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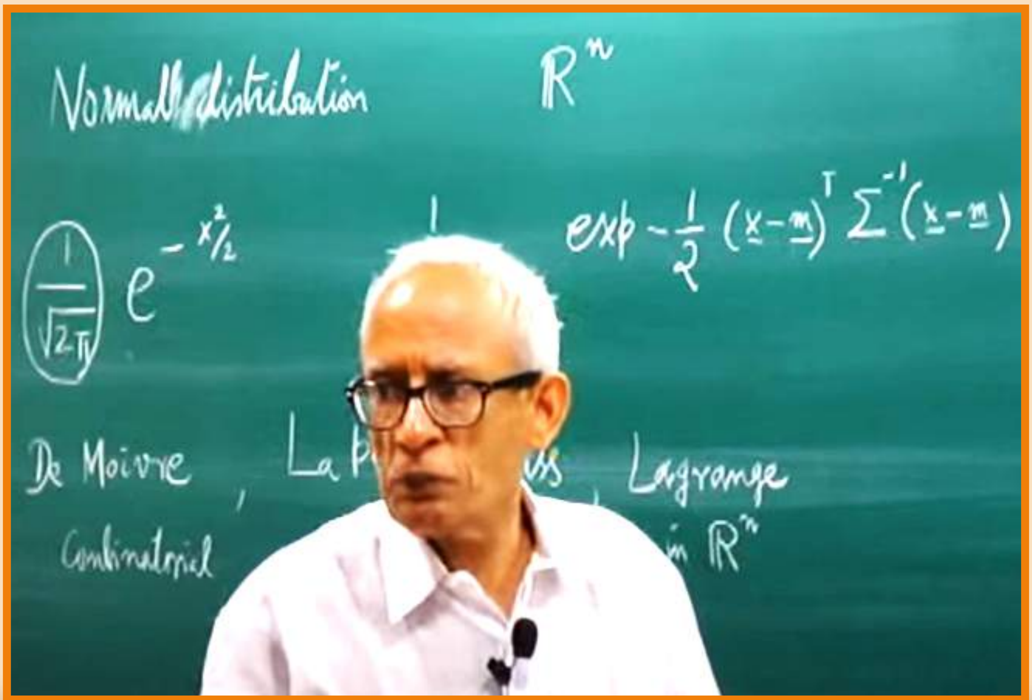
A special issue offering ardent tributes to

Professor K. R. Parthasarathy



(25 June 1936 - 14 June 2023)

Editors-in-charge

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The Mathematics Consortium

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Contents

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From the Editors' Desk

K. R. Parthasarathy, an internationally renowned doyen of mathematics from India, who has inspired generations of mathematicians, and in particular pioneered the area Quantum Stochastic Calculus, passed away on June 14, 2023. A tribute was paid in The Mathematical Consortium Bulletin (TMCB) on the occasion, in its July 2023 issue, with an article contributed by one of us (KBS), who has had the privilege of being a long-time colleague and collaborator of his. At that time it was also decided to bring out a special issue of TMCB in his memory, introducing to the readership the highlights of his contributions to mathematics and the mathematical community.

Parthasarathy (known also as Partha in the west and KRP in India) began his research career around 1960 and continued to be active almost till the end. During his career he made profound and lasting contributions on various topics including classical probability theory, Lie algebras and representations, probability distributions on Lie groups, central limit theorems and continuous tensor products, Mackey's theory of imprimitivity, perturbation theory, quantum stochastic calculus and quantum probability theory. He had a long string of students and a large number of collaborators across several countries, who all admired him as a teacher, mentor, or a mathematician sharing generously his ideas in attaining deeper understanding of mathematical issues and concepts they engaged in. There have also been others, including one of us (SGD) who have been involved in pursuing topics in which KRP left his mark, sometimes in a short span. This prompted the idea of inviting various experts associated with him in one or other way to contribute expositions on suitable topics that engaged KRP and were pursued by them. And it worked! The invitees, without exception, agreed to be enlisted for the endeavour; we may further add that they duly delivered their contributions well in time, so that we could have them processed for timely publication of the special issue as was conceived, without causing delay or warranting any last minute changes. And here is the result, which we are pleased to present to the readers.

In presenting the contributions we have tried to organize them, by and large, in the order in which the research-canvas of KRP evolved over time, chronologically; as should be clear, there are limitations to adopting such a principle, on account of the interrelations in the material discussed by various experts, and the complex connections that would have been at play when the research progressed in the first place.

The first article, by R. L. Karandikar and B. V. Rao, tells us about KRP's early years as a researcher, commencing the narrative with his first day in the Indian Statistical Institute, Kolkata. After recalling two problems discussed in his thesis, the article goes on to underline the deep influence of Kolmogorov's consistency theorem on KRP and his close learning/research collaboration with the other three of the "famous four" (V. S. Varadarajan, K. Ranga Rao and S. R. S. Varadhan). The article also discusses his work on the Lévy-Khinchine representations in the framework of locally compact groups, and its impact on the subsequent developments.

The second article by Apoorva Khare is centred on the "PRV conjecture" in KRP's paper with Ranga Rao and Varadarajan in Annals of Mathematics (1967). Following the pioneering study by Harish-Chandra of certain irreducible representations of Lie groups over Banach spaces the authors, restricting themselves to complex Lie groups, achieved a better understanding of the modules, proving existence of "minimal types", now called PRV components. The article concludes with a discussion of factorisable representations of current groups, another area that interested KRP, on which more will be said in this issue by Klaus Schmidt.

Following his earlier study on Lévy-Khinchine representations for locally compact abelian groups, KRP initiated study of infinitely divisible distributions in the wider framework of not necessarily locally compact second countable groups, and especially Lie groups, raising the question of their embeddability in one-parameter convolution semigroups. In Article 3, S. G. Dani, who has pursued the topic further, recalls the ideas introduced by KRP on the theme and their impact on later work.

The study of infinitely divisible distributions together with certain results of Araki-Woods and Streater, inspired KRP, in the mid-1970s, to look for an analogue of Lévy-Khinchine formula for limits of products of uniformly infinitesimal arrays of positive definite functions over general locally compact second countable groups. In Article 4, Klaus Schmidt, who collaborated with KRP on the theme, recounts the story of the developments, bringing out in particular its connection with factorisable representations.

Article 5 by Gadadhar Misra is, in a sense, on the dividing line of KRP's research-choreography between its "Classical" and "Quantum" phases. After preparing the reader with notions of projective unitary/anti-unitary representations and Wigner's theorem in the backdrop, the article discusses in detail KRP's generalization of Mackey's imprimitivity theorem; while Mackey's theorem concerns only projective unitary representations, the generalisation applies also to projective anti-unitary representations, and the article highlights the essential difficulties involved in the generalization.

KRP is best known in the wider world for the creation of Quantum Stochastic Calculus (QSC), jointly with Robin Hudson. Article 6 by Luigi Accardi presents a vivid personalized account of its emergence at the hands of Hudson and KRP, to which the author was initially a keen witness, until subsequently becoming a participant. The thought processes that evolved in the course of its development unfold step by step in the article. The deep influence of Kolmogorov on KRP, and that of KRP on the author are found interwoven in the narrative.

Kalyan Sinha has the distinction of being the living mathematician with the largest number of joint research papers with KRP, through a sustained collaboration lasting over two decades; it may not be out of place to mention here that Robin Hudson, the joint creator of QSC mentioned earlier with whom KRP had the largest number of joint papers, passed away in 2021 - else he would surely have been one of the contributors to this issue. In article 7, Sinha gives a glimpse of the phases and highlights of the mathematical developments through their intensive collaboration, acknowledging along the way the deep mathematical influence KRP had on him.

