(n+1)(n+2)}{6}$.

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7. International Calendar of Mathematics Events

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April 2025

- April 7-11, 2025, Interactions Between Harmonic Analysis, Homogeneous Dynamics, and Number Theory, SL Math, 17 Gauss Way, Berkeley, CA 94720 USA. *www.slmath.org/workshops/1089#overview_workshop*
- April 14-18, 2025, AIM Workshop: Integro-Differential Equations in Many-Particle Interacting Systems, American Institute of Mathematics, Pasadena, California. *aimath.org/workshops/upcoming/manyparticle/*
- April 28 - May 2, 2025, AIM Workshop: Moments in Families of L-Functions over Function Fields, American Institute of Mathematics, Pasadena, California. *aimath.org/workshops/upcoming/momentsoverff/*

May 2025

- May 1-3, 2025, SIAM International Conference on Data Mining (SDM25), Westin Alexandria Old Town, Alexandria, Virginia, US. *www.siam.org/conferences/cm/conference/sdm25*
- May 8-10, 2025, International Conference on Non-Linear Analysis and Optimization (with Workshop on Fixed Point Theory and its Applications), Kathmandu University, Dhulikhel, Kavre, Nepal. *icanopt2025.ku.edu.np/*
- May 12-16, 2025, AIM Workshop: Algorithmic Stability: Mathematical Foundations for the Modern Era, American Institute of Mathematics, Pasadena, California. *aimath.org/workshops/upcoming/algostabfoundations*
- May 19-22, 2025, Constructive Functions 2025, Vanderbilt University, Nashville, Tennessee, USA. *my.vanderbilt.edu/constructivefunctions2025/*

June 2025

- June 1-7, 2025, Spring School on Analysis 2025: Function Spaces and Applications XIII Paseky Nad Jizerou, Czech Republic. *pasekyspringschool.matfyz.cz*
- June 2-6, 2025, Workshop on Weak and Strong Lefschetz Properties Acros Mathematics Sophus Lie Conference Center, Nordfjordeid, Norway. *lefschetz2025.uken.krakow.pl/*
- June 2-6, 2025, AIM Workshop: Mathematical Foundations of Sampling Connected Balanced Graph Partitions, Amerian Institute of Mathematics, Pasadena, California. *aimath.org/workshops/upcoming/connectedbalanced/*
- June 2-6, 2025, Summer School on Modern Tools in Low-Dimensional Topology ICTP Trieste, Italy. *indico.ictp.it/event/10839*
- June 9-11, 2025, International Conference on Mathematical Sciences and Engineering (ICMSE-2025), Mizoram University, Aizawl, Mizoram/India. *drive.google.com/file/d/1ULGc4fYGUFL53Dm2LMvtyLluRdcnEDHo/view?pli=1*
- June 16-20, 2025, Advanced Course in Operator Theory and Complex Analysis Universite Clermont Auvergne, Clermont-Ferrand, France. *indico.math.cnrs.fr/e/acotca25*
- June 17-21, 2025, Optimal Transport, Heat Flow and Synthetic Ricci Bounds, Northwestern University, Evanston, IL. *sites.google.com/view/optimal-transport*
- June 23-26, 2025. 4th IMA Conference on Dense Granular FlowsCambridge, UK. *ima.org.uk/24074/4th-ima-conference-on-dense-granular-flows/*
- June 24-27, 2025, International Conference on Analysis, The 16th Romanian-Finnish Seminar University of Alba Iulia, Romania. *sites.google.com/view/rofinsem2025/accueil*

July 2025

- July 2-5, 2025, 3rd International Conference: Constructive Mathematical Analysis (ICCMA2025), Selcuk University in Konya, Turkiye. *iccma.selcuk.edu.tr/*
- July 3-5, 2025, Applied Linear Algebra 2025 - in Honor of Zhong-Zhi Bai Novi Sad, Serbia. *ala2025.pmf.uns.ac.rs*
- July 7-11, 2025, SIAM Conference on Applied Algebraic Geometry (AG25) University of Wisconsin, Madison, Wisconsin, U.S. *www.siam.org/conferences-events/siam-conferences/ag25/*
- July 8-10, 2025, 13th IMA International Conference on Modelling in Industrial Maintenance and Reliability (MIMAR), Universit'e De Lorraine, France. *ima.org.uk/24805/13th-ima-international-conference-on-modelling-in-industrial- maintenance-and-reliability-mimar2025/*
- July 9-11, 2025, Vienna Congress on Mathematical Finance (VCMF 2025), Vienna University of Economics and Business (WU Vienna), Vienna, Austria. *fam.tuwien.ac.at/vcmf2025/*
- July 13-18, 2025, Annual Meeting of The Society for Mathematical Biology, University of Alberta, Edmonton, Alberta, CN. *2025.smb.org/*
- July 15-18, 2025, SIAM Conference on Financial Mathematics and Engineering (FM25) Hyatt Regency Miami, Miami, Florida, U.S. *www.siam.org/conferences-events/siam-conferences/fm25/isaac2025.org/ www.slmath.org/summer-schools/1105#overview_summer_graduate_school*
- July 30 - August 1, 2025, SIAM Conference on Applied and Computational Discrete Algorithms (ACDA25) Montreal Convention Centre, Montreal, Quebec, Canada. *www.siam.org/conferences/cm/conference/acda25*

