The Mathematics Consortium

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Niels Henrik Abel & Quintic Polynomials

A tribute on the bicentennial of the proof of the impossibility of solving the general quintic equation in terms of radicals.

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Contents

About the Cover Page: The front cover image is designed using three images, 'Niels_ Henrik_ Abel', 'notebook of Niels Henrik Abel', and 'group corresponding to quintic polynomial $x^5 + px + q$' taken from the following sources:
https://www.cantorsparadise.com/the-mozart-of-mathematics-niels-henrik-abel-303f850139e0,
https://en.wikipedia.org/wiki/Niels_Henrik_Abel,
https://mathworld.wolfram.com/QuinticEquation.html respectively.

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*With this issue there is a change in the Chief Editorship, with Prof. S. G. Dani taking the charge. On this occasion we heartily thank Prof. Ravindra S. Kulkarni, Founding Chief Editor of the Bulletin, for his assiduous nurturing of the Bulletin until now in various ways. We continue to look forward to his advice and help.

From the Editors' Desk

It is a matter of great delight that as we set out to write this editorial introducing this current issue, the Indian Mathematical Olympiad team has just returned, from Bath, United Kingom, the venue of the 65^{th} International Mathematical Olympiad of this year, with flying colors. The performance this year far surpasses the previous record of our teams, with the 6-member team winning four gold medals, one silver medal, and one 'honourable mention', leading to India ranking fourth among 108 participating countries. This is India's best result since it first participated in 1989, both in terms of the number of gold medals and the rank achieved, the previous best rank being 7^{th}, attained twice, in 1998 and 2001.

The Homi Bhabha Centre for Science Education of the Tata Institute of Fundamental Research is in charge of selecting the team, on behalf of the National Board for Higher Mathematics, which spearheads the activity. The process works, through several steps of selection and training, through a network across the country with numerous centres. Apart from the talent of the individual students the success thus owes a great deal to the untiring efforts and selfless services offered by several mathematicians from leading academic Institutions in India, in grooming them.

We heartily congratulate the Indian team and all those involved, at various stages, in bringing about the laudable success.

It may also be mentioned here that the organizers of the Olympiad activity have also aspired to enhance the participation, and success at IMO, of girl students, which however has met with only sporadic success. On the other hand, part of the process is also being availed of to send teams for the European Girls' Mathematical Olympiad (EGMO); notwithstanding the European tag, it has participation from around the world. The competition began in 2012, and India has been sending teams since 2015 (initially of 2 girls and 4 girls since 2021). It is a pleasure to note that performance of our girls' teams has been steadily on the ascendant and they won 2 silver and 2 bronze medals at the latest EGMO held at Tskaltubo, Georgia in April 2024. We wish them further success.

Our opening article in this issue, by Prof. V. M. Sholapurkar, commemorates the 200th anniversary of a seminal paper by Niels Henrik Abel (1802- 1829), on the impossibility of finding the roots of a quintic polynomial in terms of radicals. The author motivates the issues involved, and explains a simpler version of the proof of Abel's theorem, given by V. I. Arnold in 1963.

In Article 2, Prof. S. G. Dani comments that writing of history of mathematics turns out to be a neglected subject, thereby emphasizing the need for giving adequate importance to the topic in curricular studies at various stages. In the article, he draws the life sketch of Dr. T. A. Sarasvati Amma, a renowned historian of Indian mathematics, highlighting her magisterial work on history of geometry in ancient and medieval India, which is pivotal to our knowledge of history of mathematics.

In Article 3, Dr. D. V. Shah gives an account of significant developments in the mathematical world during recent past, including the current status of Milnor's Conjecture, Pólya Conjecture on geometry of Music, and development of New AI System which can solve Complex Geometry Problems. He also describes the highlights of the work of awardees of various celebrated mathematical prizes, and pays tributes to Quant King James Simons who passed away in May 2024.

In the Problem Corner, Dr. Udayan Prajapati presents a solution to the two problems posed in the January 2024 issue. Two problems from Number Theory and Geometry, are also posed for our readers. Dr. Ramesh Kasilingam gives a calendar of academic events, planned during August, 2024 to December, 2024, in Article 6.

We are happy to bring out this first issue of Volume 6 in July, 2024. We thank all the authors, all the editors, our designers Mrs. Prajkta Holkar and Dr. R. D. Holkar, and all those who have directly or indirectly helped us in bringing out this issue on time.

Editors-in-charge

1. Abel's Theorem on Quintic Polynomials

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1.1 Introduction

The year 2024 marks as the 200^{th} anniversary of a seminal paper by Niels Henrik Abel (1802-1829), announcing the impossibility of finding the roots of a quintic polynomial in terms of *algebraic functions* of its coefficients. One of the objectives of this article is to commemorate this historic event, and explain a simpler version of the proof (without appealing to *Galois Theory*) given by a renowned Russian mathematician Vladimir Igorevich Arnold (1937-2010) in 1963.
The problem of solving a polynomial equation in a certain number system assumes a pivotal role in mathematics. In an ever unfolding drama called *algebra*, a polynomial equation plays a role of a protagonist! The quest for solving a polynomial equation has kept mathematicians intrigued and busy since antiquity. The fascination associated with the problem can be brought to the surface by taking a variety of tours. Here, we propose to choose a path that goes via the problem of solving a quintic polynomial. The three most important attributes that govern a polynomial equation are its *degree*, *coefficients* and *roots*. As we shall see, the story of solving a polynomial equation, of any degree, hinges around the tension in the relationship between coefficients and roots! In this article, we take up the tasks of narrating this story, and explaining the dramatic turn it takes as we move from lower degrees to the degree five.
We begin with a quadratic equation and take an opportunity to popularize a recent (2019) proof of the famous quadratic formula described by Po Shen Loh [10]. Though the main focus is going to be on a quintic (degree 5) polynomial, in the process we briefly describe the nature of solutions of a cubic (degree 3) and a quartic (degree 4) equation as well. This will allow us to understand a certain *break of pattern* while dealing with a quintic. In the context of understanding the relation between coefficients and roots, the passage from quadratic to quintic spans about 3000 years! A historical account of these episodes would therefore be very fascinating. However, we do not propose to dive deep into the history, but rather prefer to point to the relevant references.
In what follows, we assume that the coefficients of all the polynomials under consideration are complex numbers. We assume that the reader is familiar with *Fundamental Theorem of Algebra* which states that every non-constant polynomial has a root in the set of complex numbers. We use the following terminology in the sequel:

1. A function $f(x_1, x_2, \cdots, x_n)$ is said to be a *rational function* of its variables if f can be expressed by using the field operations (addition, subtraction, multiplication and division) on $x_1, x_2, \cdots, x_n$.

2. A function $f(x_1, x_2, \cdots, x_n)$ is said to be an *algebraic function*[2] of its variables if f can be expressed by combining the field operations on $x_1, x_2, \cdots, x_n$ and extraction of roots (square roots, cube roots, etc.).

3. A function $f(x_1, x_2, \dots, x_n)$ is called a *symmetric function* of $x_1, x_2, \dots, x_n$ if for any permutation σ of $\{1, 2, \dots, n\}$, we have $f(x_{\sigma(1),} x_{\sigma(2),} \dots, x_{\sigma(n)}) = f(x_1, x_2, \dots, x_n)$.

Recall that the coefficients of a monic polynomial (equation) can be expressed as rational, symmetric polynomials in its roots. These formulae are generally attributed to the French mathematician François Viéte (1540-1603). We quote Viéte's theorem for a ready reference.

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[1]The article is an expanded version of 'Mangala Narlikar Memorial Lecture' delivered by the author at Bhaskaracharya Pratishthana, Pune on 17th January, 2024.

[2]The algebraic functions in our sense form a subset of the set of algebraic functions in the common usage: y is an algebraic function of $x_1, x_2, \dots, x_n$ if there is a polynomial g in $n+1$ variables such that $g(y, x_1, x_2, \dots, x_n) = 0$.

Theorem 1 (Viéte). *The complex numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ are roots of the equation $x^n + a_{n-1}x^{n-1} + \cdots a_1 x + a_0 = 0$ if and only if the following relations hold:*

$$a_{n-j} = (-1)^j \sum \alpha_1\alpha_2\cdots\alpha_j, \; j = 1, 2, \ldots, n; \tag{1.1}$$

where the sum $\sum \alpha_1\alpha_2\cdots\alpha_j$ denotes the sum of all possible products of the roots, taken j at a time. In particular, $\alpha_1 + \alpha_2 + \cdots + \alpha_n = -a_{n-1}$ and $\alpha_1\alpha_2\cdots\alpha_n = (-1)^n a_0$.

The present article, in contrast with the relations (1.1), deals with exploring the possibility of finding formulae for the roots in terms of coefficients, and how the degree of a polynomial dictates the possibility of having such formulae as well as the nature of the formulae whenever such a possibility occurs.

1.2 Quadratic Equation

A quadratic equation is a polynomial equation of the form $ax^2 + bx + c = 0, a \neq 0$. The formulation of a quadratic for solving practical problems, and a formula for its solution is known since the ancient times. For general historical comments on a quadratic, the reader is referred to [8] and for the contributions of Indian mathematicians a good reference is [3]. The process of deriving a formula for the roots of a quadratic equation generally rests on the *method of completing the square*. Perhaps, this method is the first technically complicated mathematics that a student encounters in school days. At that stage, the idea could sound somewhat cumbersome and without much of motivation. As a result, students more frequently end up in mugging up the following famous formula for the roots of a quadratic equation $ax^2 + bx + c = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{2.1}$$

We may always assume that the leading coefficient a is 1, and thus write the formula for the roots of $x^2 + bx + c = 0$ as

$$x = \frac{-b \pm \sqrt{b^2 - 4c}}{2}. \tag{2.2}$$

In a recent article [10], Po Shen Loh, an American mathematician and a well known Math Olympiad trainer has devised a method of obtaining the formula by a simple idea. He essentially uses Viéte's theorem for the coefficients of a polynomial in terms of its roots. Recall that p and q are the roots of $x^2 + bx + c = 0$, if and only if

$$-b = p + q \quad \text{and} \quad c = pq. \tag{2.3}$$

Observe that the relations (2.3) can be directly obtained from the formulae ((2.2)). Now can we derive formulae ((2.2)) from the relations (2.3)? Observe that the average of p and q is $\frac{-b}{2}$. So we may assume that p and q are of the form

$$p = \frac{-b}{2} + k \quad \text{and} \quad q = \frac{-b}{2} - k$$

where k is to be determined from the other relation $pq = c$. We thus have

$$(\frac{-b}{2} + k)(\frac{-b}{2} - k) = c.$$

The values of k are given by $k = \pm\dfrac{\sqrt{b^2 - 4c}}{2}$ and hence the required values p and q match with formulae (2.2).

Though the Po Shen Loh method is not drastically different from the method of completing the square, the hurdle of completing the square is bypassed! Having said that, one must mention that the method of completing the square is a classic technique that allows one to learn algebraic manipulations, and has got historical significance as a proof technique.

Remark 1. 1. *The equation (2.2) shows that the roots of a quadratic equation can be expressed as an algebraic function of its coefficients.*

2. *Note that the relations (2.3) tell us that the coefficients are rational, symmetric functions of the roots, whereas the equation (2.2) shows that, in general the roots are not rational functions of the coefficients. A radical sign does appear in the general formula.*

3. *Observe that one radical is enough to express the roots in terms of coefficients.*

For keeping in tune with the discussion on higher degree polynomials to follow, let us record our first *Impossibility Theorem.*

Theorem 2. *(***Impossibility 1)** *The roots of a quadratic equation cannot be expressed as rational functions of its coefficients.*

Though the formula (2.2) does vouch for the claim in the above theorem, a rigourous proof demands a discussion on the (dis)continuity of the function $\sqrt{z}$. We shall communicate the ideas in the proof in Section 5.

1.3 Cubic Equation

In this section we propose to sketch a method of finding the roots of a cubic equation. We can always assume the leading coefficient to be 1, and thus deal with an equation of the form

$$x^3 + bx^2 + cx + d = 0. \tag{3.1}$$

The formulae for the roots of a cubic were obtained by a sixteenth century Italian mathematician Girolamo Cardano. Note that after the success of solving a quadratic, it took about 2500 years to get formulae for the roots of a general cubic equation. For interesting history associated with the solution of a cubic, the reader is referred to [8], [9]. For Indian contributions, refer to [3].
If α, β, γ are roots of (3.1), then we again have Viéte's formulae:

$$b = -(\alpha + \beta + \gamma), \quad c = \alpha\beta + \alpha\gamma + \beta\gamma, \quad d = -(\alpha\beta\gamma). \tag{3.2}$$

Observe that the coefficients are rational, symmetric functions of the roots. The steps involved in obtaining the roots in terms of coefficients are given below:

1. Substitute $x = y - \frac{b}{3}$ and observe that the second degree term in the given cubic gets eliminated. Thus the given cubic equation (3.1) reduces to $y^3 + py + q = 0$ where

$$p = -\frac{b^2}{3} + c \quad \text{and} \quad q = \frac{2b^3}{27} - \frac{bc}{3} + d. \tag{3.3}$$

   A cubic of the form $y^3 + py + q$ is called a *reduced cubic.*

2. The substitution for getting a reduced cubic is easy once we set up the task of eliminating the quadratic term. However, the next substitution is very non-trivial! Substitute

$$y = z - \frac{p}{3z}, \text{ or } z^2 - yz - \frac{p}{3} = 0. \tag{3.4}$$

   This equation has two roots r and s such that $r + s = y$ and $rs = -\frac{p}{3}$. The equation $y^3 + py + q = 0$ now becomes

$$z^6 + qz^3 - \frac{p^3}{27} = 0. \tag{3.5}$$

Thus, r and s are roots of (3.5).

3. Observe that the equation (3.5) is a quadratic in z^3 and therefore can be solved for z^3 by using the formula (2.2). We thus have:

$$z^3 = -\frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}.$$

If $z_1 = -\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$ and $z_2 = -\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$, then it is clear that $z_1 z_2 = -\frac{p^3}{27}$.

4. If r is a cube root of z_1, then the other cube roots are $r\omega$ and $r\omega^2$, where ω stands for a complex cube root of unity. Similarly, if s is a cube root of z_2, then the other cube roots are $s\omega$ and $s\omega^2$. It can be checked that r and s are both cube roots of z_1 or both of z_2 if and only if $\frac{q^2}{4} + \frac{p^3}{27} = 0$, i.e. (3.5) as an equation in z^3 has equal roots. Now we need to pair off the cube roots of z_1 and cube roots of z_2 in such a way that the their product is $-\frac{p}{3}$.

5. In view of the relation (3.4), the sums in each pair give a solution to $y^3 + py + q = 0$. Suppose s is the cube root of z_2 such that $rs = -\frac{p}{3}$. Then the roots $y^3 + py + q = 0$ are given by

$$\begin{aligned} y_1 &= r + s \\ y_2 &= r\omega + s\omega^2 \\ y_3 &= r\omega^2 + s\omega \end{aligned} \tag{3.6}$$

where $r = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$ and $s = \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$, the cube roots are chosen as discussed. Substituting the values of p and q from (3.3), and noting that $x_i = -\frac{b}{3} + y_i, i = 1, 2, 3.$ we get the roots of the given cubic (3.1) in terms of its coefficients.

Remark 2. 1. *The formulae (3.6) show that the roots of a cubic equation can be expressed as an algebraic function of its coefficients.*

2. *Like in case of a quadratic, note that in general, the roots are not rational functions of the coefficients. A radical sign does appear in the general formula.*

3. *A closer look at formulae reveals that there is a nested radical in the expression, unlike the formula (2.2) for the roots of a quadratic. Interestingly, as we move from a quadratic to a cubic, there seems to be an additional radical (nested) appearing in the formulae for the roots! At the same time it is worth observing that one nesting (two radical signs) is enough to express the roots of a cubic polynomial in terms of its coefficients.*

Following a similar track as in case of a quadratic, let us record our second impossibility theorem.