Article 8 is by B. V. Rajarama Bhat, one of KRP's illustrious former Ph.D. students. He writes about KRP's work on the post- Quantum Stochastic Calculus period, beginning with the construction of weak Quantum Markov processes. Contents of a few of KRP's articles on extreme points of stochastic maps on C*- algebras and extremal Quantum decision rules are also discussed, ending with KRP's later interest in Quantum computation.

In a short essay written in 2010, KRP illustrated the notion of a quantum Gaussian state as a natural generalization of the Gaussian distribution in classical probability theory. Various problems he proposed there, calling for further research, were to occupy him in the subsequent years. He contributed many papers on these questions, on the broad theme of generalized finite mode Quantum Gaussian States and their symmetries; some of these are collaborative works, but many are with him as the sole author, including his last paper published as recently as 2022; in a sense this was his last major topic of his research. In article 9 Franco Fagnola discusses the contents of these papers and the broad perspective underlying them.

Apart from the mathematical fare, we also present to the reader, reminiscences of S. R. S. Varadhan and Rajendra Bhatia - two names that are closely associated with KRP -, an introduction to KRP's long journey through mathematics, and many glimpses of the remarkable persona through photographic images.

We thank the Editorial Board of TMCB for presenting us this opportunity of bringing out this special issue in memory of KRP, which we trust will go a long way in providing a comprehensive introduction to his contributions to mathematics and to the mathematical community, especially the younger generation. We thank all the contributors and the referees for their time and efforts. We thank Mrs. Shyamala Parthasarathy and family for generously sharing with us many photographs and biographical details of KRP. We also thank Professors Vijay Pathak and S. A. Katre for their help in editing the issue, the designers Mrs. Prajkta Holkar and Dr. R. D. Holkar, and all those who have directly or indirectly helped us in the timely production of this special issue dedicated to Professor K. R. Parthasarathy.

Editors-in-charge

1. K. R. Parthasarathy - Early Years as a Researcher

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In 1956 a young man from Madras (now, Chennai) reaches ISI, Calcutta (now, Kolkata) to continue his studies. He was late by a day to join the course. When he attended the first class, there was an ordinary looking man lecturing and conducting practical session. When asked if he brought practical note book, the young man replied that he did not know and would get it later. The teacher was stern and told him that even if he is late, he should have found out before attending the class. The young man who did not like the stern warning, went to complain to the director C. R. Rao. But on entering the office, he quickly turned back. That ordinary looking teacher was C. R. Rao and the young man was K. R. Parthasarathy (KRP).

KRP contributed significantly, and established himself as a leading player in several areas. The present article covers some of his contributions in the beginning years of his career.

After completing the three-year advanced statistician's training course, he approached C. R. Rao (CRR) for a problem to work on. CRR advised him 'learn information theory'. This advice of CRR turned out to be prophetic; not only was KRP's thesis topic information theory, on problems he asked himself after reading Khinchine's book; he returned to information theory in his last years, but now to Quantum information Theory.

Here are two of the problems discussed in his thesis. Consider a finite set A and the two sided infinite product space $A^I = \mathop{\times}\limits_{-\infty}^{\infty} A_i$ where each $A_i = A$. This is a compact metric space when each coordinate is equipped with discrete topology. Points in this space are denoted by $x = (x_n : -\infty < n < \infty)$. The coordinate random variables are denoted by $(X_n : -\infty < n < \infty)$ and thus $X_n(x) = x_n$ for $x \in A^I$. The shift transformation on the space is denoted by T. Thus if $x = (x_n)$, then $T(x) = y$ where $y_n = x_{n-1}$. Let μ be a probability on A^I which is invariant under T (stationary channel). For each n-tuple $\langle x_1, \dots, x_n\rangle$, we denote $\mu(\langle x_1, \dots, x_n\rangle) = \mu(X_i = x_i; 1 \le i \le n)$. Define

$$K(\mu) = -\lim_{n\to\infty} \frac{1}{n} \sum_{\langle x_1,\dots,x_n\rangle} \mu(\langle x_1, \dots, x_n\rangle) \log \mu(\langle x_1, \dots, x_n\rangle),$$

where the sum is over all n-tuples of the alphabet A. For a stationary μ (channel) this limit exists and is known as rate per letter of the source.

KRP proved, among other things, that there is a function h on A^I such that $K(\mu) = \int h d\mu$ for any stationary μ. The analysis depends on the Kryloff-Bogoliouboff theory as explained by Oxtoby [17]. Here is the essential starting point of this latter theory. Let T be a homeomorphism of a compact metric space Ω. Firstly, using the Birkhoff Ergodic theorem and the fact that $C(\Omega)$ is separable, we can see the following: Given any ergodic (for T) probability μ, there is at least one $p \in \Omega$ such that ergodic averages of all f in $C(\Omega)$ converge. Secondly, the Riesz representation theorem now allows you to read ergodic measure μ through such points $p \in \Omega$. Finally one knows that any invariant measure is a mixture of ergodic measures. These allow one to get such a function h.

In a subsequent paper he shows, among other things, the following. Consider the bilateral product $\mathop{\times}\limits_{-\infty}^{\infty} M_i$ of a complete separable metric space $M = M_i$ (for all i) along with the shift transformation T. Then the set $\mathfrak{M}_e$ of all ergodic measures is a dense G_δ in the space of probabilities under weak topology. Further the set of periodic measures is dense in the set of ergodic measures. Such an evaluation of the size of sets is along the lines of earlier results of Halmos on the size of weakly mixing transformations and of Rokhlin on the set of strongly mixing transformations and of Kakutani on the size of the Quasiregular points (points at which the ergodic averages converge for each continuous function). Oxtoby who had earlier given an exposition of the Kryloff-Boguliouboff theory appreciated this result of KRP and showed later that the results hold in general separable Borel spaces.

V. S. Varadarajan felt that Kolmogorov consistency theorem is very fundamental and one should be able to derive many results using this theorem. He himself used it to prove the Riesz

Representation theorem for positive linear functionals on $C(X)$ for locally compact Hausdorff space X [18]. The idea is the following. For finite product of two point set $\{0,1\}$ it is hand calculation; for uncountable product X of two point space, Kolmogorov consistency theorem does the job. Then using that a compact Hausdorff space is continuous image of a closed subset of such an X one deduces the result; finally for locally compact Hausdorff spaces one argues locally on each open set with compact closure and pieces the measures defined on these compact sets together. KRP, collaborating with Bingham proved the Bochner theorem on positive definite functions on the dual of a locally compact abelian group — via Kolmogorov consistency Theorem [2]. Again the first step is to prove it for T^n, a finite product of unit circle by hand calculation and then use Kolmogorov consistency theorem for uncountable products of T. Finally the general result is deduced from this by an interesting argument. In a subsequent paper, KRP uses this probabilistic technique to prove, among other things, Pontryagin duality theorem for locally compact abelian groups.