August 2025

- August 4-8, 2025, Finite Dimensional Integrable Systems In Geometry And Mathematical Physics, CIMAT, Guanajuato, Mexico. *fdis2025.eventos.cimat.mx*
- August 4-8, 2025, Summer School: Invitation to Complex Geometry Renyi Institute, Budapest, Hungary. *erdoscenter.renyi.hu/events/summer-school-i-invitation-complex-geometry*
- August 4-8, 2025, AIM Workshop: Interactions Between Discrete and Large Topological Groups, American Institute of Mathematics, Pasadena, California. *aimath.org/workshops/upcoming/discreteandlarge/*
- August 11-15, 2025, Summer School on Singular Kählerian Metrics and Hermitian Geometry Renyi Institute, Budapest, Hungary. *erdoscenter.renyi.hu/events/summer-school-ii-summer-school-singular-kahlerian-metrics- and-hermitian-geometry*
- August 17-24, 2025, 10th International Conference on Differential and Functional Differential Equations, RUDN University, Moscow, Russia. *dfde.mi-ras.ru*
- August 18-22, 2025, The Geometric Realization of AATRN (Applied Algebraic Topology Research Network), Institute of Mathematical and Statistical Innovation (IMSI), Chicago, IL, USA. *www.imsi.institute/activities/the-geometric-realization-of-aatrn-applied-algebraic-topology-research-network/*
- August 18-22, 2025, WARTHOG: Cluster Algebras and Braid Varieties University of Oregon in Eugene, OR. *pages.uoregon.edu/belias/WARTHOG/BraidVar/*
- August 21-22, 2025, Connections Workshop: Kinetic Theory and Stochastic Partial Differential Equations, SLMath 17 Gauss Way, Berkeley, CA. *www.slmath.org/workshops/1116*
- August 25-27, 2025, International Conference on Enumerative Combinatorics and Applications, University Of Haifa, Israel (Virtual). *ecajournal.kms-ks.org/Conference/ICECA2025.html*

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Indo-European Conference in Mathematics
jointly organized by the
European Mathematical Society
and
The (Indian) Mathematics Consortium
Pune, India, January 12-16, 2026

We are pleased to announce the first Indo-European Conference in Mathematics to be held in Pune, India, during January 12-16, 2026. This event is organized by the European Mathematical Society (EMS) and The Indian Mathematics Consortium (TMC). It will be held under the joint auspices of Savitribai Phule Pune University and the Indian Institute of Science Education and Research, Pune.

The scientific committee for this event is as follows:

- R. Balasubramanian, Institute of Mathematical Sciences, Chennai, India (Co-chair)
- Sinnou David, Sorbonnne Université, Paris, France
- Oscar Garcia-Prada, Instituto de Ciencias Matemáticas, Madrid, Spain
- Sudhir R. Ghorpade, Indian Institute of Technology Bombay, Mumbai, India
- Rajesh Gopakumar, International Centre for Theoretical Sciences, Bengaluru, India
- Rajeeva L. Karandikar, Chennai Mathematical Institute, Siruseri, India
- Martine Labbé, Université Libre de Bruxelles, Brussels, Belgium
- Meena Mahajan, Institute of Mathematical Sciences, Chennai, India
- Eva Miranda, Universitat Politècnica de Catalunya, Barcelona, Spain
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- M. S. Raghunathan, Centre for Excellence in Basic Sciences, Mumbai, India
- Nadia Sidorova, University College London, UK
- Susanna Terracini, Università di Torino, Turin, Italy (Co-chair)

The organizing committee consists of the following:

- Mousomi Bhakta, Indian Institute of Science Education and Research, Pune, India
- Joaquim Correia, Universidade de Évora, Portugal
- Sudhir R. Ghorpade, Indian Institute of Technology Bombay, Mumbai, India (Co-chair)
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- Susanna Terracini, Università di Torino, Turin, Italy (Co-chair)
- Jugal K. Verma, Indian Institute of Technology Gandhinagar, India

Further information about the conference will soon be available on the webpage of EMS: https://euromathsoc.org/ and the webpage of TMC: https://www.themathconsortium.in/

Donald Knuth (10 Jan. 1938)

An American computer scientist and mathematician. Contributed to the development of the rigorous analysis of the computational complexity of algorithms. Popularized the asymptotic notation. Author of the multi-volume work, “The Art of Computer Programming”. Creator of the TeX computer typesetting system. Designed the MIX/MMIX instruction set architectures. Recipient of the prestigious ACM Turing Award.

Sir Ronald Aylmer Fisher (17 Feb. 1890 - 29 July 1962)

A British statistician and geneticist. Contributed to the theory of evolution known as the modern synthesis. Analyzed immense data from crop experiments since the 1840s, and developed the analysis of variance (ANOVA). His contributions to statistics include the maximum likelihood, fiducial inference, the derivation of various sampling distributions, founding principles of the design of experiments, and much more.

Paul Erdös (26 Mar. 1913 - 20 Sept. 1996)

A renowned Hungarian mathematician. Pursued, cracked and proposed problems in discrete mathematics, graph theory, number theory, mathematical analysis, approximation theory, set theory, probability theory and produced number of conjectures of the 20th century. Significantly contributed to Ramsey theory. Published around 1,500 mathematical papers. His prolific output with co-authors prompted the creation of the Erdös number for mathematicians.

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