Theorem 3. **(Impossibility 2)** *The roots of a cubic equation cannot be expressed as an algebraic function of its coefficients which is a combination of field operations and a single radical.*

Again, we have to wait for the proof until we discuss (dis)continuity of $z^{\frac{1}{3}}$ in Section 5.
It seems that a pattern has emerged and thus we are in a position to guess the scenario that would come up in the case of a quartic (bi-quadratic) equation. The reader is encouraged to follow the observations in remarks 1 and 2, and make the respective modifications for the quartic case.

1.4 Quartic Equation

The formulae for the roots of a quartic equation in terms of its coefficients were first given by Lodovico Ferrari (1522-1565) who was Cardano's assistant. In fact, Cardano described Ferrari's method in his book *Ars Magna* published in 1545. For interesting historic episodes associated with the solution of a quartic, the reader is referred to [8]. The method essentially relies on attempting to factorize the given quartic into two quadratic polynomials. However, one needs to solve a cubic equation in order to get these factors. The main steps that lead us to the formulae are given below:

1. Consider a quartic equation
$$x^4 + bx^3 + cx^2 + dx + e = 0. \tag{4.1}$$
Substitute $x = z - \dfrac{b}{4}$ and observe that the third degree term in the given quartic gets eliminated. Thus the given equation reduces to
$$z^4 + pz^2 + qz + r = 0 \tag{4.2}$$
where p, q and r are the rational functions of the coefficients in the equation (4.1).

2. Introduce an auxillary variable t, and write z^4 as $z^4 = (z^2+t)^2 - 2z^2t - t^2$. Substituting z^4 in equation (4.2), we get
$$(z^2+t)^2 + (p-2t)z^2 + qz + r - t^2 = 0. \tag{4.3}$$

3. Now choose t in such a way that $(p-2t)z^2 + qz + r - t^2$ becomes a perfect square. Equivalently, t is chosen in such a way that the discriminant of $(p-2t)z^2 + qz + r - t^2$ vanishes. This condition leads us to the following cubic equation:
$$8t^3 - 4pt^2 - 8tr + (4pr - q^2) = 0. \tag{4.4}$$

4. A value of t obtained by applying formulae (3.6) results into a factorization of the left side of (4.3) as a product of two quadratics.

5. Finally, apply the formulae for the roots of a quadratic to get the roots of a given quartic.

Remark 3. *1. We have not explicitly written down the formulae for four roots of a quartic. Here, we do not want to enter into the discussion of writing four roots arising out of three roots of the cubic equation (4.4). However, it is clear from the above steps that the roots of a quartic equation can be expressed as an algebraic function of its coefficients. For a detailed explanation of precisely writing the four roots, a reader may consult [8].*

2. As we are required to use the roots of a cubic and also need to solve a quadratic, it is obvious that, in general, a root of a quartic cannot be a rational function of its coefficients.

3. Now examine the steps 4 and 5 carefully. A value of t obtained by solving the cubic (4.4) will have, in general, a nested radical and after substituting this value, we solve a quadratic involving t, which yields one more radical. Therefore, in general, a doubly-nested radical is required for writing the roots of a quartic in terms of its coefficients.

4. It is worth noting that a double-nesting of radials (three radical signs) are enough to express a root of a quartic equation in terms of its coefficients.

Keeping up with the discussions in quadratic and cubic cases, we record our next impossibility theorem.

Theorem 4. **(Impossibility 3)** *The roots of a quartic equation cannot be expressed as an algebraic function of its coefficients which is a combination of rational function, a single radical or a singly-nested radical.*

Let us take a pause, and reflect on the nature of roots as we move from a quadratic to a cubic, and then from a cubic to a quartic. We find that there is something common and something different as well! The commonality is obviously the fact that in each case, roots are algebraic functions of coefficients, and the difference is that we need an extra radical as the degree increases - one radical for a quadratic, two for a cubic and three for a quartic. As for a quintic (degree five) polynomial, the most natural guess would be that the roots of a quintic would be expressible in terms of its coefficients, but an extra radical would appear in the expression. Let's explore!
I've discovered as I've grown up that life is far more complicated than you think it is (when you're a kid). It isn't just a straightforward fairytale.- Rachel McAdams

1.5 Quintic Equation

After the success of solving a quartic, it took more than 250 years to uncover a mystery about the existence of algebraic functions that would possibly represent the roots of a quintic. A first decisive and path-breaking step was taken by the legendary French mathematician Joesph-Louis Lagrange (1736-1813). In his paper *Reflection on the Algebraic Theory of Equations*, published in 1770, Lagrange analyzed the schemes for solving a cubic and a quartic and thought that such methods won't work in case of a quintic! In the process, he used the idea of permuting the roots of a polynomial, and looking for its action on a certain expression in terms of roots called a *resolvent* . For example, in case of a cubic, if the roots are x_1, x_2, x_3, then the expression $R = (x_1 + \omega x_2 + \omega^2 x_3)^3$ has a property that six permutations of three roots lead to only two values of R! Lagrange used this idea to reduce the given cubic to a quadratic and recover Candano's formulae for the roots of a cubic. Similarly, for a quartic, Lagrange used a resolvent $R = (x_1 + x_2 - x_3 - x_4)^2$ for reducing a quartic to a cubic and recovered Ferrari's formulae. For the computations on the resolvent for reducing a cubic or a quartic to a quadratic or a cubic respectively, the reader is referred to [9]. However, Langrange could not find a resolvent for a quintic that would allow him to reduce it to a lower degree polynomial. We close this section with a quote by Lagrange:
The computations in working out a resolvent of a quintic are so long and complicated that they can discourage most intrepid calculators. An uncertainty that will discourage in advance all those who might be tempted to use older methods to solve one of the most celebrated and important problem of algebra.
Lagrange was one of the prominent figures who smelled the *breaking of pattern* that emerged through roots of quadratic, cubic and quartic equations. However, he did not make any claims on impossibility of having roots as algebraic functions of coefficients.
At this stage, let us digress and discuss some group theory. A motivation for doing so lies in the fact that a key feature in understanding the subtleties of the quintic case is the notion of *permutations of roots.*

1.5.1 Commutator Subgroup

Throughout this section we assume that the reader is familiar with elementary group theory including the basic facts about the permutation group S_n. A good reference for this material is [6]. We begin with the definition that turns out to be very important in the present context.

Definition 1. *Let G be a group. For $g, h \in G$, the commutator of g and h denoted by $[g, h]$ is defined as*

$$[g, h] = ghg^{-1}h^{-1}.$$

The set of all commutators generates a subgroup of G denoted by $K(G)$. It is obvious that if G is abelian, then $K(G)$ is the trivial group. Thus $K(S_2)$ is the trivial group. It is easy to check that

$K(S_3) = \{e, (1\,2\,3), (3\,2\,1)\}$ and therefore $K(K(S_3)) = \{e\}$, the trivial group. We denote $K(K(G))$ by $K^{(2)}(G)$. As for S_4, by using a bit of group theory, it can be shown that $K(S_4) = A_4$, the alternating group and $K(A_4) = V$, the Klein-4 group, and $K(V) = \{e\}$, the trivial group. Thus $K^{(3)}(S_4) = \{e\}$.
At this stage, it is instructive to study the following table:

Quadratic	1 radical sign	$K(S_2) = \{e\}$
Cubic	2 radical signs	$K^{(2)}(S_3) = \{e\}$
Quartic	3 radical sings	$K^{(3)}(S_4) = \{e\}$

It appears that the number of nested radicals required in expressing a root in terms of coefficients of a polynomial of degree n for $n = 2, 3$ and 4 matches with the number of 'nesting of commutator subgroups of S_n' so as to reduce it to the trivial group. In what follows, an association between nesting of radicals and nesting of commutator subgroups will be made more transparent. A million dollar question that naturally emerges out of this discussions is:

Question 1. *Does there exist n such that $K^{(n)}(S_5) = \{e\}$?*

We shall come to this question later, but a reader conversant with rudimentary aspects of group theory, especially with some 'simple' properties of the alternating group A_5 is urged to tackle the question before hand!
Let us have yet another digression before we put the scattered pieces together, and make a firm statement about the possibility of having an algebraic expression for the roots of a quintic in terms of its coefficients.

1.5.2 The function $\mathbf{z^{\frac{1}{n}}}$

In this section, we discuss the continuity of the (multi-valued) function $f(z) = z^{\frac{1}{n}}$.
Let z be a non-zero complex number. If $z = r(\cos\theta + i\sin\theta)$ is represented in the polar form, then the n roots of the equation $w^n = z$ are given by

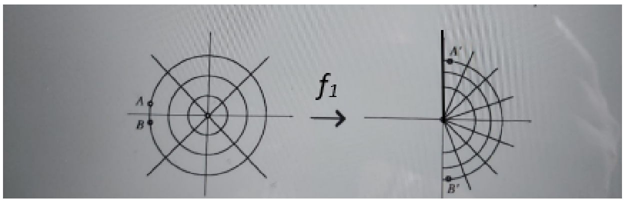
$$w = r^{\frac{1}{n}}\left(\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right),\ k = 0, 1, 2, \dots, n-1. \tag{5.1}$$

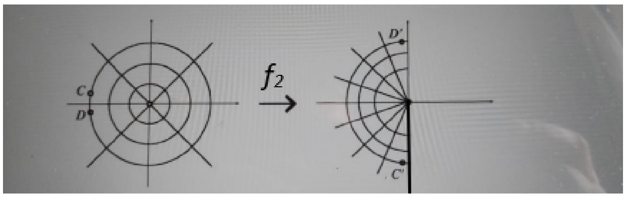
Definition 2. *The expression $\sqrt[n]{z}$ denotes a multi-valued function which assigns to every $z \neq 0$, all n roots of the equation $w^n = z$ as given by the formulae in (5.1).*

As a particular case, it is worth understanding the function $f(z) = \sqrt{z}$. Let z be a non-zero complex number and let θ be an argument of z such that $-\pi < \theta \leq \pi$. If $z = r(\cos\theta + i\sin\theta)$, then $f(z) = \sqrt{z}$ has two values

$$w_1 = r^{\frac{1}{2}}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right) \quad \text{and} \quad w_2 = r^{\frac{1}{2}}\left(\cos\left(\frac{\theta+\pi}{2}\right) + i\sin\left(\frac{\theta+\pi}{2}\right)\right).$$

Note that $w_2 = -w_1$. Now consider the function $f_1(z) = w_1$. Then f_1 is a single-valued function defined on the entire complex plane ($f_1(0)$ is defined as 0). Observe that f_1 maps the complex plane onto the right half plane Re$(z)>0$ union non-negative Imaginary axis. Now consider two points z_1, z_2 (arbitrarily) close to each other and close to *negative real axis* (from 0 to $-\infty$), but z_1 in the upper half plane Im$(z)>0$, and z_2 in the lower half plane Im$(z)<0$. Note that $f_1(z_1)$ and $f_1(z_2)$ are distant and cannot be controlled to be at a desired distance. In other words the continuity of f_1 is lost as we *cross* the negative real axis. However, it is clear that f_1 is continuous at all points in the plane except for the points on the negative real axis. Similarly, $f_2(z) = w_2$ is a single-valued function continuous at all points except the points on the negative real axis.
A consequence of discontinuity of (any one of the single-valued branches) $\sqrt{z}$, very crucial in the present context, is stated below:





Observation: *As z moves along a closed path around the origin, $\sqrt{z}$ does not follow a closed path; instead, it is shifted to the other value of the square-root.*
Note that, as z completes one revolution around the origin along a closed path, the argument of z changes from θ to $\theta + 2\pi$ and thus the argument of $\sqrt{z}$ changes by π.
In general, for a nonzero complex number z, no single-valued branch of $\sqrt[n]{z}$ is continuous throughout the complex plane, and as z traverses a loop once (i.e. the argument of z changes by 2π), then the argument of $\sqrt[n]{z}$ varies by $\dfrac{2\pi}{n}$. In other words, $\sqrt[n]{z}$ moves to a nearest neighbour on the circle of radius $r^{\frac{1}{n}}$ where all the values of $\sqrt[n]{z}$ are located at the vertices of a regular $n-$gon inscribed in it.

1.5.3 Impossibilty Theorems Revisited

Now we are in a position to sketch the ideas involved in convincing the validity of the *impossibility theorems* that we stated earlier. However, a careful analysis is required in writing the actual proofs. We begin with Theorem 2, the first impossibility theorem that occured in the context of a quadratic equation.
Theorem 2.2(Impossibility 1)
Rather than a detailed proof, an idea is given below.

- Let α_1 and α_2 be distinct roots of a quadratic equation $x^2 + bx + c = 0$.
- Suppose there exist rational functions F_1 and F_2 such that
$$\alpha_1 = F_1(b, c), \quad \alpha_2 = F_2(b, c).$$
- By Viéte's theorem (Theorem 1.1), we know that the coefficients b, c are rational, symmetric functions of α_1 and α_2.
- If we permute the roots (action of the transposition $(1\,2)$), the coefficients b, c, and in turn their rational functions F_1, F_2 remain unchanged. Indeed, suppose γ_1 is a continuous path from α_1 to α_2 and γ_2 is a continuous path from α_2 to α_1. As α_1 traverses from α_1 to the position of α_2 along γ_1 and α_2 traverses from α_2 to the position of α_1 along γ_2, we assume that the denominators of $F_1(b, c)$ and $F_2(b, c)$ remain non-zero for suitable choices of γ_1 and γ_2. Once this is granted, at the end α_1 and α_2 get interchanged but b and c remain the same and so do $F_1(b, c)$ and $F_2(b, c)$. This contradicts our assumption that $\alpha_1 \neq \alpha_2$. For a detailed analysis of this process leading to a rigourous proof, the reader is referred to [4], [7], [11].

Remark 4. 1. *The statement of Theorem 2 remains valid for any polynomial of degree $n \geq 2$. The same proof as in case of a quadratic would work by choosing any two distinct roots.*

2. *As α_1 and α_2 traverse to α_2 and α_1 respectively as above (this corresponds to the transposition $(1\,2)$), both b, c follow a loop (closed path) and thus $F_1(b, c)$ and $F_2(b, c)$ also follow a loop.*

3. *The argument used in proving Theorem 2 applies to other classes of functions such as the class of functions continuous throughout the complex plane! This leads to the version of Theorem 2 as given below:*

   Theorem 5. *The roots of a quadratic equation cannot be expressed as continuous functions of its coefficients.*

4. *Observe that the permutation* $(1\,2)$*, the only non-trivial permutation in* S_2 *played its role in the proof.*

We now take up the task of proving Theorem 3, the second impossibility theorem that occurred in the context of a cubic equation.
If we closely observe the proof of Theorem 1.5.3 outlined above, we notice that the idea hinges around the permutation of roots. In case of a cubic, we shall have six permutations of the roots. Recall that a major difference (in the present context) in the structure of S_2 and S_3 that we have noted earlier is $K(S_2) = \{e\}$ whereas $K(S_3) = \{e, (1\,2\,3), (1\,3\,2)\}$. A Reader is encouraged to think about the role of a non-trivial commutator in the proof of Theorem 3 before going through it!
Theorem 3.2 (Impossibility 2) The roots of a cubic equation cannot be expressed as an algebraic function of its coefficients, which is a combination of field operations and a single radical.
Again, we outline the main steps.