Suppose we have a process X_t for $0 \leq t \leq 1$ which is relatively stationary in the following sense: If $n \geq 1$ and $(t_1, \dots, t_n)$ and $(t_1 + h, \dots, t_n + h)$ are all in $[0,1]$ then $(X_{t_1}, \dots, X_{t_n})$ and $(X_{t_1+h}, \dots, X_{t_n+h})$ have the same distribution. Question: Can we extend this as a stationary process to all of the real line? In other words can we find a stationary process $(Y_t : t \in R)$ such that $(Y_t, 0 \leq t \leq 1)$ and $(X_t, 0 \leq t \leq 1)$ have the same distribution? KRP, collaborating with Varadhan, showed in [9] that the answer is affirmative when the given process is real valued and is continuous in probability. They first prove a Hahn-Banach type theorem on extension of a positive linear functional which is invariant under action by the group of rationals. Interestingly enough as a consequence they obtain, using Gaussian processes, the following theorem of M. G. Krein: A continuous positive definite function given on $[-1,1]$ can be extended as a continuous positive definite function on all of R. This problem of extending stationary processes was taken up nearly thirty years later by Kamm in her thesis (see [4]) in the context of Hausdorff space valued process. However the basic questions whether the extension is unique, whether there is an Ergodic extension, or whether similar result holds for random fields etc. appear unanswered.

From the deep study, by KRP, Ranga Rao and Varadhan, of probability measures on locally compact second countable abelian groups [8], especially the Lévy-Khinchine Representation, it follows that an infinitely divisible distribution can be embedded in a convolution semigroup. KRP takes up this embedding problems for not necessarily abelian groups in [11] and [14] and frees the problem from the existence of Lévy-Khinchine representation. For instance he proves that this can be done after a shift, on a compact second countable group if the measure has no idempotent factors. This problem, of embedding an infinitely divisible probability in a one parameter semigroup of probabilities, was later taken up by several authors including S. G. Dani and M. McCrudden.

In an appendix to the book of Kolmogorov and Gnedenko [5], Doob makes an interesting statement. Since the limit distributions for sums of independent random variables are usually discussed through convolutions of characteristic functions, it is possible to develop a theory, not mentioning the word random variable, but restricting to positive definite functions. KRP, collaborating with K. Schmidt took this up in a wider context of tensor products of Hilbert spaces and discussed limits of products of uniformly infinitesimal families of positive definite kernels.

The work on Locally Compact Abelian groups, jointly with Ranga Rao and Varadhan is a seminal contribution. There were some studies earlier on extending the study of probability distributions and weak convergence on structures beyond the reals. In [8] the authors provide definitive extensions, for LCA groups, of (i) Lévy-Khinchine Representation theorem for infinitely divisible distributions, (ii) criteria for weak convergence of i.d.laws and (iii) Khinchine's theorem on sums of infinitesimal summands.

The book *Probability Measures on Metric Spaces* [10], published in 1967 is an ever green and lasting contribution for both researchers and students. This book has over 4500 citations on Google Scholar, unusual for a research level book in mathematics that is not a text book. While visiting Sheffield, KRP gave a course for PhD students and wrote notes on the subject. Eugene Lukacs, visiting the department, saw the notes and apparently said, "I am the editor of a series run by

Academic Press, can I have these notes for publication".

The book is based on work of the group of four- V. S. Varadarajan, Ranga Rao, Varadhan and KRP. The four had met when they were working towards their PhD at Indian Statistical Institute, Kolkata, and were there for some time subsequently. Though the book is called *Probability Measures on Metric Spaces*, it has lot of material going beyond metric spaces. Perhaps the name was suggested after the contents were written.

In the 1950s, going beyond Euclidean spaces, the focus on measures was on locally compact topological spaces. It was clear that from probability theory point of view, when dealing with stochastic processes, one had to go beyond locally compact spaces, as even $C[0, 1]$ is not locally compact, while Wiener had constructed a measure on this space (later called the Wiener measure).

Doob, Donsker and others had started working on ways to extend approximation theorems such as Central limit theorem to i.i.d. observations from a stochastic process, which required them to deal with $C[0, 1]$ and $D[0, 1]$ - the space of right continuous functions having left limits. The Russian school of probability had started working on convergence in distribution for random variables taking values in a metric space.

Varadarajan had worked on convergence in distribution of stochastic processes, presumably unaware of Prohorov's work published few years earlier in Russian. Varadarajan's work appears with details in KRP's book, for $C[0, 1]$ and also for $D[0, 1]$ valued stochastic processes. Around the same time as publication of this book (1967), Billingsley's book Convergence of Probability Measures [1] was also published. While Billingsley's focus in the book was weak convergence of measures, KRP's book connects weak convergence to topological aspects of the set of measures on a metric spaces.

As Dudley says in his review [3] of KRP's book this was written at just the right time when the subject matter reached the appropriate degree of maturity; carefully and cogently done. As Tom Liggett points out in his review [16], much of the material presented is made available here for the first time in book form.

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□ □ □

With Prof. C. R. Rao (↑) and with Prof. Ranga Rao (↓) (in later days)

(Photos courtesy of Mrs. Shyamala Parthasarathy)

2. KRP's Contributions in Lie Theory

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In Lie theory, perhaps the most well-known contribution of K. R. Parthasarathy (henceforth termed "KRP") is the paper [11] with Ranga Rao and Varadarajan, following their announcement [10]. The initial part of this paper was worked out also with S.R.S. Varadhan (see [14] for an account of the development of this work); the paper has subsequently often been referred to as the "PRV paper".

The PRV paper arose out of the grand program of Harish-Chandra on the representation theory of real connected semisimple Lie groups. Let G be such a group and K a maximal compact subgroup. Harish-Chandra [3, 4] pioneered the study of certain irreducible Banach-space representations of G that are "K-finite", i.e., direct sums of finite-dimensional K-modules. In a historical sense, this program follows other landmark works on representations of groups: the Peter–Weyl theorem on (unitary) representations of compact groups; Weyl's connecting compact and complex Lie groups; and the work of Gelfand and Naimark, to name a few.

2.1 Motivation; minimal type

In Harish-Chandra's aforementioned program lies his famous subquotient theorem, which connects every irreducible Banach-space G-representation V that is "admissible" (i.e., when restricted to the action of K, the multiplicity of every K-module is finite), to the principal series representations. Following these works for real Lie groups, KRP et al. revisited the situation with G a (connected, simply-connected, semisimple) complex Lie group. To discuss their motivation, first let $\mathfrak{g}$ denote the complex Lie algebra of G, let $U\mathfrak{g}$ denote its universal enveloping algebra, and define

$$\widehat{\mathfrak{g}} := \mathfrak{g} \times \mathfrak{g} \quad \supset \quad \overline{\mathfrak{g}} := \{(X, X) : X \in \mathfrak{g}\}. \tag{2.1}$$

It was known thanks to Harish-Chandra that the irreducible V as above – which are moreover equipped with a character of the center $Z(U\mathfrak{g})$ – are essentially the same as irreducible modules in the category $\mathcal{C}(\widehat{\mathfrak{g}}, \overline{\mathfrak{g}})$ (defined below). The goal of KRP et al. was to study these latter modules, and hence obtain a better understanding of the former G-modules V. The strategy that the authors adopted was to use "minimal types".

Definition 1. *Given a complex semisimple Lie algebra $\mathfrak{g}$, a Cartan subalgebra $\mathfrak{h}$, and a fixed choice of simple roots $\Pi = \{\alpha_i : i \in I\}$, define $\widehat{\mathfrak{h}}$ and $\overline{\mathfrak{h}}$ similar to (2.1), and let P^+ denote the set of dominant integral weights (inside the weight lattice P) – these parametrize the irreducible finite-dimensional representations of $\mathfrak{g} \cong \overline{\mathfrak{g}}$, with $V_{\overline{\mathfrak{g}}}(\lambda)$ denoting the $\overline{\mathfrak{g}}$-module corresponding to $\lambda \in P^+$. (The corresponding module over $\mathfrak{g}$ will simply be denoted by $V(\lambda)$; these and other basics of semisimple Lie algebras can be found e.g. in [5].) We will denote the $\overline{\mathfrak{h}}$-weights of $V_{\overline{\mathfrak{g}}}(\lambda)$ by* $\mathrm{wt} V_{\overline{\mathfrak{g}}}(\lambda)$.