- Let $\alpha_1, \alpha_2, \alpha_3$ be distinct roots of a cubic equation $x^3 + bx^2 + cx + d = 0$.
- Suppose there exist functions G_1, G_2 and G_3 such that
$$\alpha_1 = G_1(b, c, d), \quad \alpha_2 = G_2(b, c, d), \quad \alpha_3 = G_3(b, c, d) \tag{5.2}$$
where G_1, G_2, G_3 are algebraic functions which are combinations of field operations and a single radical.
- By Viéte's theorem (Theorem 1.1), we know that the coefficients b, c, d are rational, symmetric functions of $\alpha_1, \alpha_2, \alpha_3$.
- We know that the permutation $(1\,2\,3) \in K(S_3)$. Specifically, we have
$$(1\,2\,3) = [(1\,2), (2\,3)] = (1\,2)(2\,3)(1\,2)^{-1}(2\,3)^{-1} = (1\,2)(2\,3)(1\,2)(2\,3)$$
In general $(i\,j\,k) = [(ij), (j\,k)]$. (Operations on symbols are performed from left to right.)
- The action of $(1\,2\,3)$ on roots will take α_1 to α_2, α_2 to α_3 and α_3 to α_1.
- The permutation of roots will not alter the coefficients b, c, d as well as their rational combination. However, we need to check its effect on the expressions containing a radical.
- Suppose we have a function involving a radical of the type $\sqrt[k]{f(z)}$ where f is a rational function of b, c, d. Under the action of the permutation $(1\,2\,3)$ the function f will follow a loop around origin. As f traverses a loop around origin, $\sqrt[k]{f(z)}$ will go to one of the other roots out of k roots of $\sqrt[k]{f(z)}$.
- The most crucial idea in the proof is to act the permutation $(1\,2\,3)$ not directly, but as a commutator $(1\,2\,3) = (1\,2)(2\,3)(1\,2)^{-1}(2\,3)^{-1}$. If applied this way, $\sqrt[k]{f(z)}$ the action of $(1\,2)$ on a root gets nullified by the action of $(1\,2)^{-1}$ (or simply by one more action of $(1\,2)$). Similarly, the actions of $(2\,3)$ and $(2\,3)^{-1}$ *cancel each other.*
- The net effect of the action of a commutator on expression of the type $\sqrt[k]{f(z)}$ results in traversing a loop! Thus if we assume the validity of equations 5.2, then the scenario that G_1, G_2, G_3 stand unchanged whereas $\alpha_1, \alpha_2, \alpha_3$ get permuted is contradictory.

Remark 5. *1. Theorem 3 remains valid for any polynomial of degree $n \geq 3$.*

2. *A fact that the permutation $(1\,2\,3)$ is a commutator is crucially used in the proof. Such a permutation does not exist in S_2. At the same time, it is worth noting that $(1\,2\,3) = [(1\,2), (2\,3)]$ but the transpositions $(1\,2)$ and $(2\,3)$ cannot be written as commutators. We can therefore rule out only the expressions involving single radicals. Of course, we already have a formula involving a nested radical. See equations 3.6.*

3. *At this stage, the reader is urged to pause, and relate the fact that $K(S_3)$ is nontrivial but $K^{(2)}(S_3)$ is trivial, with ruling out the possibility of having formulae with only single radicals, and keeping a scope for having formulae containing nested radicals.*

The next task is naturally to prove Theorem 4, the third impossibility theorem occurred in the context of a quartic. In view of the third point in Remark 5.8, we have a good hint for proving Theorem 4. Unlike in case of S_3, we know that $K^{(2)}(S_4)$ is not trivial. Thus we can expect a *nested commutator* to play its role in the proof.

Theorem 4.2 (Impossibility 3) As the pattern of the proof is now evident, we propose to shorten the proof and leave a few gaps, to be filled in by the reader.

- Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ be distinct roots of a quartic equation $x^3 + bx^3 + cx^2 + dx + e = 0$.
- Suppose there exist functions H_1, H_2, H_3 and H_4 such that

$$\alpha_1 = H_1(b, c, d, e), \quad \alpha_2 = H_2(b, c, d, e), \quad \alpha_3 = H_3(b, c, d, e), \alpha_4 = H_4(b, c, d, e) \tag{5.3}$$

  where H_1, H_2, H_3, H_4 are algebraic function which are combinations of field operations, a single radical or a nested radical (at most singly-nested).
- By Viéte's theorem (Theorem 1.1), we know that the coefficients b, c, d, e are rational, symmetric functions of $\alpha_1, \alpha_2, \alpha_3, \alpha_4$.
- We know that the permutations $(1\,2\,3)$ and $(2\,3\,4)$ belong to $K(S_4)$ because $(1\,2\,3) = [(1\,2), (2\,3)]$ and $(2\,3\,4) = [(2\,3), (3\,4)]$.
- Now the fact that $K^{(2)}(S_4)$ is nontrivial is reflected in the equation

$$[(1\,2\,3), (2\,3\,4)] = (1\,4)(2\,3) \tag{5.4}$$

  In general $[(i\,j\,k), (j\,k\,l)] = (i\,l)(j\,k)$.
- The action of $(1\,2\,3)$ on roots will take α_1 to α_2, α_2 to α_3 and α_3 to α_1.
- On the basis of the argument in the proof of Theorem 3.2, the action of the permutations $(1\,2\,3)$ and $(2\,3\,4)$ will not alter the coefficients b, c, d, e as well as their algebraic functions which are combinations of field operations and expressions involving single radicals. However, we need to check its effect on the expressions containing a nested radical.
- Suppose we have a function involving a nested radical of the type $\sqrt[k]{f(z)}$ where f is an algebraic function of b, c, d, e involving a single radical. Under the action of the permutations $(1\,2\,3)$ and $(2\,3\,4)$ the function f will follow a loop around origin. As f traverses a loop around origin, $\sqrt[k]{f(z)}$ will go to one of the other roots out of k roots of $\sqrt[k]{f(z)}$.
- Now imitating the argument in the proof of Theorem 3, we get that the action of the permutation $(1\,4)(2\,3)$ interchanges α_1 with α_4, and α_2 with α_3. However, each of the functions H_1, H_2, H_3, H_4 follow a loop because of the commutator relation 5.4. This is clearly a contradiction.

Finally, we turn our attention to the case of a quintic polynomial, the most awaited part of the story.

1.5.4 The Grand Finale

We have thoroughly discussed the nature of the roots as well as impossibility results in the cases of quadratic, cubic and quartic polynomials. In this section we shall deal with a quintic polynomial and exhibit the *breaking of pattern* that we have been hinting through out. Clearly, there is a pattern that comes up through the discussion on quadratic, cubic and quartic. The most important observation that has surfaced is:
The nesting of n commutators rules out a formula containing nesting of n radicals.
Now consider S_5 and look at the following commutator relation:

$$[(i\,j\,k),(k\,l\,m)] = (j\,k\,m) \tag{5.5}$$

This curious relation actually destroys all the hopes of having a formula for the roots of a quintic as an algebraic functions of its coefficients. Note that a $3-$cycle is a commutator of another two $3-$cycles! Therefore one can apply the formula again to the cycles $(i\,j\,k)$ and $(k\,l\,m)$ and express $(j\,k\,m)$ as a nested commutator. Now keep repeating the same strategy indefinitely. We now have an answer to the question 1: there is no natural number n such that $K^{(n)}(S_5) = \{e\}$. In fact, it can be shown that $K(S_5) = A_5$ and $K(A_5) = A_5$. In other words, every element of A_5 is a product of commutators (in fact a commutator in the case of A_5). This certainly rules out a formula as desired, for the roots of a quintic. This is precisely the Abel's celebrated theorem.
It is clear that a formula is not possible for the roots of a polynomial of degree n for any $n \geq 5$. We find it appropriate to quote Abel's theorem separately.

Theorem 6. (Abel) *The roots of a polynomial of degree five (or more) cannot be expressed as algebraic functions of its coefficients.*

The charm of Abel's theorem lies in its surprise element which breaks a well defined trajectory. The theorem is surely one of the cornerstones in the history of algebra.

1.6 Epilogue

We conclude the article by noting some of the important facets of the present theme.

- Abel published his first version of the work on quintic polynomials in 1824 and a more elaborate version in 1826 [1], [2]. Abel's original proof uses the method of contradiction. The proof essentially rests on the following steps:

  1. He assumes the solvability of a quintic and proves that the solution has to be of the form
  $$y = p_1 + R^{\frac{1}{5}} + p_2R^{\frac{2}{5}} + p_3R^{\frac{3}{5}} + p_4R^{\frac{4}{5}}$$
  where $p_1, \ldots, p_4$ are finite sums of polynomials and radical and $R^{\frac{1}{5}}$ is an irrational function of the coefficients of the quintic.
  2. Abel proves that the solution, as expressed above is a rational function of its roots. (One can easily verify the statement for a quadratic. If the solutions of $x^2 + bx + c = 0$ are expressed in the form $p + R^{\frac{1}{2}}$, then we have $p = -\dfrac{b}{2}$ and $R = -c + \dfrac{b^2}{4}$. If α_1, α_2 are roots of the quadratic, then we have $R^{\frac{1}{2}} = \alpha_1 - \alpha_2$, which is a rational function of the roots. Though one can verify the claim for a cubic and a quartic, the computations become more complicated. The notion of *discriminant* of a polynomial turns out to be useful in these computations). Abel proved that $R^{\frac{1}{5}}$ is a rational function of the roots of the given quintic.

3. Abel then used the following result by Cauchy and arrived at a contradiction:
If a rational function of five variables takes fewer than five values when the variables are permuted, it can take only two different values (equal in magnitude and opposite in sign), or one value, but never three or four values. Abel proved that both the possibilities can be ruled out: five values as well as two values and thus concluded that a quintic cannot be solved by radicals. For an elaborate discussion on Abel's original proof and for detailed historical accounts related to the solvability of a quintic, the excellent reference is [9]. In his scholarly article [12], Michael Rosen presented a proof of Abel's theorem in the spirit of Abel's original proof, but in a more sophisticated and modern language.

- V. Arnold delivered a series of lectures on Abel's theorem and presented a new proof in 1963. However, he never published the solution. Later, V. Alekseev who attended these lectures published the solution in a friendly problem-solution format [4]. Arnold's argument is topological in nature and uses the idea of a *mondromy group* associated to a multi-valued function. In the process one needs to go through the construction of Riemann surfaces of a multi-valued function.

- Evariste Galois (1811-1832), gave a complete solution of the problem, developing a criterion for the solvability of a polynomial in terms of radicals. Galois Theory is arguably one of the most beautiful and elegant pieces of work. Galois submitted his work in 1830, but it got published only in 1846. Galois associated a group to a polynomial and the *solvability* of the group determines the existence of the solution by radicals. Solvability of a group G essentially means the existence of n such that $K^{(n)}(G) = \{e\}$. We have seen that S_2, S_3 and S_4 are solvable whereas $S_n, n \geq 5$ is not solvable.

- The first person who thought of the impossibility of expressing the roots of a quintic in terms of algebraic functions of its coefficients was the Italian scholar (a surgeon and a mathematician) Paolo Ruffini. His work got rather neglected by the contemporary mathematicians. There was considerable overlap in Abel's and Ruffini's work. Therefore many mathematician address the quintic theorem as *Abel-Ruffini Theorem*. Abel himself acknowledged Ruffini's work:
The first, and if I am not mistaken the only one, who before me had tried to show the impossibility of the algebraic solution of general equations is the geometer Ruffini. But his paper is so complicated that it is very difficult to decide the correctness of his reasoning. It seems to me that his reasoning is not always satisfactory.

- The present article closely follows the treatment in [7] and [11] which are the simplified versions of Arnold's proof.

- For the arguments in the proofs of impossibility theorems, we have used the actions of permutations on roots, and its effect on the radicals. An excellent visual tool developed by Leo Stein is extremely useful for the understanding of these geometric ideas. The reader may refer to duetosymmetry.com/tool/polynomial-roots-toy

- The book [5] is an extremely lucid introduction to Galois Theory which carries out a detailed discussion on cubic and quartic polynomials. For all the algebraic prerequisites as well as for a proof of insolvability of a quintic, the standard reference is [6].

References

1. N. Abel, Dèmonstration de l'impossibilitè de la rèsolution algèbrique des èquations gènèrales qui passent le quatrième degrè, J. Reine Angew. Math., 1824, 1:65-96.

2. N. Abel, Oeuvres Complètes, Two Volumes, (L. Sylow and S. Lie ed.,) Grondhal and Christiana, 1881.

3. Amartya Kumar Dutta, Mathematics in India, Part 6: The Foundations of Algebra-Glimpses, Bhavana, Vol. 7(2), April 2023.

4. V. Alekseev, Abel's Theorem in Problems and Solutions, Kluwer Academic Publishers, Moscow, 2004.

5. D. Cox, Galois Theory, 2nd edition, John Wiley and Sons, 2012.

6. J. Fraleigh, A First Course in Abstract Algebra, 5th edition, Addison -Wesley, 1999.

7. L. Goldmakher, Arnold's elementary proof of the unsolvability of the quintic, *web.williams.edu/Mathematics/lg5/ArnoldQuintic.pdf*

8. R. Irving, Beyond the Quadratic Formula, Math. Asso. Amer., 2013.

9. P. Pesic, Abel's Proof, The MIT Press, Cambridge, England, 2003.

10. Po Shen Loh, A simple proof of the quadratic formula, arXiv:1910.06709.

11. P. Ramond, The Abel-Ruffini Theorem: Complex but Not Complicated, Amer. Math. Monthly, March 2022, 231-245 *https://doi.org/10.1080/00029890.2022.2010494*.

12. Michael I. Rosen, Niels Hendrik Abel and equations of the fifth degree, Amer. Math. Monthly 102, no. 6 (1995), 495–505.

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JUBILANT INDIAN TEAM AT 13^{th} EGMO-2024 TSKALTUBO, GEORGIA April 11-17, 2024

Left-Right: Saee Vitthal Patil (Pune, Maharashtra), Larissa (Hisar, Haryana), Sanjana Philo Chacko (Thiruvananthapuram, Kerala), Gunjana Aggarwal (Gurgaon, Haryana).

Name	P1	P2	P3	P4	P5	P6	Σ	Award
Gunjan Aggarwal	3	7	7	7	2	2	28	Silver Medal
Larissa	7	7	2	0	2	0	18	Bronze Medal
Saee Vitthal Patil	3	7	3	2	2	1	18	Bronze Medal
Sanjana Philo Chacko	4	7	3	7	2	0	23	Silver Medal

We Congratulate the Indian Team for its commendable performance and Shri Sahil Mhaskar (Leader), Ms. Aditi Muthkhod (Deputy Leader), and Ms. Ananya Ranade (Observer), all from the Chennai Mathematical Institute, and all others who are directly or indirectly involved in motivating and training the Indian Team.

2. T. A. Sarasvati Amma: A Leading Light on Early Geometry in India

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2.1 Introduction

In India while there is a deep sense of pride of a rich tradition in mathematics, writing of history of mathematical developments nevertheless turns out to be a neglected subject. Correspondingly, historians of mathematics seldom receive attention in the public sphere. T. A. Sarasvati Amma, whose magesterial work on history of geometry in ancient and medieval India is pivotal to our knowledge of history of mathematics, would be a case in point.

Sarasvati Amma (1908 - 2000)
Courtesy R. C. Gupta

Along with the two-part work *History of Hindu Mathematics: A Source Book*, by Bibhutibhushan Datta and Avadhesh Narayan Singh [2], Sarasvati Amma's book *Geometry in Ancient and Medieval India* [16] (GAMI, for short, in the sequel) serves as a foundational work to our present corpus of history of Indian mathematics. And though it has now been around for well over four decades, it has retained its charm, and is unsurpassed in the comprehensive and insightful nature of its contents.

The two parts of Datta and Singh mentioned above were published in 1935 and 1938 respectively; the first one deals with the history of Numerals and Arithmetic, and the second one with Algebra. The authors had planned a third part (as mentioned in the Preface to the first part) which was to include the history of various other topics including Geometry[1], but that was not to see the light of day; a (possibly incomplete) manuscript of theirs that was left behind in this respect, was with Kripa Shankar Shukla, another acclaimed historian of Indian mathematics contemporary to Sarasvati Amma. It was later edited and published by him in a series of papers in the *Indian Journal of History of Science*. The part on Geometry was published in 1980 [3]; however, the contents are quite meagre compared to Sarasvati Amma's extensive work. GAMI thus stands out as a pioneering work on the theme; interestingly, she herself introduces the book in her Preface as "the third in a series of books on Indian Mathematics", the first two being the two parts authored by Datta and Singh, as above, published more than 40 years earlier.

Apart from bringing out the contents of various ancient manuscripts in an accessible and easy to read form, the book also presents a fresh perspective on Indian mathematics, in the domain of geometry. While doing so, she also maintains scholarly objectivity in reasoning out her assertions. The book systematically traces the development of geometric ideas as they evolved in India, since the very old times. As it unfolds, it displays a great deal of mathematical insight on the part of the author, in explaining the principles and the proofs.

2.2 Family background and education

Sarasvati Amma was born in Cherpulassery in the Palakkad district, in the Malabar region of Kerala[2], of parents Thekkath Amayathodu Kuttimalu Amma and Marath Achutha Menon. She

[1] Some of the other topics mentioned will turn up later in the discussion in this article.

[2] Incidentally, this region in the basin of river Nila was where the Kerala school of Madhava had flourished for well over two centuries, starting sometime in the second half of the 14th century.

is known to have been born in the year 1094 of the Kollam (Kolamba) era, which corresponds to 1918-19 CE; the specific date of birth within the year is, however, not known. Sarasvati (or Saraswathi, spelt in the South Indian style) was the second daughter of her parents, and had two younger brothers and a (much) younger sister. The renowned Kerala historian K. P. Padmanabha Menon, who is regarded as the first modern historian of Kerala, was her maternal uncle. Her younger sister Rajalakshmi, though she had done M. Sc. in Physics and served as a Lecturer in that subject at a College, became an accomplished novelist, short story writer and poet, with many works to her credit (in Malayalam), and in particular won the Kerala Sahithya Academy Award in 1960 for her novel *Oru Vazhiyum Kure Nizhalukalum*[3] - unfortunately she committed suicide in 1965, at the age of 34.

Little is known about Sarasvati's childhood and early education. She was married, presumably at a rather young age, but got divorced soon after giving birth to a son[4]; the circumstances around this development are also not known. The situation would evidently have been very challenging for her, but while she remained reticent about various details, she determinedly and couragiously pursued higher education, and followed it with a distinguishing career.

She did her graduation from the University of Madras, in Physics and Mathematics, with a first class, and did M.A. from Banaras Hindu University in Sanskrit, in the first division. She also followed it with a M.A. from the Bihar University in English literature.

After acquiring the degrees and spending some years in teaching, Sarasvati joined the Department of Sanskrit at the Madras University, as a Research Scholar, in 1957, when she was approaching the age of 40. Though not everything would turn out to be hunky dory from this point on, as we shall see below, this was nevertheless an opportune development for her, and for the subject of history of mathematics! She was to spend three years there, until 1960. Her research work was carried out under the supervision of the late Prof. V. Raghavan. Professor Raghavan, who headed the Sanskrit Department of the University, was a Sanskrit scholar and musicologist of great eminence. He was proficient in reading and deciphering palm-leaf manuscripts in Sanskrit, Prakrit, and Pali, and discovered, edited, and published numerous previously unpublished ancient works. He had toured Europe, during 1953 - 1954, in search of Indian manuscripts in libraries, museums, and research institutions, and discovered and catalogued about 20,000 previously uncatalogued manuscripts. He had been awarded the P. V. Kane Gold Medal by the Asiatic Society, in 1953, and went on to be bestowed the national honour of Padmabhushan in 1962, not long after Sarasvati's association with him as Research Scholar.

In selecting a topic for Sarasvati to pursue for her Ph.D. he is seen to have been very perceptive. In this respect it bears recalling the following part from his Foreword to GAMI:

> When the author of the present Thesis came to me to do research, I did not want her to take up any subject in the overworked fields of Alamkara, Vedanta or general literature and wanted to know if she was prepared to work in the fields which were neglected or in which few young scholars were inclined to put forth their efforts. On further enquiry I found that she was qualified in mathematics, having taken her first degree in physics and mathematics and decided that she should speciliase in the field of Indian contributions to mathematics, algebra and geometry.

Apart from an eminent teacher, Sarasvati was also fortunate enough to have an atmosphere at the university which was very conducive for studies in the chosen topic. In [4] Divakaran makes the point that "Madras University of the 1950s was home to a remarkable gathering of scholars who were either already making outstanding contributions to the history of Indian mathematics or

[3]An English translation of the book by R. K. Jayasree, bearing the title "A path and many shadows and twelve stories" was published in 2016 (Orient Longmans). The work has also been adapted into a tele-series as well as into a play.

[4]The son grew up to be an engineer and settled in Australia, but he does not feature in anything in the limited details that are commonly known about her later life.

were soon to do so.". He goes on to mention C. T. Rajagopal from the Department of Mathematics, who had been publishing articles on the mathematics of the Madhava school since the mid-1940s[5], K. Kunjunni Raja and K. V. Sarma from the Department of Sanskrit itself, and Frits Staal from the Department of Philosophy, and also Kuppanna Sastry in the Sanskrit College, not far from the university campus. While their respective pursuits were unique and novel, there was much by way of commonalities between them. There is no doubt that there would have been a vibrant atmosphere conducive to research in the broad area encompassing them. All the individuals mentioned by Divakaran, with the exception of Frits Staal, have been explicitly thanked by Sarasvati Amma in the Preface of her book, for help "in procuring books and manuscripts and in unravelling the meaning of obscure mathematical passages".

In her research Sarasvati Amma used, apart from printed books and journals from the rich collection of the libraries there, also handwritten manuscripts, which included Parameswara's commentary on Lilavati, Bhaskara I's Bhasya on Aryabhatiya and also Sankara Variar's Kriyakramakari, which is considered the best commentary on Lilavati.

Apparently the Thesis was essentially written up before her departure from Madras in 1960; this is quite remarkable, given the novel and substantive content of the thesis (which was later turned into the book GAMI), and the fact that the Guide himself did not have a mathematical background. However, for some unknown reason, perhaps having to do with her unsettled life, the thesis was not submitted to the University of Madras at that time. It was submitted only in 1963, about three years later, that too at Ranchi University (then in Bihar). The *viva voce* examination was however conducted in Madras, in February 1964; Prof. R. S. Mishra from University of Allahabad and Prof. A. Narasinga Rao from IIT Madras formed the Committee, together with Prof. V. Raghavan, the Guide; this is borne out by a letter from R. S. Mishra to the Registrar of Ranchi University, which has been reproduced in [8].

2.3 Publication of the thesis as a book

While the submission of the Thesis was delayed by three years, its publication as a book was delayed by much longer. Apparently, she began her efforts to publish the thesis soon after her Ph.D. result was announced. In this respect, to begin with she chose to elicit comments from an expert professional mathematician on the mathematical treatment in the thesis. This was done through Prof. K. M. Saxena, the Head of the Department of Mathematics of Ranchi University; the task involved was carried out by one of the latter's Research Students. R. C. Gupta states in [8] that this happened in July 1965. He goes on to add that when he contacted her about a year later, for research work, she lent him the thesis to go through, and at that time she was seriously keen to publish the thesis.

The Research Scholarship granted to her carried also a provision for covering 50% of the cost of publication of the thesis; in her Preface to GAMI she thanks the Government of India for the provision, adding "though I could not make use of the promised help in time and so forfeited it". The reasons for the delay, to the extent of causing the lapse of the grant, are unclear. After some attempts to secure support from other sources for the publication, she decided to get it published on her own. Accordingly it was printed at a local press, G.E.L. Press, Ranchi. However, she found that a lot of printing errors had remained in the final print, having escaped proof-reading. As a result she had to abandon the whole lot of the printed copies; Prof. R. C. Gupta mentions in [8] that he posesses a copy of that "edition"(!), acquired from the printers, G.E.L. Press, in 1974,

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[5]It may be worthwhile to recall here that Rajagopal was largely responsible in rejuvenating the studies on mathematics of the Kerala school of mathematics, which is now acclaimed as a major development in Indian mathematics, especially for contribution of ideas in the stream of Calculus. Though the work had been brought to the attention of the western scholarship by Charles Whish as far back the 1830's - his two major papers in this respect are from 1832 and 1834 - but the exposure had a prejudicial and hostile response, and eventually it was pretty much forgotten, until the work of C. T. Rajagopal and his collaborators in the 1940s, over a hundred years later; the reader is referred to [17] for an account of the early developments in this respect.

"just for a look, and for the record".

The idea of publishing the thesis seems to have got a fresh impetus, presumably after a hiatus of a few years, in the early 1970s. In his Foreword to GAMI Prof. Raghavan writes

> The efforts of the Ministry of Education dealing with the History of Science in India and the Association for the History of Science and their Journal have been helpful to the development of researches in the field. Special emphasis was laid by the First International Sanskrit Conference held recently by the Ministry of Education on Sanskrit in Science and Technology and it revealed the talent available for tracking subjects in the area.

The Journal that is alluded to is the *Indian Journal of History of Science*, which brought out its first volume in 1966. Volume 4 of the Journal, of the year 1969, includes a paper by Sarasvati Amma [14], which in some ways is a summary of her Thesis, together with a perspective on the development of mathematical ideas in India since the ancient times. The First International Sanskrit Conference that is mentioned was held in March 1972 at New Delhi, led by Professor Raghavan, and Sarasvati Amma presented a paper in the conference on "Sanskrit and Mathematics". On the whole there would have been a very encouraging atmosphere then, inspiring fresh efforts towards publication of the Thesis, which many in the field would by then have known for its worth in the area. There is a tell-tale sign corroborating this: the Foreword of Prof. Raghavan is dated 1.10.1972!

At this time she was able to secure a grant from Ranchi University, subsidising publication of the book by well-known publishers Motilal Banarasidass, Delhi. This seems to have happened sometime in 1972, but apparently there was considerable delay in the funds being released. In her letter to Prof. R. C. Gupta of 16 April 1973 (see Figure 1; the letter was first reproduced in [8]), she is seen to complain "The Ranchi University is taking a long time releasing the aid they have sanctioned for the publication of the thesis and my publishers are waiting for the subsidy."

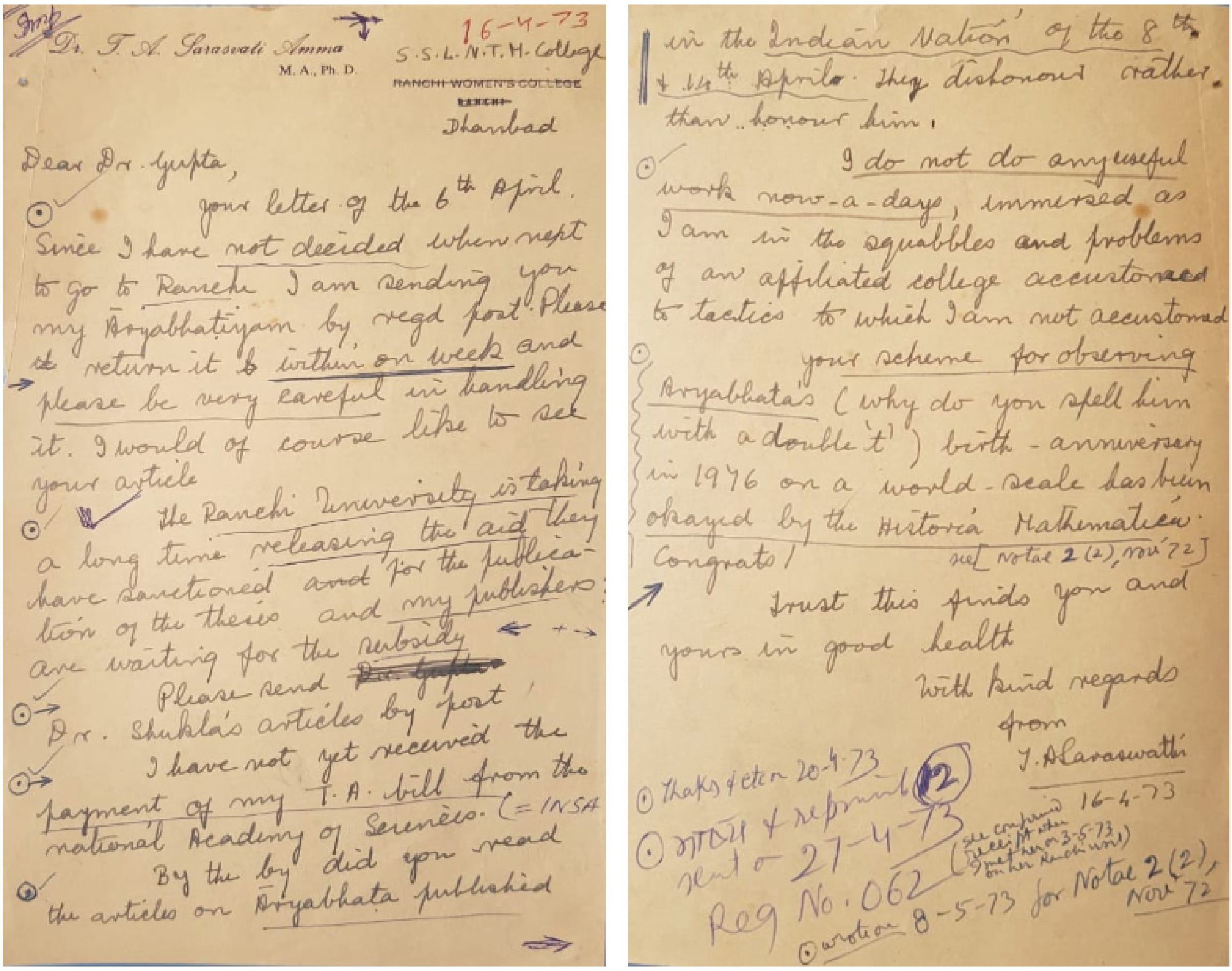
It is not clear when the subsidy finally reached the publishers, but it is seen to have taken a long time even after the reference in the letter, for the process to move on. Thankfully it all went well in the end, and the book came out, in good form, though only in January 1979; by then it was over 20 years since an early version of the text was in place, and she herself had either already attained the age of 60, or was quite close to it.

2.4 Reception of the book

The book, GAMI, was greatly welcomed in the academic circles.

In *Historia Mathematica*, which is a unique international journal of history of mathematics, associated with the International Commission on History of Mathematics, the book was reviewed (cf. [18]) by Michio Yano, a well-known historian of mathematics with a long record of engagement with ancient Indian mathematics. Attesting to Sarasvati Amma's referring to it in her Preface as "the third in a series of books on Indian Mathematics" the reviewer comments "Datta and Singh's supposed original plan has been better achieved by another person a generation later." The review contains a chapterwise account describing the strengths, as also some weaknesses, of the contents. The reviewer remarks in particular that "Chapter VII, the most remarkable chapter of GAMI, shows the outstanding aspect of Indian mathematics–the discovery of the infinite series of π and of sine and cosine series." Indeed, notwithstanding the long passage of time since its publication, the description in the Chapter stands as one of the most convenient references to the work involved. The review concludes with the statement "...the book has established a firm foundation for the study of Indian geometry, and it will very surely give stimulus to the students of history of mathematics."

For *Mathematical Reviews*, which is a review journal with universal appeal in the mathematical community, GAMI was reviewed by A. I. Volodarskiĭ, another well-known historian of mathematics who has also contributed to studies on history of mathematics from India. He comments "As pointed out by the author in the preface this book is the third in a series of books on Indian


Dr. T. A. Sarasvati Amma
M.A., Ph. D.
16-4-73
S.S.L.N.T.H. College
~~RANCHI WOMEN'S COLLEGE~~
~~RANCHI~~
Dhanbad

Dear Dr. Gupta,

your letter of the 6th April. Since I have not decided when next to go to Ranchi I am sending you my Aryabhatiyam by regd post. Please return it within on week and please be very careful in handling it. I would of course like to see your article

The Ranchi University is taking a long time releasing the aid they have sanctioned for the publication of the thesis and my publishers are waiting for the subsidy

Please send Dr. Shukla's articles by post

I have not yet received the payment of my T.A. bill from the national Academy of Sciences. (= INSA

By the by did you read the articles on Aryabhata published in the Indian Nation of the 8th & 15th April. They dishonour rather than honour him.