Next, define $\mathcal{C}(\widehat{\mathfrak{g}}, \overline{\mathfrak{g}})$ to be the full subcategory of $\widehat{\mathfrak{g}}$-modules V that can be decomposed as direct sums of finite-dimensional (irreducible) $\overline{\mathfrak{g}}$-modules $V_{\overline{\mathfrak{g}}}(\lambda)$, each with at most finite multiplicity – denoted $[V : V_{\overline{\mathfrak{g}}}(\lambda)]$.

Given a simple object V in $\mathcal{C}(\widehat{\mathfrak{g}}, \overline{\mathfrak{g}})$ (an irreducible "Harish-Chandra module"), a weight $\lambda \in P^+$ is said to be a minimal type of V if the multiplicity $[V : V_{\overline{\mathfrak{g}}}(\lambda)] > 0$, and

$$[V : V_{\overline{\mathfrak{g}}}(\mu)] > 0, \ \mu \in P^+ \implies \lambda \in \mathrm{wt} V_{\overline{\mathfrak{g}}}(\mu).$$

Note that if V has a minimal type λ, then λ is unique. Now in the PRV-paper, the authors construct a family $\{\widehat{\pi}_{\lambda,\nu} : \lambda \in \mathfrak{h}^*, \nu \in P\}$ of simple modules in $\mathcal{C}(\widehat{\mathfrak{g}}, \overline{\mathfrak{g}})$ with minimal types, and obtain a better understanding of them through their minimal types. It is also clear that for $\lambda, \mu \in P^+$, the $\widehat{\mathfrak{g}}$-module $V(\lambda) \otimes V(\mu)$ is a simple object in $\mathcal{C}(\widehat{\mathfrak{g}}, \overline{\mathfrak{g}})$. Thus, the first question is whether $V(\lambda) \otimes V(\mu)$ has a minimal type. (For $\mathfrak{g} = \mathfrak{sl}_2$, in which case $\lambda, \mu \in \mathbb{Z}_{\geq 0}$, consideration of the Clebsch–Gordan coefficients shows that the minimal type is $|\lambda - \mu|$.) The authors showed that the minimal type always exists (now termed the *PRV component* following their paper):

Theorem 1. *Let W be the Weyl group of G and $w_\circ$ denote the longest element of W, and $\lambda, \mu \in P^+$. The minimal type of $V(\lambda) \otimes V(\mu)$ is $\overline{\lambda + w_\circ \mu}$, where $\overline{\nu}$ for an integral weight $\nu \in P$ is the unique W-translate of ν which is dominant (i.e., $P^+ \cap W\nu = \{\overline{\nu}\}$).*

2.2 Tensor product multiplicities and the (K)PRV conjecture

Another famous result in the PRV-paper along these lines also concerns the module $V(\lambda) \otimes V(\mu)$. The study of the minimal type involves decomposing this module over $\overline{\mathfrak{g}}$; in other words, we are considering the Littlewood–Richardson coefficients

$$m^{\nu}_{\lambda,\mu} := [V(\lambda) \otimes V(\mu) : V(\nu)].$$

From Theorem 1 we know that $m^{\overline{\lambda+w_\circ\mu}}_{\lambda,\mu} = 1$. More generally, KRP et al. showed:

Theorem 2. *Given weights $\mu, \nu \in P^+$ and a weight $\gamma \in \mathfrak{h}^*$, define*

$$V^+(\mu; \gamma, \nu) := \{v \in V(\mu)_\gamma : e_i^{\nu(h_i)+1} v = 0 \ \forall i \in I\},$$

where $V(\mu)_\gamma$ is the γ-weight space for $\mathrm{ad}(\mathfrak{h})$ and e_i a Chevalley generator. Then,

$$m^{\nu}_{\lambda,\mu} = \dim V^+(\mu; \nu - \lambda, \lambda) = \dim V^+(\nu; \lambda + w_\circ\mu, -w_\circ\mu), \quad \forall \lambda, \mu, \nu \in P^+.$$

This result provides an exact formula for the tensor product multiplicity, and does not involve cancellations – this follows other formulas by Steinberg and by Brauer (which involve cancellations), as well as work of Kostant, among others. Both the theorems above have been widely used and generalized in the literature; the reader is referred to the survey [6] for a detailed overview of the PRV-paper, its past inspirations, contemporary works, and future applications.

Here is a second widely-explored follow-up involving tensor product multiplicities. As mentioned above, $V(\lambda) \otimes V(\mu)$ always has a minimal type $\overline{\lambda + w_\circ \cdot \mu}$. It is not hard to see that there is also always a "maximal type" – the weight $\lambda + \mu = \overline{\lambda + 1 \cdot \mu}$ – and both of these weights (i.e., the corresponding simple finite-dimensional modules) have multiplicity 1 in $V(\lambda) \otimes V(\mu)$. Thus, a natural question would be if the same holds when $w_\circ, 1$ are replaced by an arbitrary element of W; it was conjectured that this holds, and it was called the *PRV conjecture*.

This conjecture was significantly strengthened by Kostant, and is now called the KPRV conjecture. It was settled by Kumar [7] and by Mathieu [8] (with later proofs by Polo, Rajeswari, Littelmann, and Lusztig, among others), asserting that *for any $\lambda, \mu \in P^+$ and $w \in W$, the module $V(\overline{\lambda + w\mu})$ occurs with multiplicity 1 in the $U\mathfrak{g}$-submodule of $V(\lambda) \otimes V(\mu)$ generated by the one-dimensional $(\lambda, w\mu)$-weight space $V(\lambda)_\lambda \otimes V(\mu)_{w\mu}$.*

A final digression, for completeness, is that KRP et al. introduce a set of matrices $\mathbf{K}'_\mu$ of size $d_\mu \times d_\mu$, where $d_\mu = \dim V(\mu)_0$, such that if $d_\mu > 0$, then $\det \mathbf{K}'_\mu$ (now called the *PRV determinant*) splits into a product of linear factors, which are related to the Shapovalov form and to the annihilators of Verma modules. These PRV determinants have also been much studied in the subsequent literature; one notable application mentioned here is in (re)proving Duflo's remarkable result that the annihilator in $U\mathfrak{g}$ of any Verma module – which is a left-ideal in $U\mathfrak{g}$ – is generated by the annihilator in $Z(U\mathfrak{g})$. This was done by Joseph, with Letzter, over quantum groups and also classically over semisimple Lie algebras, and later by Gorelik for strongly typical Verma modules over basic classical Lie superalgebras. The PRV determinants are central in these proofs.

2.3 Irreducible admissible representations and their minimal types

Returning to the original motivation, for each $\xi \in \mathfrak{h}^*$ and integral weight $\nu \in P$, Harish-Chandra had constructed G-representations $\pi_{\xi,\nu} \subset L^2(K, \mu_{\text{Haar}}; \mathbb{C})$, with irreducible admissible G-modules corresponding to subquotients of $\pi_{\xi,\nu}$. Now let ρ denote the half-sum of the positive roots, and

$$\lambda = \lambda_{\xi,\nu} := \tfrac{1}{2}(\xi + \nu) - \rho.$$

In [11], the authors constructed G-subquotients of $\pi_{\xi,\nu}$, which they denoted by $\hat{\pi}_{\lambda,\nu}$. They then obtained detailed information about these modules – the following points are collected together from [6], and are either contained in [11] or can be deduced from it.

Theorem 3. *Fix $\xi \in \mathfrak{h}^*$ and an integral weight $\nu \in P$, and let $\lambda = \lambda_{\xi,\nu}$ as above.*

1. *$\hat{\pi}_{\lambda,\nu}$ is an irreducible subquotient of Harish-Chandra's module $\pi_{\xi,\nu} \subset L^2(K, \mu_{\text{Haar}}; \mathbb{C})$, so it too is defined on a Hilbert space. Moreover, $\pi_{\xi,\nu}$ is irreducible if and only if $\hat{\pi}_{\lambda,\nu} \cong \pi_{\xi,\nu}$, if and only if $[\hat{\pi}_{\lambda,\nu} : V_{\overline{\mathfrak{g}}}(\mu)] = \dim V_{\overline{\mathfrak{g}}}(\mu)_\nu$ for all $\mu \in P^+$.*
2. *$\hat{\pi}_{\lambda,\nu}$ is an object of $\mathcal{C}(\hat{\mathfrak{g}}, \overline{\mathfrak{g}})$, with minimal type component $\overline{\nu} \in P^+ \cap W\nu$. Moreover, $[\hat{\pi}_{\lambda,\nu} : V_{\overline{\mathfrak{g}}}(\overline{\nu})] = 1$.*
3. *$\hat{\pi}_{\lambda,\nu}$ has the same infinitesimal character as $\pi_{\xi,\nu}$. This character is $\chi(\lambda, \nu - \lambda - 2\rho)$, where $\chi(\lambda, \lambda')$ is the central character of $Z(U\hat{\mathfrak{g}}) \cong Z(U\mathfrak{g}) \otimes Z(U\mathfrak{g})$ corresponding to the $\hat{\mathfrak{g}}$-Verma module $M(\lambda, \lambda') \cong M_{\mathfrak{g}}(\lambda) \otimes M_{\mathfrak{g}}(\lambda')$.*
4. *The modules $\hat{\pi}_{\lambda,\nu}$ include the finite-dimensional irreducible modules: $V(\lambda) \otimes V(\mu) \cong \hat{\pi}_{\lambda,\lambda+w_\circ\mu}$ for all $\lambda, \mu \in P^+$.*
5. *Moreover, if $\bullet$ denotes the twisted W-action ($w \bullet \lambda = w(\lambda + \rho) - \rho$) then $\hat{\pi}_{\lambda,\nu} \cong \hat{\pi}_{w\bullet\lambda, w\nu}$ for all $w \in W$, while $\hat{\pi}_{\lambda,\nu}$ and $\hat{\pi}_{\lambda',\nu'}$ are not equivalent if $\nu' \notin W\nu$. If $\nu = \nu' = 0$, then the converse to the first assertion is also true: If $\lambda' \notin W \bullet \lambda$, then $\hat{\pi}_{\lambda,0} \ncong \hat{\pi}_{\lambda',0}$.*

The final point to be made here is a partial resolution of the "isoclasses question", which has been resolved by now. Thus, we know that every simple object in $\mathcal{C}(\hat{\mathfrak{g}}, \overline{\mathfrak{g}})$ is isomorphic to some $\hat{\pi}_{\lambda,\nu}$. Moreover, given $(\lambda, \nu), (\lambda', \nu') \in \mathfrak{h}^* \times P$, the converse to the above result of KRP et al. holds:

$$\hat{\pi}_{\lambda,\nu} \cong \hat{\pi}_{\lambda',\nu'} \quad \Longleftrightarrow \quad \exists w \in W : (\lambda', \nu') = (w \bullet \lambda, w\nu).$$

2.4 Factorisable representations of current groups

While the primary goal of this section was to elaborate in an informal way on the contents of the PRV paper, let us add another area in which KRP had a sustained interest: current groups and their factorisable representations. These notions were introduced by Araki (together with Woods) [1, 2] in the 1960s in the context of quantum field theory, to help understand the current commutation relations. In [12], KRP and Schmidt proved some fundamental results towards understanding factorisable multiplier representations, in addition to working via measure theory, as opposed to the techniques of Araki and Woods (so they provide novel technology as well).

Given a locally compact second countable group G, a standard Borel space $(T, \sigma(T))$, an Araki multiplier $S : \sigma(T) \times G \times G \to \mathbb{R}$, and an Araki S-function $\phi : \sigma(T \times G \to \mathbb{C}$, the authors show the existence of a direct integral Hilbert space $H = \int_T^\oplus H_t \, d\mu(t)$ with respect to a totally finite measure μ on $\sigma(T)$, together with a continuous unitary representation of G in H, a projection-valued measure on $\sigma(T)$, and an H-valued function δ on G satisfying certain technical conditions. The converse also holds. Once this measure and associated items are shown to exist, the authors then show – for G as above and moreover connected – that factorisable multiplier representations of the weak current group $F(T, G)$ are intimately linked to an "Araki pair" (S, ϕ). This helps them obtain a complete description (for connected locally compact second countable topological groups G) of the factorisable multiplier representations $\tilde{W}$ of the weak current group $F(T, G)$. In particular, the authors show how to construct $\tilde{W}$ from the direct integral Hilbert space mentioned above; this also yields the Araki–Woods imbedding theorem.

The study of multipliers and of factorisable representations engaged the attention of KRP and his collaborators for several years. The work [9] is KRP's expository survey about multipliers, covering results by Bargmann, Mackey, Varadarajan, and Simms. With Schmidt in [13], KRP also provided a novel method to construct factorisable representations over $\mathbb{R}^n$ when G is a connected simply-connected Lie group.

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3. KRP and the Embedding Problem for Distributions

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Consider a locally compact second countable topological group G. By a distribution on G we mean a probability measure on G defined on the σ-algebra of Borel subsets of G. We denote by $P(G)$ the class of distributions on G, and consider it equipped with the weak* topology. On $P(G)$ there is also a multiplication arising as convolution product of distributions, making $P(G)$ a topological semigroup. The (convolution) product of any $\mu, \nu \in P(G)$ will be denoted by $\mu\nu$ and correspondingly for $n \in \mathbb{N}$ the nth (convolution) power of any λ will be denoted λ^n. By a continuous one-parameter semigroup $\{\mu_t\}$ in $P(G)$ we mean a family of μ_t's, $t \geq 0$, such that $\mu_s\mu_t = \mu_{s+t}$ for all $s, t \in \mathbb{R}$, such that $t \mapsto \mu_t$ is continuous.

Given a $\mu \in P(G)$ it is of interest to know whether it lies in a continuous one-parameter semigroup, namely whether there exists a continuous one-parameter convolution semigroup $\{\mu_t\}$ in $P(G)$ such that $\mu_1 = \mu$; when this holds we say that μ is *embeddable*. When μ represents the transition probability of a random walk on G, embeddability corresponds to the question whether there is a continuous time stochastic process on G, of which the walk is the unit-time specialization. One reason for the interest is that continuous one-parameter semigroups of distributions are amenable to what can be broadly termed "methods of calculus". These one-parameter semigroups admit what is called the Lévy-Khinchine representation, which help understanding them; extending the classical studies, such a theory was developed by G. A. Hunt, in a paper in 1956, for the case of (connected) Lie groups; it was extended to the general case of locally compact groups later, by Hazod and Siebert, in 1973; see [4], p. 331.

It is easy to see that a necessary condition for embeddability of a distribution μ is that it must admit convolution roots of all orders; μ with this property is said to be *infinitely divisible*. For various classes of distributions of interest such a property be derived by other means. The question then is which infinitely divisible distributions are embeddable.