I do not do any useful work now-a-days, immersed as I am in the squabbles and problems of an affiliated college accustomed to tactics to which I am not accustomed

your scheme for observing Aryabhata's (why do you spell him with a double 't') birth-anniversary in 1976 on a world-scale has been okayed by the Historia Mathematica. Congrats!
see Notae 2(2), Nov '72

Trust this finds you and yours in good health

With kind regards
from
T. A. Sarasvathi
16-4-73

Thanks letter 20-4-73
Offprint & reprint (2) sent on 27-4-73
Reg No. 062
Question 8-5-73 for Notae 2(2), Nov 72

Figure 1: The letter from Sarasvati Amma to Prof. R. C. Gupta, Courtesy R. C. Gupta.

mathematics. .... In fact, this book considerably enlarges our knowledge of the history of Indian geometry."

Not surprisingly, the reviews published in India were also similarly euphoric and lauded it for being "an almost exhaustive survey" and for its impartial scholarly attitude to the study (see Figure on the inner back cover page and [8] for some details).

2.5 A brief sketch of the contents of the book

The book, which has close to 300 pages, gives a comprehensive exposition of various geometrical constructions and principles enunciated in works from India from the very ancient times to until about 1700 CE, with the periods divided suitably according to their context (see below). The author also includes at various points critical evaluation, with clarification or rebuttal, of various claims or beliefs held by earlier scholars, especially from the west.

The contents of GAMI are divided into 10 Chapters. Chapter I, titled Introduction, sets out the perspective and the plan for the book. The Chapter includes also a general survey of the history of Geometry in India. The history is divided into three distinct periods, the pre-Aryan period of the Indus valley civilization, the Vedic period of the millennium BCE and the CE period (which she refers as "post-Christian period"), and they are discussed, dealing individually with their background and summarizing the major features of the periods; the discussion on the pre-Aryan period is mostly limited to this Chapter.

Chapters II and III are devoted to details on the middle period from the above division. Chapter II concerns geometry of the Vedic Sulbasutras, associated with the activity of construction of *Vedis* (altars) and *agni*s (fireplaces) for performing the *yajna*s. Various geometric constructions and principles enunciated in the Sulbas are recalled. In particular it is highlighted that the statement of what is known as Pythagoras theorem is found in the Sulbasutras from long before Pythagoras. It is followed by a relatively short Chapter III on early Jaina Geometry. The Jainas had, in particular, interesting approximate formulae relating lengths of arcs of circles to the corresponding chords (see [1] for an exposition on this), which have been recalled in this Chapter, describing the context around them.

Chapters IV to VIII concern the CE period, focussing on the geometric knowledge witnessed in the interrelated works of numerous exponents, starting with Aryabhata, continuing with Bhaskara I, Brahmagupta, Sridhara, Mahavira, Aryabhata II, Bhaskara II, Narayana Pandita, Madhava, Nilakantha, Paramesvara and numerous others along the way, until the 17th century. The Chapters are divided thematically according to the geometric figures on which they focus on, dealing respectively with the trapezium, the quadrilateral, the triangle, the circle, and solids; while the order in which the figures occur may seem confusing, it has to do with the level of sophistication of the results appearing in their context. The part is rich in content with numerous insightful discussions - some of these, highlighted in the review of Michio Yano mentioned earlier in the article.

Chapters IX is on Geometrical Algebra, dealing with problems of algebraic nature through geometric understanding. It traces the practices along the theme, from the ancient time of the Sulbasutras, until all the way to mathematics from the Kerala School that flourished from the 14th 16th centuries. Chapter X is on "Shadow problems" and other problems, these being of special significance in Astronomy.

2.6 Other research and guidance

Sarasvati Amma did not publish many research papers. In a way, as Divakaran comments in [4], "That does not much matter. The one book she did write was the equal of any number of research communications, so broad is it in the material covered, so rich in new insights and so analytically rigorous in its working out of details." However, such a comment could be rather misleading, if not taken in the right spirit, as it may suggest that she felt so fulfilled with the thesis, leading to disinterst in further research. As an academic, she was not self-satisfied or smug to be content having authored the work, even though she was quite conscious of its extraordinary worth. It was never as if the thesis was an end in itself. She lived with a perennial urge to accomplish more to bring out the essence of ancient Indian mathematics, though the circumstances she faced subsequently were, unfortunately, not quite conducive to her fulfilling it well.

A list of her papers is included in [6]. It has 10 titles authored by her, apart from GAMI and some reviews. Five of them are from the period 1958 - 63, including one from 1958 - 59 from her Research Scholar days at Madras. Three more, other than the two from 1969 and 1972 that were mentioned earlier in this article, are from 1975 - 76. While some of the papers are related to the topics covered in the book, some are on topics other than Geometry. Here are some highlights on some of the papers.

Brahmagupta's results on the area, and the diagonals, of cyclic quadrilaterals (quadrilaterals for which there is a circle passing through its four vertices) has been one of the major highlights of 7th century mathematics. This topic, discussed in her thesis, was also brought into focus in her paper [12]. Commenting on this work in his review of GAMI, Michio Yano writes "Sarasvati's discussion of the cyclic quadrilaterals treated by Brahmagupta reveals her remarkable competence in dealing with mathematical Sanskrit texts."

Apart from Geometry, she is seen to have a fascination for the topic of mathematical series. In her Preface to GAMI she mentions having collected some materials for a history of Series mathematics in India, which she hoped to present in a book form[6] At least three of her papers

[6]Coming in the Preface of a book whose publication was extraordinarily delayed, this sounds quite an optimistic statement - unfortunately, that is not how it was to be.

concern this topic, an early one from the Research Scholar days [9] concerning diagrammatic representation used in aid of series summations especially in the work of Sridharacharya, one from 1962 [12] on Mahavira's treatment of series, and another in 1963 [13] on development of the topic in India after Bhaskara II. There was a common belief at one time that mathematics in India stagnated after Bhaskara II, in terms of original contributions, especially as the work of Narayana Pandita and the Kerala School had not been known well enough. The last paper mentioned above which was published in the Bulletin of the National Institute of Sciences of India (which was later remodelled into the present Indian National Science Academy) made a significant difference in this respect.

She is also seen to have keenly studied the works on ancient Jaina mathematics, which has been a rather neglected area, even within the studies of ancient Indian mathematics. Her enthusiasm for the topic is witnessed by the Chapter on it in her thesis, and more so by the later long paper [11] on the ancient Jaina work *Trilokaprajnapti*.

The programme that Datta and Singh set out to follow, towards producing a comprehensive exposition of Indian mathematics, also included, apart from Arithmetic and Algebra, which they covered in the first two parts as mentioned, and Geometry which was ably taken care of by Sarasvati Amma, the topic of Trigonometry[7]. Mathematical astronomy has been one of the driving forces for mathematical developments in India, at least since the time of Aryabhata from the fifth century, if not a few centuries earlier, with the onset of the *Siddhanta astronomy*. On account of its close association with astonomy, and the considerable progress that was made on the topic, Trigonometry is an important component of ancient Indian mathematics, since the early centuries of the common era. Sarasvati Amma was also instrumental in furthering the objective of exposition of ancient Trigonometry, by guiding the Ph.D. thesis of R. C. Gupta on the topic. The latter, who was an assistant professor of mathematics at the Birla Institute of Technology (BIT) Mesra, also known as BIT Ranchi, joined her in the mid-1960s for doing Ph. D. under her supervision, which he completed in 1970-71, with a thesis on "Trigonometry in ancient and medieval India", based primarily on astronomical works in Sanskrit. It may not be out of place to mention here that Prof. Gupta went on to become one of the illustrious historians of mathematics internationally, and in particular he was awarded in 2009 the Kenneth O. May medal, a coveted international recognition in the field of history of mathematics, awarded once in four years under the aegis of the International Commission for History of Mathematics (ICHM). He has an impressive oeuvre on the subject, and a selection of his papers, entitled "Ganitananda"[8] [7] was brought out by the Indian Society for History of Mathematics, in 2015, on the occasion of his 80th birthday; the volume was released at the annual conference of the Society held at the Indian Institute of Technology, Bombay, which was also dedicated to him. Professor Gupta holds Sarasvati Amma in high esteem, as may be seen from his articles [5], [6], [8], and various other writings and references.

2.7 Teaching and administration

Following her departure from Madras in 1960, until about 1980 she seems to have been engaged in teaching, and later also in administration, in Bihar. The main affiliations that are known from the period are Women's College at Ranchi, and Shree Shree Lakshmi Narain Trust Mahila Mahavidyalaya (SSLNT-MM) (Women's College), Dhanbad. Women's College, Ranchi, is mentioned

[7]Specifically the authors declare "It has been decided to publish the book in three parts. ... The third part contains the history of geometry, trigonometry, calculus, and various other topics such as magic squares, theory of series and permutations and combinations." From the later-mentioned topics, the calculus part is covered in Sarasvati Amma's book, as an extension of geometry of the circle. She might have also covered Trigonometry in a similar spirit, if she were to have more time at Chennai. She is likely to have saved it for later, to take it up leisurely, and passed on the task to R. C. Gupta when he joined her as a Ph.D. student. I may also recall here, on the other hand, that all the topics mentioned above are covered in the series of papers published by K. S. Shukla in the *Indian Journal of History of Mathematics*, succeeding [3], based on the notes of Datta and Singh, listing the latter two as authors.

[8]The title was inspired by the pen-name under which he wrote many papers.

in her paper [10] published in 1961; it is noted in [8] (in a footnote) that at that time she had recently been appointed there as Lecturer in Sanskrit. It is also mentioned in her paper [14] from 1969.[9] From 1973 to about 1980 she was the Principal of SSLNT-MM College, Dhanbad. While the former institution seems to have suited her academically, at least to a reasonable extent when one considers her work during the period, the latter assignment seems to have sapped her energies, leaving little scope for pursuing research with any vigour. In her letter of 16 April 1973 to R. C. Gupta (see Figure 1) she expressed her frustration, saying "I do not do any useful work nowadays, immersed as I am in the squabbles and problems of an affiliated college accustomed to tactics to which I am not accustomed."[10] The letter is from an initial period of her term as the Principal, and perhaps she may have been able to make peace with the status quo there, at a later stage, having continued there for several more years, until 1980. There is nothing to confirm this however. On the other hand, it is believed in some quarters that her exit from the college in (or around) 1980 was by resignation, following disenchantment, rather than retirement[11]. That however does not seem to be the case; while disenchantment may have continued to be a part of the predominant disposition, the departure followed retirement in the normal course; indeed, she herself refers to it as retirement, in her letter to Prof. R. C. Gupta written in 1986 (see below, for more about this letter).

2.8 The final years

After retirement she returned to Kerala, her native state, joining her mother in Ernakulam at their family house. Around then the Indian Society for History of Mathematics (officially listed as Society for History of Mathematics, India) was founded, in Delhi, by Prof. U. N. Singh of the University of Delhi, with Prof. R.C. Gupta as one of the founding members. Its Bulletin, named Ganita Bharati, began to be published, starting from 1979, under the Editorship of Prof. R. C. Gupta.[12] Sarasvati Amma joined as a member of the Society and was keeping in touch with the contents of Ganita Bharati. It would evidently have elated her spirit that a journal devoted exclusively to history of mathematics had come up, and was being fostered by her student. At the request of Prof. Gupta she also contributed a review of the book "Geometry according to Sulbasutras" by R. P. Kulkarni, which was published in Ganita Bharati in 1986.

1986 was also the year when Sarasvati Amma moved out, with her mother, from the Ernakulam house, following its sale, to a smaller house in Ottapalam in Palakkad district. Shortly after that her ailing mother passed away, at the age of 90.

In a letter to Prof. R. C. Gupta written in January 1986 Sarasvati Amma wrote "When I retired from Principalship of Dhanbad College, I was hoping to get some time for research work. But I have been mistaken. With a big house to manage, my mother down with a fractured thigh bone, I have my hands full with household work…". While this manifests her dominant scholarly academic spirit, it would indeed have been difficult for her to pursue research without access to a good library, especially in her old age.

Unfortunately, she suffered from bone cancer, for quite long, and after 27 August 1999, she was bedridden due to a bone-fracture in her leg. In a happy incidence, the publishers of GAMI, Motilal Banarasidass, brought out its second edition in 1999, which moreover had rectified some of the errors and misprints in the 1979 edition. Hopefully it would have brought her some cheer

[9] It may perhaps also be there in some of the other papers, but this is not ascertained.

[10] It may be noted, however, that notwithstanding the expression of frustration, the letter as seen in Figure 1 shows a certain sense of engagement with academic developments.

[11] See, for instance, the online article of S. Dhawani, dated 12 March 2019, at *https://feminisminindia.com/2019/03/12/t-a-saraswathi-amma-ancient-indian-geometry/*

[12] Prof. R. C. Gupta assiduously nurtured the journal until 2005, bringing out 27 volumes, publishing a wide range of articles on the history of mathematics, both Indian as well as from around the world. The present author has the privilege of being the current Editor of Ganita Bharati, since 2010.

in what had by then been difficult times for her. She passed away on 15 August 2000.

Sarasvati Amma was a major exponent of the 20th century on the history of Indian mathematics, with deep commitment to the subject. Her book continues to be one of the major and unique references in the area, and her ideas will no doubt live on, inspiring the coming generations of enthusiasts of history of Indian mathematics.

References

1. S. G. Dani, Some Approximate Formulae from the Ancient Times, The Mathematics Consortium Bulletin, Vol. 3, Issue 3, January 2022, pp 23-32.
2. Bibhutibhushan Datta and Avadhesh Narayan Singh, History of Hindu Mathematics, Part I, Motilal Banarsidass, Lahore, 1935; Part II, Motilal Banarsidass, Lahore, 1938; available also in other later editions.
3. Bibhutibhusan Datta and Avadhesh Narayan Singh, Hindu geometry, Revised by Kripa Shankar Shukla, Indian J. Hist. Sci. 15 (1980), no. 2, 121-188.
4. P. P. Divakaran, T. A. Sarasvati Amma: a centennial tribute, Ganita Bharati 40 (2018), no. 1, 1-16.
5. R. C. Gupta, T. A. Sarasvati Amma (c. 1920 - 2000): a great scholar of Indian geometry, Ganita Bharati 23 (2001), no. 1-4, 125-127.
6. R. C. Gupta, Obituary T. A. Sarasvati Amma (1918-2000), Indian J. Hist. Sci. 38 (2003), no. 3, 317 - 320.
7. Radha Charan Gupta, Ganitananda – selected works of Radha Charan Gupta on history of mathematics, Edited by K. Ramasubramanian, Indian Society for History of Mathematics, Delhi, 2015; Second edition by Springer, Singapore, 2019, pp. xxvii+639.
8. R. C. Gupta, T. A. Sarasvati Amma (c. 1920 - 2000): a Great Scholar of Early Indian geometry, Bhavana 3, no. 2, April 2019; available at: *https://bhavana.org.in/t-a-sarasvati-amma-1918-2000/*
9. T. A. Sarasvati Amma, Sredhi ksetras or diagrammatic representation of mathematical series, Journal of Oriental Research, 28 (1958-59), 74-85.
10. T. A. Sarasvati Amma, Cyclic Quadrilateral in Indian Mathematics, Proceedings of the All-India Oriental Conference. 1961. 21: 295-310.
11. T. A. Sarasvati Amma, The mathematics of the first four mahadhikaras of the Trilokprajnapti, J. of the Ganganath Jha Research Institute, 18 (1961-62), 27-51.
12. T. A. Sarasvati Amma, Mahavira's treatment of series, Journal of Ranchi University, 1 (1962), 39-50.
13. T. A. Sarasvati Amma, Development of mathematical series in India after Bhaskara II, Bulletin of National Institute of Sciences in India 21 (1963), 320-343.
14. T. A. Sarasvati Amma, Development of Mathematical Ideas in India, Indian Journal of History of Science. 4 (1969), 59-78.
15. T. A. Sarasvati Amma, Sanskrit and Mathematics, Paper read at the First International Conference, New Delhi, 1972, Published in the Proceedings (of the Conference), Volume III, Part 1, pp. 196 -200 (New Delhi, 1980). Also reprinted in the *Souvenir of the World Science Conference*, New Delhi, 2001, pp. 63-78.
16. T. A. Sarasvati Amma, Geometry in Ancient and Medieval India, Motilal Banarsidass, Delhi, 1979, pp. vii+277. Second revised edition brought out in 1999.
17. U. K. V. Sarma, Vanishri Bhat, Venketeswara Pai, and K. Ramasubramanian, The discovery of Madhava series by Whish: an episode in historiography of science, Ganita Bharati 32 (2010), no. 1-2, 115-126 (2012).
18. Michio Yano, Review of the book "Geometry in Ancient and Medieval India" by T.A. Sarasvati Amma, Historia Mathematica 10 (1983) 467-470.

□□□

3. What is Happening in the Mathematical World?

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3.1 Smooth Fractal Snowflakes Disprove 50-Year-Old Milnor's Conjecture

In 1968, *John Milnor*, a renowned mathematician, then at Princeton University, conjectured that an average sense of a complete shape's curvature was enough to tell us that it could not have infinitely many holes. For the next 50 years, many results partially supported his claim and it was tempting to believe it was true.

To understand Milnor's conjecture, it helps to first consider how topologists and geometers think about curvature. Both study manifolds - spaces (in dimension 2) that look flat when you zoom in on them. A tiny ant on the surface of a sphere, doughnut or other two-dimensional manifolds will perceive its immediate neighborhood to be no different from a two-dimensional plane. But if the ant moves around a little, it might notice that the experience is different, in terms of distances covered, suggesting the space being "curved" in some way. But curvature is more intricate to define mathematically.

Consider first a one-dimensional object such as a circle. In this case the space cannot, in a mathematical sense, be intrinsically curved.

But if you embed a circle in a two-dimensional plane, it is apparent that it has constant, positive extrinsic curvature. Smaller circles bend more quickly as you move around them, and therefore have higher extrinsic curvature; bigger circles have lower curvature. (A straight line, in this sense, is like an infinitely big circle, its curvature is zero.)

We can also apply this idea to more complicated shapes that have changing curvature, by considering how big a circle you would need to match the shape at any given point. In this way, curvature is a local property: Every point on a manifold has an associated curvature.

For a surface - a two-dimensional manifold - there are many ways to place circles so that they match the surface's curves. At a given point, you can measure curvature in any direction by placing an appropriately sized circle in that direction. intrestingly, it is possible to define the surface's curvature at that point with just one number. If you find the directions that give you the biggest and smallest curvature values, and multiply those values together, you get a number called the *Gaussian curvature*. This number summarizes the information about how the surface bends in a useful way. Even more surprisingly, the Gaussian curvature turns out to be an intrinsic property: It does not depend on any higher-dimensional background space the surface might be placed into. In this sense, paradoxically, cylinders are not intrinsically curved (as Gaussian curvature at every point on the cylinder is zero), though spheres are.

Figure 1

In three or more dimensions, it is generally no longer possible to capture useful information about curvature with a single number. One instead keeps track of the curvature using "tensors", which can be thought of as arrays of numbers that transform according to particular mathematical rules. There are several different ways to describe a manifold's curvature using tensors, but one of the most important is what is called the Ricci tensor. Like Gaussian curvature, it refines essential information into a (comparatively) simpler form.

Unlike numbers, tensors can not be neatly sorted into order - but like numbers, they can be categorized as "nonnegative" if they satisfy a certain property. In 1968, Milnor conjectured that "complete manifolds whose Ricci tensor is nonnegative at every point, can not have an infinite number of holes" (as shown in the Figure 1).

The conjecture has been known to be true in two dimensions, since the 1930s. And in 2013, it was proved for three-dimensional manifolds. Under some additional constraints, that the manifold is closed and bounded, like a sphere, or that the volume grows at a particular rate - the conjecture holds in all dimensions.

More than half a century later, now, *Aaron Naber* of Northwestern University (left), *Daniele Semola* of the Swiss Federal Institute of Technology Zurich (center) and *Elia Bruè* of Bocconi University in Italy (right) found a counterexample to the 50-year-old conjecture - and built an entirely new kind of topological shape in the process - after two years of failed attempts.

This trio figured out how to construct a strange seven-dimensional manifold. They built it by gluing together infinitely many seven- dimensional pieces in subtle and intricate ways, assembling the entire manifold bit by bit. All the while, they had to make sure that the Ricci curvature always stayed nonnegative. And they had to avoid getting satisfied accidentally any of the many properties for which Milnor's conjecture was already known to be true. They ended up with what they called a *smooth fractal snowflake* - an infinite and delicate self-similar structure. It has nonnegative Ricci curvature at every point and has an infinite number of holes. Thus, they have disproved Milnor's conjecture.

Using similar techniques, the trio has been able to build analogous counterexamples in higher-dimensional spaces, as well as in six dimensions. It is not known if a counterexample exists in four or five dimensions.

Source: *https://www.quantamagazine.org/strangely-curved-shapes- break- 50-year-old-geometry-conjecture- 20240514/*

3.2 Decoding the Geometry of Music: 70-Year-Old Pólya Conjecture Solved For Discs

Researchers have made a significant advance in spectral geometry by proving a special case of Pólya's conjecture related to the eigenvalues of a disk. Their work, blending theoretical elegance with potential practical applications, highlights the universal value and artistic beauty of mathematical research.

Iosif Polterovich, a Professor in the Department of Mathematics and Statistics at Université de Montréal, likes to address questions like: Is it possible to identify the shape of a drum, just by hearing the sounds it makes?

To understand this type of physical phenomenon related to wave propagation, recently, *Iosif Poltrovic* (left), in collaboration with his international collaborators *Nikolai Filonov* (second), *Michael Levitin* (third) and *David Shear* (right), proved a special case of a famous conjecture in spectral geometry formulated in 1954 by a prominent Hungarian-American mathematician *George Pólya*, in a paper entitled "Pólya's conjecture for Euclidean balls" which appeared in Inventiones mathematicae (2023) 234:129-169.

The celebrated conjecture of Pólya states that the eigenvalue counting functions of the Dirichlet and Neumann Laplacian on a bounded Euclidean domain can be estimated from above and below, respectively, by the leading term of Weyl's asymptotics.

Suppose $\Omega \sqsubset \mathbb{R}^d$ be a bounded domain. The Dirichlet eigenvalue problem for the Laplacian

$$-\Delta := -\sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2} \text{ in } \Omega \text{ is } : -\Delta u = \lambda u \text{ in } \Omega \text{ and } u = 0 \text{ on } \partial\Omega \qquad (1)$$

Then It is well known that the spectrum of (1) is discrete and consists of isolated eigenvalues of finite multiplicity accumulating to $+\infty, 0 < \lambda_1(\Omega) \leq \lambda_2(\Omega) \leq \cdots \leq \lambda_n(\Omega) \leq \ldots$, which is enumerated with account of multiplicities.

Similarly, assuming additionally that $\delta\Omega$ is Lipschitz, consider the Neumann eigenvalue problem:

$$-\Delta u = \mu u \text{ in } \Omega \text{ and } \partial_n u = 0 \textbf{ on } \partial\Omega, \qquad (2)$$

where $\partial_n u = \langle \nabla u, n\rangle_{\partial\Omega}$ denotes the normal derivative of u with respect to the exterior unit normal n on the boundary. The spectrum of (2) again consisits of isolated eigenvalues of finite multiplicity accumulating to $+\infty, 0 < \mu_1(\Omega) \leq \mu_2(\Omega) \leq \cdots \leq \mu_n(\Omega) \leq \dots$, which ic enumerated with account of multiplicities.

For $\lambda \in \mathbb{R}, \mathcal{N}_\Omega^D(\lambda) := \#\left\{n|\lambda_n(\Omega) \leq \lambda^2\right\}$ and $\mathcal{N}_\Omega^N(\lambda) := \#\left\{n|\mu_n(\Omega) \leq \lambda^2\right\}$ denotes the counting function of the Dirichlet and Neumann eigenvalue problems on Ω. It follows from the variational principles for (1) and (2) that $\mathcal{N}_\Omega^D(\lambda) \leq \mathcal{N}_\Omega^N(\lambda)$ for any $\lambda \geq 0$.

Under the assumptions stated above, the leading term asymptotics of the counting function is given by Weyl's law, $\mathcal{N}_\Omega(\lambda) = C_d|\Omega|_d\lambda^d + R(\lambda)$, where $\mathcal{N}_\Omega(\lambda)$ denotes either $\mathcal{N}_\Omega^D(\lambda)$ or $\mathcal{N}_\Omega(\lambda), |\cdot|_d$ denotes the d-dimensional volume, $R(\lambda) = o(\lambda^d)$ as $\lambda \to +\infty$, and $C_d := \frac{1}{(4\pi)^{\frac{d}{2}}\Gamma(\frac{d}{2})+1}$, is the so-called Weyl constant.

In 1954, G. Pólya conjecture that the inequalities $\mathcal{N}_\Omega^D(\lambda) \leq C_d|\Omega|_d\lambda^d \leq \mathcal{N}_\Omega^N(\lambda)$ holds for all $\lambda \geq 0$.

Pólya's conjecture is known to be true for domains which tile the Euclidean space, and, in addition, for some special domains in higher dimensions. Polya himself confirmed his conjecture in 1961 for tiling domains Ω: that is, domains such that $\mathbb{R}^d$ can be covered, up to a set of measure zero, by a disjoint union of copies of Ω (such as triangles or rectangles). Until last year, the conjecture was known only for these cases, while the disk, despite its apparent simplicity, remained elusive. In fact, a disk is actually not a good shape for tiling.

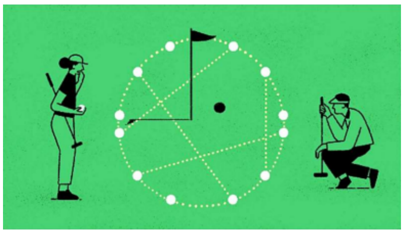
The above team of researchers thus showed that Pólya's conjecture is correct for the disk, a particularly challenging case. Though their result is essentially of theoretical value, their method has applications in computational mathematics and numerical computation. The authors are now investigating this avenue.

Sources:

1. *https://scitechdaily.com/decoding-the-geometry-of-music-70-year-old-math-problem-solved/*
2. *https://www.awanireview.com/an-open-mathematics-question-for-70-years-has-finally- been-solved/*
3. *https://doi.org/10.1007/s00222-023-01198-1*

3.3 The Arrangement of Large but Finite Integer Distance Sets

Mathematicians have illuminated how the sets of points can look like if the distances between them are all whole numbers. If a large (but non-infinite) set of points are whole-number distances away from each other, how can they be arranged? A new result proves that a circle is the only option.



On the plane, we can describe an infinite set of points that are all integer distances apart - just take a line, and use some or all of the points corresponding to whole numbers. It turns out that, this is the only way to construct an infinite integer distance set in the plane, as *Norman Anning* and *Paul Erdős* showed in 1945; thus, proved that an infinite number of points in the plane R can have mutual integer distances only if all the points lie on a straight line.

In place of infinite sets one may ask such a question, about possible restrictions, for large but finite sets. Consider first points on a circle. If you want an integer distance set with, say, a trillion points, there are ways to come up with a x trillion points on a circle of radius 1 whose distances apart are all rational (fractions). Then you can scale up the circle (by a factor of the least common denominator of the distances) so that all the fractional distances are whole numbers.

For the general case, the possibilities are a little more varied. Large sets have been constructed lying on either a line or a circle, with three or four extra points that are off the main drag.

Since Anning and Erdős' work in 1945, there has been little progress on understanding integer distance sets, until now in terms of characterizing such sets.

Now, *Rachel Greenfeld* of the Institute for Advanced Study in Princeton (left), *Sarah Peluse* of University of Michigan (middle) and *Marina Iliopoulou* of National and Kapodistrian University of Athens (right) have proved that all the points in a large integer distance set - except for a small proportion of outlier points - must lie on a single line or circle. More specifically they prove the following result:

If $S \subset [-N, N]^2$ is an integer distance set then either $[S] \ll (\log N)^{O(1)}$ or there exists a line or circle $C \subset \mathbb{R}^2$ such that $[S \backslash C] \ll (\log \log N)^2$.

The new approach uses ideas and techniques from three distinct areas of mathematics: combinatorics, number theory and algebraic geometry. This joining together of different fields could be a real psychological breakthrough. They have laid down a very solid foundation for a very broad set of problems and will find even deeper applications, including some problems in Harmonic Analysis.

Sources:

1. *https://www.quantamag On integer distance sets azine.org/merging-fields-mathematicians-go- the-distance-on-old-problem-20240401/*
2. *https://arxiv.org/abs/2401.10821v1*

3.4 Google Deepmind's New AI System can solve Complex Geometry Problems

Google DeepMind has created an AI system that can solve complex geometry problems. Designing AI tools for tackling mathematical, more specifically geometric problems, has been challenging as solving such problems requires logical reasoning, something that most current AI models are not well equipped to deal with due to lack of adequate training data.

DeepMind's program, named *AlphaGeometry*, combines a language model with a type of AI tool called a symbolic engine, which uses symbols and logical rules to make deductions. Language models excel at recognizing patterns and predicting subsequent steps in a process. However, their reasoning lacks the accuracy required for mathematical problem-solving. The symbolic engine, on the other hand, is based purely on formal logic and strict rules, which allows it to guide the language model toward rational decisions.

These two approaches, responsible for creative thinking and logical reasoning respectively, work together to closely mimic how humans work through geometry problems, combining their existing understanding with explorative experimentation.

DeepMind says it tested AlphaGeometry on 30 geometry problems at the same level of difficulty found at the International Mathematical Olympiads. It completed 25 within the time limit. The previous state-of-the-art system, developed by the Chinese mathematician *Wen-Tsün Wu* in 1978, completed only 10. DeepMind says this system demonstrates AI's ability to reason and discover new mathematical knowledge.

When presented with a geometry problem, AlphaGeometry first attempts to generate a proof using its symbolic engine, driven by logic. If it cannot do so using the symbolic engine alone, the language model adds a new point or line to the diagram. This opens up additional possibilities for the symbolic engine to continue searching for a proof. This cycle continues, with the language model adding helpful elements and the symbolic engine testing new proof strategies, until a verifiable solution is found.

To train AlphaGeometry's language model, the team of researchers including *Thang Luong*, a Senior Staff Research Scientist at Google DeepMind (left), *Floris van Doorn*, a mathematics

Professor at the University of Bonn (middle) and *Quoc V. Le*, a scientist at Google DeepMind (right) had to create their own training data to compensate for the scarcity of existing geometric data. They generated nearly half a billion random geometric diagrams and fed them to the symbolic engine. This engine analyzed each diagram and produced statements about its properties. These statements were organized into 100 million synthetic proofs to train the language model. AlphaGeometry's ability shows a significant advancement toward more sophisticated, human-like problem-solving skills in machines.

Beyond mathematics, AlphaGeometry can be effectively used in fields that rely on geometric problem-solving, such as computer vision, architecture, and even theoretical physics.
Source: *https://www.technologyreview.com/2024/01/17/1086722/google-deepmind-alphageometry/*

3.5 Awards

3.5.1 Michel Talagrand Wins Abel Prize 2024 for Studies of Universe's Randomness

A French mathematician and a retired researcher at France's National Center for Scientific Research (CNRS) *Prof. Michel Talagrand* is the recipient of Abel Prize 2024, equivalent of the Nobel prize, for advances in understanding randomness in the universe, work that has found use in mathematical physics and statistics. Dr. Talagrand received 7.5 million Norwegian kroner, or about US $7,00,000.

The citation mentions - “Talagrand is an exceptionally prolific mathematician whose work has transformed Probability theory, Functional analysis and Statistics. His research is characterized by a desire to understand interesting problems at their most fundamental level, building new mathematical theories along the way”. He had also won the Shaw Prize, another prestigious award, in 2019.