In the classical framework for distributions on the group of real numbers such a question was considered in a somewhat different form. The distributions were considered equivalent if they were translates of each other, focusing on their centered versions in a way; such a formulation arose from the context of the distributions which had barycenters, which it was considered appropriate to suppress. Thus a distribution μ was considered infinitely divisible if for all n there exists a distribution whose n-th power is a translate of μ and embeddable if a translate of μ lies in a one-parameter convolution semigroup. Embeddability of infinitely divisible distributions in this sense was established by P. Lévy, in 1954. A generalization of the result for locally compact abelian groups was proved in one of KRP's early papers, jointly with R. Ranga Rao and S. R. S. Varadhan [10], in 1963, in a similar formulation of the concepts, with the distributions considered equivalent if they are translates of each other; note that the group being abelian the left and right translates of a distributions are the same.

The version of the embedding problem not involving equivalence upto translations, as formulated above, was introduced by KRP in his 1967 paper [7]. The modification in the formulation has been important in furthering the study on the general theme; in [4] (see p. 245) Herbert Heyer refers to it as "a more generalizable approach to the theorem…proposed by Parthasarathy".

KRP motivated this question in [7] by recalling Hunt's representation theorem mentioned above, on the one hand, and embeddability results proved in the older formulation, on the other hand; the latter included his result with Ranga Rao and Varadhan as in [10] and also a result proved independently R. A. Gangolli and V. N. Tutubalin (in 1964 and 1962 respectively), which is an analogue of the embeddability assertion for distributions for symmetric spaces, in place of groups. He emphasizes exploring a direct interrelationship, confirming embeddability within the framework of probability theory on groups, rather than via harmonic analysis which had been the fulcrum of earlier work. In his words, "The method adopted in these papers is to obtain a formula for the "Fourier transform" and then deduce the imbeddability. However, Hunt [4] has obtained directly the representation of one-parameter convolution semigroups in any connected Lie group.

The question naturally arises whether one can directly imbed an infinitely divisible distribution in a one-parameter convolution semigroup and deduce the Lévy-Khinchine representation theorem."

Apart from introducing the problem which has been a source of much further activity, KRP led the way, in the paper, by introducing certain techniques towards its resolution, which have played a crucial role in the subsequent developments. In the paper he proved the embedding theorem (in the new formulation) for $\mathbb{R}$, with the method extending also to $\mathbb{R}^n$ for $n \geq 2$, and compact connected (not necessarily abelian) Lie groups, giving expression to the techniques. We shall discuss below some details of the ideas introduced and their impact in a wider context. Before doing so it would be convenient to note certain points to put the matter in perspective.

We note that there are some exceptions to embeddability being implied by infinite divisibility, in the renewed formulation. Consider the group $G = \mathbb{Q}$, of rational numbers equipped with the discrete topology, and a point-mass distribution $\mu = \delta_x$ where $x \in \mathbb{Q}$. Then it is infinitely divisible since $\delta_{x/n}$ is a convolution root of order n, for all $n \in \mathbb{N}$. However it is easy to see that it can not be embeddable, unless $x = 0$. A similar phenomenon arises in the case of p-adic numbers. A conjecture emerged in the aftermath of KRP's above mentioned paper that if G is a connected Lie group then every infinitely divisible distribution on G is embeddable.

The 1967-paper is from his Manchester years. Though on returning to India in the 1970s he largely moved on to other areas, he did revisit the embedding problem, and contributed the papers [8] and [9], and the latter especially was to have major influence on further progress.

In a major development on the question, it was proved by M. McCrudden in 1981 that if G is a connected Lie group and μ is an infinitely divisible distribution with the further property that its support is not contained in a proper closed subgroup of G then μ is embeddable. The review of the paper in Mathematical Reviews is by KRP. In it he attests it saying "This settles a long-standing problem of probability theory on Lie groups." and adds, with an evident sense of satisfaction, "As the author points out the method used is a modification of the argument employed by the reviewer." Indeed the reference here is to his paper [9].

McCrudden, incidentally, worked at the University of Manchester. He did not actually overlap with KRP there, but got introduced to the problem and the area, on account of a student there, Quentin Burrell, to whom KRP had suggested the problem. His first work on the problem was with Burrell, extending KRP's result for connected nilpotent Lie groups. Though KRP evidently did not think of the condition on the supports of the distributions as being a serious restriction, McCrudden was passionate about eliminating it. In 1985 McCrudden and I were both at a conference at Oberwolfach, Germany, where he introduced me to the problem. That was to lead to a long collaboration between us, leading to proof of the embedding theorem for all infinitely divisible distributions for a large class of connected Lie groups, though not all. I shall not go into the details here; the reader is referred to [2] for an exposition on the theme, including also some of my later work with Yves Guivarc'h and Riddhi Shah, and the current status on the question.

While a good deal of technicalities and new ideas have gone into further developments there has always been a core part of the strategy which can be traced back to KRP's 1967 paper. It may be briefly described as follows. Let μ be an infinitely divisible distribution. To get an embedding $\{\mu_t\}$ one aims at identifying candidates for μ_r, r any rational number, which would give a "rational embedding" and ensuring that it would extend to a continuous embedding with real parameter. For $r = \frac{p}{q}$, p, q natural numbers, the candidate would have to be of the form λ_q^p where λ_q is a qth root of μ. When μ is infinitely divisible one gets an abundant supply of qth roots; if $m = kq$ and λ_m is an mth root then λ_m^k is a qth root; (they may not be distinct, but one does not need to worry about it). For any $r = \frac{p}{q}$ the set say P_r of possible λ_q^ps that can be produced in this way is a closed set. It turns out that if it is actually compact then, taking limits suitably, one can arrive at rational embeddings, which can then be seen to extend to continuous embeddings; when the P_r's are not compact the endeavour is to find suitable compact subsets of them to which also the strategy can be applied.

A strategy to arrive at compactness of P_rs as above was introduced by KRP in [7] via an interesting property of convolution products of distributions proved in [10] (see also [6] for an exposition); this property has in fact been useful in a variety of other contexts as well (see [1] in particular): Let G be a locally compact second countable group. Let $\{\xi_n\}$ and $\{\eta_n\}$ be sequences of distributions such that $\{\xi_n\eta_n\}$ is relatively compact. Then there exists a sequence $\{g_n\}$ in G such that the sequences $\{\xi_n g_n\}$ and $\{g_n^{-1}\eta_n\}$ are relatively compact. If $\{\xi_n\}$ is a sequence in P_r as above with $r < 1$ then we readily get a sequence $\{\eta_n\}$ in P_{1-r} such that $\xi_n\eta_n = \mu$. The theorem therefore implies that $\{\xi_n g_n\}$ is relatively compact for a sequence $\{g_n\}$ in G. In the case of $\mathbb{R}$ (and $\mathbb{R}^n$) with some further arguments KRP was able to prove that sequence $\{g_n\}$ is relatively compact, proving in this case that P_r is compact.

Numerous variations of the theorem and strategies are involved in various subsequent works. The legacy lives on.

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4. From Central Limit Theorems to Continuous Tensor Products

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Dedicated to the memory of Kalyanapuram Rangachari Parthasarathy

4.1 Introduction

After reading K. R. Parthasarathy's (KRP's) monograph *Probability measures on metric spaces* [8] in 1968, I decided to apply for a post-doc position at Manchester University to work with him. When I eventually arrived there in 1969 KRP's interests had begun to focus on some intriguing connections between central limit theorems in classical probability theory and analogous limit theorems for unitary representations of locally compact groups. The fact that he was starting on a new mathematical venture, combined with his extraordinary willingness to share problems and ideas, allowed me to begin working with him on these problems almost immediately after my arrival.