At the age of 5 Talagrand lost his right eye due to a detached retina caused by a genetic disease. By the age of 15, he had suffered multiple retinal detachments and lived in terror of going blind. His high school education was interrupted. His father, a mathematics teacher at a preparatory school, gave him lessons while he was blindfolded because of the operation. Unable to run around with friends, Talagrand immersed himself in his studies and mentalized mathematics.

In 1974, Talagrand was admitted to the French National Centre for Scientific Research (CNRS), and he later proceeded to get Ph.D. from the University of Paris VI. He remained affiliated with CNRS until he retired in 2017. He spent a decade studying Functional analysis before going deep in to probability. It was then that Talagrand developed his influential theory about “Gaussian processes”, which made it possible to study some random phenomena like, temperatures, water level in a river, market swings. Talagrand studied how to estimate the maximum of such random measurements. He was able to use ideas of geometry to analyze what could be said about random measurements, with striking results.

Later, Dr. Talagrand became interested in a physics problem known as spin glasses, where there is a complicated interaction between individual magnets - an example of a spin glass would be iron atoms randomly mixed into a grid of copper atoms. Based on intuition, a physicist, Giorgio Parisi, had come up with a detailed description of how these disordered magnetic materials should behave. For this work Parisi shared a Nobel prize in Physics in 2021. After working on the problem for five years Dr. Talagrand made a simple observation providing a theoretical basis for Parisi's description.

His work, deepening the understanding of random phenomena, has become essential in today's world. His algorithms are now the basis of weather forecasts and major linguistic models. Interestingly, Dr. Talagrand also offers monetary prizes for solving problems posed by him. On his website, he proclaims, "Become RICH with my prizes," listing five problems.

Sources:

1. *https://www.sciencealert.com/abel-prize-winner-who-tamed-randomness-turned-to-math -out-of-necessity*
2. *https://www.nytimes.com/2024/03/20/science/abel-prize-mathematics-randomness.html*
3. *https://www.lemonde.fr/en/france/article/2024/03/25/michel-talagrand-an- unlikely-mathematical- superstar-i-wanted-to-take-risks_6650876_7.html*

3.5.2 Jorge Nocedal Awarded John Von Neumann Prize for work in Nonlinear Optimization

Prof. Jorge Nocedal has been awarded the 2024 John von Neumann Prize, the highest honor bestowed by the Society for Industrial and Applied Mathematics (SIAM), for his fundamental work in nonlinear optimization.

Prof. Nocedal, Professor of Industrial Engineering and Management and director of the Center for Optimization and Statistical Learning, was recognized for his fundamental work in nonlinear optimization, both in the deterministic and stochastic settings.

Nocedal's main area of research is optimization, with applications in machine learning, engineering design, and the physical sciences. His research activities range from the design of new algorithms to their software implementation and mathematical analysis. Nocedal tackles large-scale problems (with millions of variables), optimization under uncertainty, and machine learning.

A prolific author of academic papers, Nocedal was elected to the National Academy of Engineering in 2020. His career achievements, including vast contributions to the theory of nonlinear optimization methods and the creation of new, widely applied algorithms, earned him the George B. Dantzig Prize in 2012 from the Mathematical Optimization Society and the 2017 John von Neumann Theory Prize of the Institute for Operations Research and the Management Sciences (INFORMS).

SIAM awards the John von Neumann Prize annually to an individual for outstanding and distinguished contributions to the field of applied mathematics and for the effective communication of these ideas to the community. It is one of SIAM's most distinguished prizes.

Source: *https://www.mccormick.northwestern.edu/news/articles/2024/04/jorge-nocedal-awarded-john-von- neumann-prize-by-the-society-for-industrial-and-applied-mathematics/*

3.5.3 Peter Sarnak Awarded the 2024 Shaw Prize for Mathematical Sciences

Prof. Peter Sarnak, Eugene Higgins Professor of Mathematics at Princeton University has been awarded the 2024 Shaw Prize for Mathematical Sciences for developing the arithmetic theory of thin groups and the affine sieve, by bringing together number theory, analysis, combinatorics, dynamics, geometry and spectral theory.

The award is given annually by the Shaw Prize Foundation, which was founded in 2002 by the Hong-Kong-based philanthropist Run Run Shaw, in three disciplines, namely, Astronomy, Life Science and Medicine, and Mathematical Sciences, and comes with a monetary prize of 1.2 million US dollars.

Prof. Sarnak is currently Gopal Prasad Professor in the School of Mathematics at the Institute for Advanced Study and the Eugene Higgins Professor of Mathematics at Princeton University in the United States.

Prof. Sarnak was born in Johannesburg, completed his B.Sc. degree from The University of the Witwatersrand, Johannesburg, commonly known as Wits University or Wits, in 1974, and was awarded the Herbert Le May Prize for applied mathematics and the William Cullen Medal for the best graduate in the Faculty of Science in 1974. He did his Ph.D. at Standford University in 1980. He also received an honorary doctorate from Wits in 2014.

Prof. Sarnak is a member of the editorial boards of the top journals and has published over 120 research articles, several books, and has supervised numerous Ph.D. students. He is a member of the US National Academy of Sciences and Fellow of the Royal Society of London.

We may mention here that the Shaw Prize for Astronomy, of this year went to Shrinivas R Kulkarni, George Ellery Hale Professor of Astronomy and Planetary Science at the California Institute of Technology, an Indian-origin US scientist and brother of philanthropist Sudha Murty, for his discoveries about millisecond pulsars, gamma-ray bursts, supernovae, and other variable or transient astronomical objects.

Sources:

1. *https://www.wits.ac.za/news/latest-news/research-news/2024/2024-05/peter-sarnak-awarded-the- 2024-shaw-prize.html*
2. *https://www.newindianexpress.com/states/karnataka/2024/May/24/sudha-murtys- brother - gets-shaw-prize*

3.5.4 Prof. Simon Brendle Awarded the 2024 Breakthrough Prize in Mathematics

Prof. Simon Brendle, a professor of mathematics at Columbia University, New York has been awarded the Breakthrough Prize in Mathematics. Prof. Brendle was recognized for a series of remarkable leaps in differential geometry, a field that uses the tools of calculus to study curves, surfaces and higher dimensions manifolds.

Prof. Brendle is a German-American mathematician working in differential geometry and nonlinear partial differential equations. He received his Doctor Rerum Naturalium (Doctor of Natural Sciences) from Tübingen University under the supervision of Gerhard Huisken in 2001. He is currently a professor at Columbia University.

His numerous achievements in geometry include results on the Yamabe compactness conjecture, the differentiable sphere theorem, the Lawson conjecture and the Ilmanen conjecture, as well as singularity formation in the mean curvature flow, the Yamabe flow and the Ricci flow. He has received numerous awards and recognitions for his work, including the 2012 EMS Prize, the 2014 Bôcher Prize, and the 2017 Fermat Prize.

The Breakthrough Prizes, founded in 2012, are sponsored by foundations established by Sergey Brin, Priscilla Chan and Mark Zuckerberg, Julia and Yuri Milner, and Anne Wojcicki.

The Breakthrough Prizes recognize the research achievements of the world's top scientists, awarding approximately \$15 million annually in prizes. Each prize is \$3 million and presented in the fields of Life Sciences, Fundamental Physics and Mathematics. In addition, up to three New Horizons in Physics Prizes (\$1,00,000), up to three New Horizons in Mathematics Prizes (\$1,00,000), and up to three Maryam Mirzakhani New Frontiers Prizes (\$50,000) are given out to early-career researchers each year.

Sources:

1. *https://breakthroughprize.org/News/86*
2. *Simons Investigator Simon Brendle Awarded Breakthrough Prize in Mathematics (simons-foundation.org)*

3.5.5 Prof. Claire Voisin Awarded the 2024 Crafoord Prize in Mathematics

Prof. Claire Voisin, a prominent mathematician based at the Jussieu Institute of Mathematics in Paris, has been awarded the 2024 Crafoord Prize in Mathematics for her exceptional work in algebraic geometry. Notably, she is the first woman to receive this prestigious prize in mathematics. The Royal Swedish Academy of Sciences recognized Voisin's significant contributions to the field and her influence on the development of new mathematical theories and methods.

Voisin's work primarily revolves around the still-unsolved *Hodge conjecture*, which is one of the Millennium Prize Problems. Her expertise in this field has significantly advanced the understanding of complex algebraic varieties and their applications in various fields of mathematics and physics.

Besides the Hodge conjecture, Voisin has also delved into questions related to string theory. In particular, she has worked on the mirror symmetry conjecture inspired by string theory.

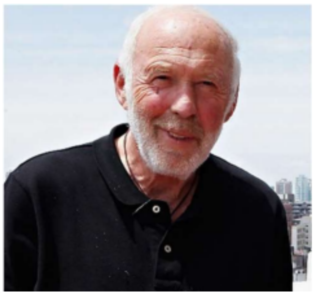
Voisin's journey in mathematics was not without personal influence. Her father played a crucial role in shaping her interest in traditional mathematics, developing a deep-seated love for the subject. Initially, she also had an interest in philosophy, which gradually merged with her passion for mathematics. The confluence of these two fields has perhaps contributed to her unique approach to problem-solving in algebraic geometry.

Voisin has always preferred working independently and asking her own questions. This independent streak and intellectual curiosity have undoubtedly contributed to her groundbreaking work in algebraic geometry, and her status as a leading figure in the field.

Source: *https://www.mathunion.org/cwm/news-and-events/2024-01-30/crafoord-prize-mathematics-2024-awarded-claire-voisin*

3.6 Obituary

3.6.1 Jim Simons, Mathematician, Philanthropist and Hedge Fund Founder, Passes away at 86



James "Jim" Simons, also known as the "Quant King", a renowned mathematician and pioneering investor who built a fortune on Wall Street and then became one of the biggest philanthropists, died on May 10, 2024 at the age of 86.

He was an exceptional leader who did transformative work in mathematics and developed a world- leading investment company. Together with his wife *Marilyn Simons*, Jim created an organization that has had enormous impact in mathematics, basic science, and our understanding of autism.

Simons' first career was in mathematics, for which he won praise. But in 1978, he moved over to Wall Street where he and his wife built over the years an estimated net worth of more than $30 billion, making him one of the 50 richest people in the world at the time of his death.

James Harris Simons was born in Newton, Massachusetts. He showed an attraction for mathematics and numbers at an early age. He got his undergraduate degree from MIT in 1958 and Ph.D. from the University of California, Berkeley, in 1961.

After spending some time teaching at MIT and Harvard University Simons took a job at the Institute for Defense Analyses in Princeton, New Jersey, as a code breaker for the National Security Agency. From 1968 to 1978, he was Chairman of the Mathematics Department of the State University of New York at Stony Brook. In 1976, Simons received the American Mathematical Society's Oswald Veblen Prize in Geometry, for research that was to prove to be influential in Physics, especially in string theory.

In 1978, Simons started his investment firm. He retired as CEO of the hedge fund in 2010, following which he focused on philanthropic work, through the foundation he and his wife founded in 1994 to support research in science, mathematics and education. Over the years, the couple donated billions of dollars to hundreds of philanthropic causes.

In 2023, their Foundation donated $500 million to the State University of New York at Stony Brook to support scholarships, professorships, research and clinical care.

In the Chronicle of Philanthropy's list of the biggest charitable donations from individuals or their foundations in 2023, Simon's name appears just at the second place, following Warren Buffet.
Source: *https://ca.finance.yahoo.com/news/james-simons-mathematician-philanthropist- hedge-222005991.html*

□□□

JUBILANT INDIAN TEAM AT 65th IMO-2024 BATH, UNITED KINGDOM July 11-22, 2024

Left-Right: Arjun Gupta (Grade 12 from Delhi), Adhitya Mangudy Venkata Ganesh (Grade 11 from Pune), Rushil Mathur (Grade 12 from Mumbai), Siddharth Choppara (Grade 12 from Pune), Kanav Talwar (Grade 10 from Noida), Ananda Bhaduri (Grade 12 from Guwahati)

Contestant [♀♂][←]	P1	P2	P3	P4	P5	P6	Total	Rank		Award
								Abs.	Rel.	
Ananda Bhaduri	7	7	2	7	7	2	32	14	97.86%	Gold medal
Siddharth Choppara	7	1	0	7	0	0	15	327	46.38%	Honourable mention
Arjun Gupta	7	7	2	7	0	1	24	100	83.72%	Silver medal
Adhitya Mangudy Venkata Ganesh	7	7	4	7	7	3	35	5	99.34%	Gold medal
Rushil Mathur	7	5	2	7	7	2	30	29	95.39%	Gold medal
Kanav Talwar	7	7	1	7	7	2	31	19	97.04%	Gold medal

We Congratulate the Indian Team for its outstanding performance and

The National Coordinator, Mathematical Olympiad: Prof. Prithwijit De, Homi Bhabha Centre for Science Education (HBSCE), TIFR, Mumbai,

Mentors: Prof. Krishnan Sivasubramanian of IIT Bombay, and former IMO medallists Rijul Saini of HBCSE and Rohan Goyal currently a Ph D student at MIT, USA, and

all others who are directly or indirectly involved in motivating and training the Indian Team.

4. Problem Corner

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In the January 2024 issue of TMC Bulletin, we posed two problems: a problem from Algebraic inequality and a problem from Number theory. We received three solutions to the problem on inequality, one by Prof. M. R. Modak which is elementary, accessible to high school students, and the two other solutions by calculus method by the problem proposer Dr. Vinay Acharya and Prof Jagannath Salunke. We present below two solutions one by Prof. Modak and other by Dr. Acharya as these solutions are more elegant as compared to the third solution.

The problem from Number Theory was proposed by Prof. Rajendra Pawale from Mumbai University, Mumbai. We also received two solutions to this problem, a short solution by Prof. Jagannath Salunke and the other by Prof. Pawale himself. Both these solutions are also presented below.

Prof. Salunke also provided a solution to the second problem posed in the October 2023 Issue of TMCB. We will present that solution in the next issue.

In this issue we pose two problems, one from Number Theory and one from Geometry. The geometry problem is proposed by a 10^{th} standard student Shubhankar Dod of Pune. Readers are invited to email their solutions to Dr. Udayan Prajapati (udayan.prajapati@gmail.com), Coordinator, Problem Corner, before 1^{st} September, 2024. Most innovative solution will be published in the subsequent issue of the bulletin.

1. Problem posed in the previous issue by Dr. Vinay Acharya, Fergusson College, Pune:

Prove that $\frac{1}{\sqrt{1+x^2}}+\frac{1}{\sqrt{1+y^2}} \leq \frac{2}{\sqrt{1+xy}}$, (1)

for $0 \leq x, y \leq 1$.

First Solution (by Prof. M. R. Modak, Pune): Let $x, y \in [0,1]$. By Root Mean Square inequality,
$\frac{1}{\sqrt{1+x^2}}+\frac{1}{\sqrt{1+y^2}} \leq \sqrt{2\left(\frac{1}{1+x^2}+\frac{1}{1+y^2}\right)}$

Now as $x, y \geq 0$, $\sqrt{2\left(\frac{1}{1+x^2}+\frac{1}{1+y^2}\right)} \leq \frac{2}{\sqrt{1+xy}} \Leftrightarrow \frac{1}{1+x^2}+\frac{1}{1+y^2} \leq \frac{2}{1+xy}$

$\Leftrightarrow (2+x^2+y^2)(1+xy) \leq 2(1+x^2)(1+y^2)$

$\Leftrightarrow 2+x^2+y^2+2xy+x^3y+xy^3 \leq 2+2x^2+2y^2+2x^2y^2$

$\Leftrightarrow x^2+y^2-2xy-x^3y-xy^3+2x^2y^2 \geq 0$

$\Leftrightarrow (x-y)^2-xy(x-y)^2 \geq 0$

$\Leftrightarrow (x-y)^2(1-xy) \geq 0$,

which is true since $0 \leq x, y \leq 1 \Rightarrow xy \leq 1$. Hence (1) follows.