One of the important problems in classical probability theory is to determine which probability distributions on the real line arise as weak limits of sums of *uniformly infinitesimal* independent random variables. By using the correspondence between probability measures and positive definite functions on $\mathbb{R}$ this problem becomes equivalent to finding all continuous positive definite functions which occur as limits of *uniformly infinitesimal arrays of positive definite functions* (explained below). All such limits can be described by the *Lévy-Khinchine formula* (cf. e.g., [5]). In 1963, KRP, Ranga Rao and Varadhan proved a version of the Lévy-Khinchine formula for limits of uniformly infinitesimal arrays of positive definite functions on locally compact second countable abelian groups (cf. [11], Theorem 7.1 or [8], Corollary IV.7.1).

A few years later R. Streater [19], motivated by papers by Araki and Woods ([1], [2]), showed that analogous limits of products of uniformly infinitesimal arrays of positive definite functions on more general groups can be used to construct so-called *factorisable representations* of certain current groups, leading to interesting mathematical models of quantum fields. KRP was immediately captivated by the problem of finding an analogue of the Lévy-Khinchine formula for such limits on general locally compact second countable groups.

On the following pages I will attempt to give a brief account of what progress we made on this problem in [13]. I hasten to emphasize that the lion's share of this progress should be credited to my former friend and collaborator KRP.

 

KRP at Warwick, England - relaxing moments.
(Photos courtesy of Prof. Klaus Schmidt)

Limit theorems for positive definite functions on groups

Let G be a Polish (i.e. complete, separable, metric) group with identity element 1_G, $\mathcal{H}$ a complex, separable Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, and let $V: g \mapsto V_g$ be a (weakly) continuous, unitary, cyclic representation of G on $\mathcal{H}$ with a cyclic unit vector v. If we set

$$\phi(g) = \langle V_g v, v\rangle \tag{4.1}$$

for every $g \in G$, then the function $\phi: G \to \mathbb{C}$ is continuous and *positive definite*, i.e.

$$\sum\nolimits_{i,j=1}^{m} c_i \bar{c}_j \phi(g_j^{-1} g_i) = \left\|\sum\nolimits_{i=1}^{m} c_i V_g v\right\|^2 \geq 0$$

for every $m \geq 1$ and every choice of $g_1, \dots, g_m$ in G and $c_1, \dots, c_m$ in $\mathbb{C}$. Furthermore, ϕ is *normalized* in the sense that $\phi(1_G) = 1$.

Conversely, if $\phi: G \to \mathbb{C}$ is a continuous normalized positive definite function, then there exists a continuous unitary representation V of G on a complex separable Hilbert space $\mathcal{H}$ with a cyclic unit vector v satisfying (4.1) for every $g \in G$ (cf. e.g. [3] Construction 2.B.4 or [13] Theorem 1.2).

If (V, v) and (V', v') are two continuous, unitary, cyclic representations of G on Hilbert spaces $\mathcal{H}, \mathcal{H}'$, respectively, and if

$$\phi(g) = \langle V_g v, v\rangle, \qquad \phi'(g) = \langle V'_g v', v'\rangle,$$

for every $g \in G$, then the tensor product representation $V \otimes V'$ on $\mathcal{H} \otimes \mathcal{H}'$, restricted to the cyclic subspace of $\mathcal{H} \otimes \mathcal{H}'$ generated by $v \otimes v'$, satisfies that

$$\langle (V_g \otimes V'_g)(v \otimes v'), (v \otimes v')\rangle = \phi(g)\phi'(g), \tag{4.2}$$

for every $g \in G$. This shows that the pointwise product $\phi\phi'$ of two continuous, positive definite functions is again positive definite and arises from the tensor product of the representations arising from ϕ and ϕ'.

Assume now that the group G is locally compact, second countable, and abelian. We denote by $\widehat{G} = \mathrm{Hom}(G, \mathbb{S})$ the *dual* or *character group* of G, consisting of all continuous homomorphisms from G into the group $\mathbb{S} = \{z \in \mathbb{C} \mid |z| = 1\}$. Under the topology of uniform convergence on compact sets $\widehat{G}$ is again a locally compact and second countable abelian group (cf. [17], or [8] Chapter IV).

For every $\gamma \in \widehat{G}$ and $g \in G$, we write $\langle g, \gamma\rangle \in \mathbb{S}$ for the value at $g \in G$ of the character $\gamma \in \widehat{G}$. *Bochner's Theorem* allows one to find, for every continuous positive definite function $\phi: G \to \mathbb{C}$, a unique Borel probability measure μ_ϕ on $\widehat{G}$ such that

$$\phi(g) = \int_{\widehat{G}} \langle g, \gamma\rangle d\mu_\phi(\gamma) \tag{4.3}$$

for every $g \in G$. Conversely, if μ is a Borel probability measure on $\widehat{G}$, then the function $\hat{\mu}: G \to \mathbb{C}$ defined by

$$\hat{\mu}(g) = \int_{\widehat{G}} \langle g, \gamma\rangle d\mu(\gamma) \tag{4.4}$$

is continuous, positive definite, and normalized. One can reconcile the equations (4.1) and (4.3) by setting $\mathcal{H} = L^2(\widehat{G}, \mu_\phi)$, and by considering the unitary representation V of G on $\mathcal{H}$ given by

$$(V_g f)(\chi) = \langle g, \gamma\rangle f(\gamma)$$

for every $f \in \mathcal{H}$, $g \in G$ and $\gamma \in \widehat{G}$. The constant function $v = 1 \in \mathcal{H}$ is cyclic for V, and

$$\langle V_g v, v\rangle = \int_{\widehat{G}} \langle g, \gamma\rangle \, d\mu_\phi(\chi) = \phi(g)$$

for every $g \in G$.

If μ, μ' are two Borel probability measures on $\widehat{G}$, the product of the positive definite functions $\widehat{\mu}, \widehat{\mu}'$ in (4.4) satisfies that

$$\widehat{\mu}\widehat{\mu}' = \widehat{\mu * \mu'}, \tag{4.5}$$

where $\mu * \mu'$ is the *convolution* of μ and μ', i.e. the probability measure on $\widehat{G}$ with

$$\int_{\widehat{G}} f d(\mu * \mu') = \int_{\widehat{G}} \int_{\widehat{G}} f(\chi\chi')\, d\mu(\chi)\, d\mu'(\chi')$$

for every bounded Borel function $f\colon \widehat{G} \to \mathbb{C}$.

Infinitely divisible probability measures

A probability measure μ on $\widehat{G}$ is *infinitely divisible* if there exists, for every $n \geq 1$, a probability measure ν_n on $\widehat{G}$ with $\mu = \nu_n^{*n} = \nu_n * \cdots * \nu_n$. If μ on $\widehat{G}$ is infinitely divisible, then (4.4) – (4.5) imply that $\widehat{\mu}$ is a continuous *infinitely divisible positive definite function*, i.e. that there exists, for every $n \geq 1$, a continuous normalized positive definite function ϕ_n with $\widehat{\mu} = \phi_n^n$. The infinitely divisible continuous normalized positive definite functions on G are thus in one-to-one correspondence with the infinitely divisible Borel probability measures on $\widehat{G}$.

For $G = \mathbb{R}$, the celebrated *Lévy-Khinchine formula* states that a continuous, positive definite function $\phi\colon \mathbb{R} \to \mathbb{C}$ is infinitely divisible if and only if

$$\log \phi(t) = it\beta - \tfrac{t^2\sigma^2}{2} + \int_{\mathbb{R}} (\exp itx - 1 - \tfrac{itx}{1+x^2}) \tfrac{1+x^2}{x^2}\, d\nu(x) \tag{4.6}$$

for some $\beta \in \mathbb{R}$, $\sigma^2 \geq 0$, and some finite Borel measure ν on $\mathbb{R}$ with $\nu(\{0\}) = 0$ (cf. [5] §18, Theorem 1).