Second Solution (by Dr. Vinay Acharya): (1) is true if either $x=0$ or $y=0$. Now assume x and y are positive and chose real numbers u and v such that $x=e^{-u}$ and $y=e^{-v}$. Because x and y are both at most 1, u and v must both be non-negative. Substituting for x and y in terms of u and v, we see that it suffices to prove that

$\frac{1}{\sqrt{1+e^{-2u}}}+\frac{1}{\sqrt{1+e^{-2v}}} \leq \frac{2}{\sqrt{1+e^{-(u+v)}}}$, for non-negative u and v.

Consider the function $f:\mathbb{R}\to\mathbb{R}$ given by $f(t)=\frac{1}{\sqrt{1+e^{-2t}}}$.

Then we need to prove that $f(u)+f(v) \leq 2f((u+v)/2)$, for $u, v \geq 0$.

To do this, all we need to show is that f is a concave function on the interval $[0,\infty)$.

We thus find the second derivative of f and show that it is negative on $[0,\infty)$.

$$\begin{aligned}
f'(t) =& (1+e^{-2t})^{-\frac{3}{2}}e^{-2t} \\
f''(t) =& 3(1+e^{-2t})^{-\frac{5}{2}}e^{-4t} - 2(1+e^{-2t})^{-\frac{3}{2}}e^{-2t} \\
=& \frac{3-2e^{2t}(1+e^{-2t})}{(1+e^{-2t})^{\frac{5}{2}}e^{4t}} = \frac{1-2e^{2t}}{(1+e^{-2t})^{\frac{5}{2}}e^{4t}}
\end{aligned}$$

The denominator of the last expression is certainly positive, while the numerator is negative because $e^{2t} \geq 1$ for $t \geq 0$. Thus, $f''(t) < 0$ for $t \geq 0$ and f is indeed concave for $t \geq 0$. This completes the proof.

2. Problem posed in the previous issue by Prof. Rajendra Pawale, Mumbai University, Mumbai:

Show that $(q^t+1)^2 - 4eq^t(q+1)(q^{t-1}+1) \qquad (2)$
is not a perfect square, where q is a positive odd prime power and $e > 0, t \leq 2$ are integers.

First Solution (by Jagannath Salunke): For a prime number $(q \geq 3)$ and integers $e \geq 1$ and $t \geq 2$ we have
$4eq^t(q+1)(q^{t-1}+1) > q^t(q+1)(q^{t-1}+1) = q^{2t} + q^{2t-1} + q^{t+1} + q^t > q^{2t} + 2q^t + 1 = (q^t+1)^2.$
So, $(q^t+1)^2 - 4eq^t(q+1)(q^{t-1}+1)$ is a negative integer and hence not a perfect square.

The assumption that q is a prime is not required in this proof.

Second Solution (by Prof. Rajendra Pawale): Take $D = (q^t+1)^2 - 4eq^t(q+1)(q^{t-1}+1) = x^2$ for some positive integer x, then

$$x^2 - 1 = (q^{2t} + 2q^t) - 4eq^t(q+1)(q^{t-1}+1) = q^t(q^t + 2 - 4e(q+1)(q^{t-1}+1))$$

As q is an odd integer, observe that q^t divides either $x-1$ or $x+1$. (If $q = p^a$, p prime, then p divides exactly one of $x-1$ and $x+1$.)

If $x = q^t z + 1$, for some positive integer z, then $D - x^2 = q^t f(z)$, where

$$f(z) = -4eq^{t-1} - 4eq^t + q^t - 4eq - 4e - z^2q^t - 2z + 2.$$

Observe that $f(z) \leq f(1) = -4e(q+1)(q^{t-1}+1) < 0$, which implies that $D - x^2 < 0$, which is a contradiction.

If $x = q^y z - 1$, for some positive integer z, then $D - x^2 = q^t g(z)$, where

$$g(z) = -4eq^{t-1} - (z^2 + 4e - 1)q^t - 4eq - 4e + 2z + 2.$$

As in the previous case $g(z) \leq g(1) = 4 - 4e(q+1)(q^{t-1}+1) < 0$, which Implies that $D - x^2 < 0$, which is a contradiction.

Problems for this issue

Problem 1: Show that the equation $x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 = 23456789$ has no integer solution.

Problem 2: Let ABC be a triangle and Γ_{ab} and Γ_{ac} be the circles touching BC and passing through A, B and A, C respectively, intersect at A and P. Similarly, let Q and R be the points of intersection of Γ_{ba}, Γ_{bc} and Γ_{ca}, Γ_{cb} respectively. Prove that P, Q, R, G, H are concyclic and their center lies on the Euler Line. Here G and H are the centroid and the orthocenter of the triangle ABC respectivly.

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5. International Calendar of Mathematics Events

Ramesh Kasilingam
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August 2024

- August 12-16, 2024, International Workshop on Operator Theory and its Applications (IWOTA), University of Kent, Canterbury, United Kingdom. *blogs.kent.ac.uk/iwota2024*
- August 14-16, 2024, The Fifth Information-Theoretic Cryptography (ITC) Conference, Stanford University, California, USA. *itcrypto.github.io/2024/index.html*
- August 19-23, 2024, Topology of Moduli Spaces, University of Copenhagen, 5 DK-2100 Copenhagen. *https://www.math.ku.dk/english/calendar/events/topology-of-moduli-spaces/*
- August 26-30, 2024, The International Conference on Free Boundary Problems: Theory and Application, Universidade Federal Da Paraíba, João Pessoa, Paraíba/Brazil. *www.mat.ufpb.br/fbp2024*
- August 26-30, 2024, Introductory Workshop: New Frontiers in Curvature, SLMath 17 Gauss Way, Berkeley, CA 94720 US. *www.slmath.org/workshops/1093#overview_workshop*

September 2024

- September 2-7, 2024, XIV Annual International Conference of The Georgian Mathematical Union, Batumi, Georgia, Batumi Shota Rustaveli State University. *gmu.gtu.ge/conferences/*
- September 3-6, 2024, Path Integrals and Friends, Helsinki, Finland. *www.helsinki.fi/en/conferences/path-integrals-and-friends*
- September 4-7, 2024, IFSCOM-E 2024 International IFS and Contemporary Mathematics and Engineering Conference, Mersin University, Mersin, Türkiye. *ifscom.com/*
- September 9-13, 2024, Ramification in Geometric Langlands and Non-Abelian Hodge Theory Universität Heidelberg, Germany. *sites.google.com/view/ramificationheidelberg2024/*
- September 30 - October 2, 2024, Workshop on Analysis and PDE, Leibniz University Hannover, Germany. *www.math-conf.uni-hannover.de/anapde24.html*

October 2024

- October 2-4, 2024, Kocaeli Science Congress, Kocaeli University, Umuttepe Campus, Kocaeli /Turkey. *fefkongre.kocaeli.edu.tr/en*
- October 5-6, 2024, 2024 Fall Southeastern Sectional Meeting, Georgia Southern University, Savannah, Georgia. *www.ams.org/meetings/sectional/2315_program.html*
- October 26-27, 2024, 2024 Fall Western Sectional Meeting, University of California, Riverside, Riverside, CA. *www.ams.org/meetings/sectional/2304_deadlines.html*
- October 28-31, 2024, 2^{nd} International Conference on Differential Geometry (ICDG- FEZ'2024), Sidi Mohamed Ben Abdellah University, Faculty of Sciences, Dhar El Mahraz, Fez, Morocco. *www.fsdm.usmba.ac.ma/ICDGFEZ2024/*
- October 30 - November 2, 2024, National Diversity in STEM Conference 2024 (NDiSTEM2024), Phoenix Convention Centre, 100 North Third Street, Phoenix, AZ 85004. *s6.goeshow.com/sachn/ndsc/2024*

November 2024

- November 13-14, 2024, 7^{th} International Conference on Frontiers in Industrial and Applied Mathematics (FIAM-2024), Central University of Punjab, Bathinda, Punjab, India. *fiam2024.cup.edu.in*

- November 29 - December 1, 2024, International Conference on Matrix Analysis and Mathematical Modelling (MAMM 2024), Dr B R Ambedkar National Institute of Technology Jalandhar, Punjab, India. *www.nitj.ac.in/MAMM2024/*

December 2024

- December 2-6, 2024, 46^{th} Australasian Combinatorics Conference, The University of Queenslande, Brisbane, Australia. *46acc.github.io/*
- December 5-7, 2024, CIPMSL (Interdisciplinary Perspectives in Mathematics, Science, and Linguistics, VIT Bhopal University, Bhopal, India. *drive.google.com/file/d/1VpmJWMTg5P-oG_mdmgxCu3-CRL86xD7J/view?usp=shar ing*
- December 8-11, 2024, Asian Technology Conference in Mathematics, Universitas Negeri Yogyakarta, Yogyakarta, Indonesia. *www.atcm.mathandtech.org/*
- December 9-13, 2024, Harmonic Analysis and Convexity, Institute for Computational and Experimental Research in Mathematics, Brown University, Providence, RI, USA. *icerm.brown.edu/topical_workshops/tw-24-hac/*
- December 9-13, 2024, Hot Topics: Life After The Telescope Conjecture, SL Math, 17 Gauss Way, Berkeley, CA 94720. *www.slmath.org/workshops/1103#overview_workshop*
- December 9-13, 2024, Foliations and Diffeomorphism Groups Feuilletages et Groupes de Difféomorphismes, *https://conferences.cirm-math.fr/3082.html*
- December 10-12, 2024, (MathConnect 2024) International Conference of Mathematics and its Applications, King Fahd University of Petroleum & Minerals, Dhahran, Saudi Arabia. *events.kfupm.edu.sa/event/217/*
- December 26-30, 2024, International Conference on Fractional Calculus and Applications, Sousse - Tunisia. *icofca.com/*

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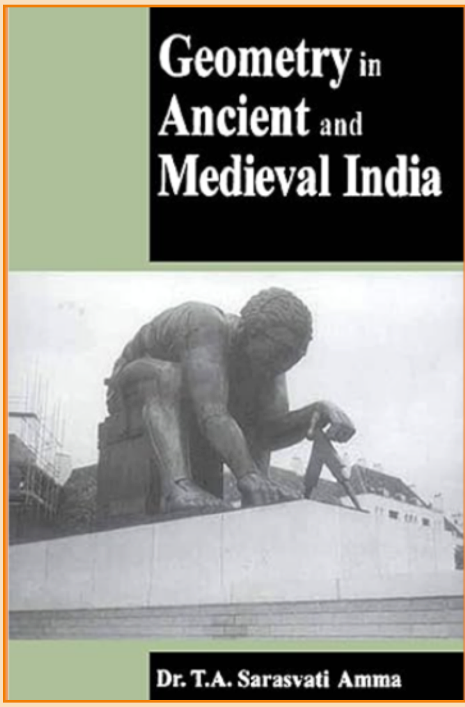
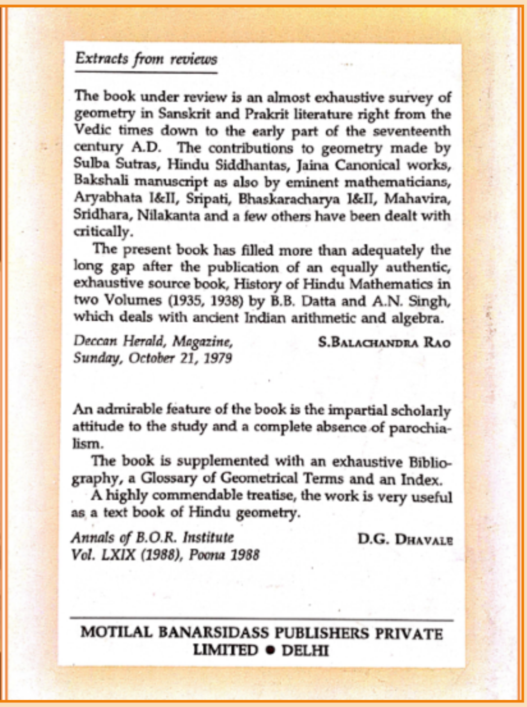
Sarasvati Amma's book (1999-Edition)





Extracts from reviews

The book under review is an almost exhaustive survey of geometry in Sanskrit and Prakrit literature right from the Vedic times down to the early part of the seventeenth century A.D. The contributions to geometry made by Sulba Sutras, Hindu Siddhantas, Jaina Canonical works, Bakshali manuscript as also by eminent mathematicians, Aryabhata I&II, Sripati, Bhaskaracharya I&II, Mahavira, Sridhara, Nilakanta and a few others have been dealt with critically.

The present book has filled more than adequately the long gap after the publication of an equally authentic, exhaustive source book, History of Hindu Mathematics in two Volumes (1935, 1938) by B.B. Datta and A.N. Singh, which deals with ancient Indian arithmetic and algebra.

Deccan Herald, Magazine, Sunday, October 21, 1979 — S. Balachandra Rao

An admirable feature of the book is the impartial scholarly attitude to the study and a complete absence of parochialism.

The book is supplemented with an exhaustive Bibliography, a Glossary of Geometrical Terms and an Index.

A highly commendable treatise, the work is very useful as a text book of Hindu geometry.

Annals of B.O.R. Institute Vol. LXIX (1988), Poona 1988 — D.G. Dhavale

MOTILAL BANARSIDASS PUBLISHERS PRIVATE LIMITED ● DELHI

The book has been a pioneering work in the history of geometry in ancient and medieval India, based on Sanskrit and Prakrit sources. It has greatly influenced the studies in the area in the last about half a century since it emerged, and continues to be a very relevant and unique reference in the area. It represents the author's endeavour for a doctoral thesis, as a Research Scholar at the University of Madras during 1957 - 60, under the guidance of the renowned Sanskrit scholar Prof. V. Raghavan; the thesis was however submitted only in 1963 to the Ranchi University (then in Bihar), securing her the degree in 1964. Its accessibility to wider readership got delayed, however, for various reasons, until it was published by Motilal Banarasidass Publishers, in 1979. A second edition, whose cover is reproduced above, was brought out by the publishers in 1999, which has been much cited in literature; the publishers have also brought out another edition recently, in 2017.

The book provides an excellent introduction to the topic, to students and interested readers, and a rich source of historical material, with deep insightful comments, to researchers. It contains a systematic and lucid exposition of the mathematical contents of various sources, including the Sulbasutras from the Vedic literature, the Jaina Canonical works, works in the long-lived Siddhanta tradition of mathematical astronomy by Aryabhata, Brahmagupta, Bhaskara I, Bhaskara II, Narayana Pandita and numerous others from the tradition, works of Sridharacharya and Mahaviracharya, with its climax in the works from the Kerala school of mathematics, of Madhava, Paramesvara, Nilakantha, Jyesthadeva and a host of others, known for their contributions along the general theme which is now recognized as calculus.

Georges Henri Joseph Édouard Lemaître (17 July 1894 - 20 Jun. 1966)

A Belgian mathematician and astronomer who worked on the theory of the expanding universe. He was the first to derive the Hubble–Lemaître law, and the first to estimate the Hubble constant. He also proposed "hypothesis of the primeval atom" which later became known as the "Big Bang theory". With Manuel Sandoval Vallarta, he worked on a theory of primary cosmic radiation. Lemaître was also an early adopter of computers for cosmological calculations.

Sir George Gabriel Stokes (13 Aug. 1819 - 01 Feb. 1903)

An Anglo-Irish physicist and mathematician who was a Lucasian Professor of Mathematics (1849-1903) at Cambridge. He made seminal contributions to fluid mechanics and physical optics. Also worked on polarization and fluorescence. Known for the Navier– Stokes equations, Stokes' theorem in vector calculus, and his contribution to the theory of asymptotic expansions. He was made a baronet (hereditary knight) by the British monarch in 1889.

Constantin Carathéodory (13 Sept. 1873- 02 Feb. 1950)

A Greek mathematician who spent most of his professional career in Germany. He made significant contributions to real and complex analysis, the calculus of variations, and measure theory. He also created an axiomatic formulation of thermodynamics. Known for Carathéodory's theorem and conjecture in convex geometry, Carathéodory extension theorem in measure theory, and his work in Optics. Edited two volumes of Euler's complete works.

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