The formula on the right hand side of (4.6) has three parts: After exponentiation, the first part ($t \mapsto it\beta$) determines a character of $\mathbb{R}$ and amounts to a translation of the probability measure μ_ϕ in question. The second part comes from a homomorphism ($t \mapsto t\sigma/2$) from $\mathbb{R}$ to $\mathbb{R}$ and corresponds to the *Gaussian part* of our measure μ_ϕ. The third, and most intimidating looking, part refers to the *Poisson part* of μ_ϕ: It is - in essence - a fancy linear combination of terms of the form $t \mapsto e^{itx} - 1$ which will reappear much later in this account (cf. (4.9)).

In the derivation of analogous Lévy-Khinchine formulas for infinitely divisible positive definite functions on arbitrary locally compact, second countable abelian groups G in [11] or [8], one obtains the same three parts as in (4.6) with the possible addition of a fourth term coming from potential nontrivial open subgroups of the group G.

Ever since [5], the discussion of infinitely divisible positive definite functions has been broadened to include limits of uniformly infinitesimal arrays of probability measures on $\mathbb{R}$ and, more generally, on locally compact second countable abelian groups.

Definition 2. *Let $\mathcal{M} = \{\mu_{n,j} \,|\, n \geq 1, 1 \leq j \leq k_n\}$ be a triangular array of probability measures on a locally compact, second countable abelian group G. The array $\mathcal{M}$ is* uniformly infinitesimal *if*

$$\lim_{n\to\infty} \sup_{1\leq j\leq k_n} \mu_{n,j}(G \setminus \mathcal{N}) = 0 \tag{4.7}$$

for every neighbourhood $\mathcal{N}$ of the identity in G. Equation (4.7) is equivalent to the condition that

$$\lim_{n\to\infty} \sup_{1\leq j\leq k_n} \sup_{g\in K} |\widehat{\mu}_{n,j}(g) - 1| = 0$$

for every compact set $K \subset G$.

A uniformly infinitesimal array $\mathcal{M}$ converges to a probability measure μ on G if the sequence of probability measures $(\mu_n := \prod_{j=1}^{k_n} \mu_{k,j},\ n \geq 1)$ converges to μ in the weak topology (cf. [8] Section II.6).

The following result follows from [8] Theorem V.5.2 and [13] Theorem 12.5:

Theorem 4. *Let G be a locally compact, second countable abelian group. If a probability measure μ on G is the limit of a uniformly infinitesimal triangular array of probability measures on G, then μ is infinitely divisible.*

Conversely, if G is connected and locally connected, and if μ is an infinitely divisible probability measure on G, then μ is the limit of a uniformly infinitesimal triangular array of probability measures on G.

For refinements and extensions of the convergence results for uniformly infinitesimal triangular arrays of positive definite functions in [13] the reader may wish to consult the follow-up paper [10] by KRP.

From infinitely divisible positive definite functions to cocycles

If G is an arbitrary Polish group G, then (4.2) implies that every infinitely divisible positive definite function ϕ on G defines a continuous, unitary, cyclic representation V of G on a complex separable Hilbert space $\mathcal{H}$ which can, for every $n \geq 1$, be written as the tensor product of n copies of some other cyclic representation V_n of G on some Hilbert space $\mathcal{H}_n$. According to R. Streater [19], such "infinitely divisible representations" of G lead to the construction of factorisable representations of current groups and to interesting mathematical models of quantum fields. KRP's earlier interest in a Lévy-Khinchine formula for infinitely divisible probability measures on locally compact abelian groups led him to be immediately captivated by the problem of finding an analogous formula for infinitely divisible positive definite functions and, more generally, for limits of uniformly infinitesimal arrays of positive definite functions on general Polish groups.

We start with two definitions.

Definition 3. *A* triangular array *of continuous normalized positive definite functions $\Phi = \{\phi_{n,j} | n \geq 1, 1 \leq j \leq k_n\}$ on a Polish group G is* uniformly infinitesimal *if*

$$\lim_{n\to\infty} \sup_{1\leq j\leq k_n} \sup_{g\in K} |\phi_{n,j}(g) - 1| = 0$$

for every compact set $K \subset G$.

Such a uniformly infinitesimal triangular array Φ converges *to a continuous positive definite function ϕ if ϕ is the limit of the sequence $(\phi_n := \prod_{j=1}^{k_n} \phi_{k,j}, n \geq 1)$ in the topology of uniform convergence on compact subsets of G.*

As in Theorem 4 one can show that limits of uniformly infinitesimal arrays of continuous positive definite functions on a group G are infinitely divisible, and that the reverse implication holds under certain hypotheses on the group G.

When I arrived in Manchester in 1969, KRP had just proved the following striking result:

Theorem 5. *([13], Theorem 11.2) Let G be a Polish group, and let $\phi\colon G \to \mathbb{C}$ be a limit of a uniformly infinitesimal array of continuous normalized positive definite functions. Then there exist an open subgroup $G_0 \subset G$, a continuous unitary representation U of G_0 on a complex separable Hilbert space $\mathcal{K}$ with inner product $\langle\cdot,\cdot\rangle$, and a continuous map $\delta\colon G_0 \to \mathcal{K}$ such that*

$$U_g\delta(h) = \delta(gh) - \delta(g), \tag{4.8}$$

$$\phi(gh)\phi(g)^{-1}\phi(h)^{-1} = \exp\langle\delta(h), \delta(g^{-1})\rangle \tag{4.9}$$

for all $g, h \in G_0$.

Any continuous function $\delta\colon G_0 \to \mathcal{K}$ satisfying (4.8) is called a (1-)*cocycle* for the representation U. Such a cocycle is a *coboundary* (or *trivial*) if there exists a $z \in V$ such that

$$\delta(g) = V_g z - z \tag{4.10}$$

for every $g \in G_0$. If two cocycles differ by a coboundary they are called *cohomologous*.

From cocycles to infinitely divisible projective representations

After having proved Theorem 5, KRP tried to obtain a converse to this result: Given a continuous unitary representation V of a Polish group G on a Hilbert space $\mathcal{K}$ and a continuous cocycle $\delta\colon G \to \mathcal{K}$ satisfying (4.8), can one find a continuous infinitely divisible positive definite function $\phi\colon G \to \mathbb{C}$ satisfying (4.9)?

At that time I had just read KRP's notes [9] and noticed that the map $L\colon G \times G \to \mathbb{C}$ defined by

$$L(g, g') = \langle \delta(g'), \delta(g^{-1}) \rangle \tag{4.11}$$

is a *2-cocycle*:

$$L(g_1, g_2) + L(g_1 g_2, g_3) = L(g_1, g_2 g_3) + L(g_2, g_3) \tag{4.12}$$

for all $g_1, g_2, g_3 \in G$. Since the real part $\Re L$ of L satisfies the equation

$$\Re L(g, g') = \tfrac{1}{2}(\|\delta(g)\|^2 + \|\delta(g')\|^2 - \|\delta(gg')\|^2)$$

for all g, g', $\Re L$ is a *2-coboundary*, i.e., there exists a continuous map $b\colon G \to \mathbb{C}$ such that

$$\Re L(g_1, g_2) = b(g_1 g_2) - b(g_1) - b(g_2).$$

The imaginary part $\Im L$ of L may or may not be a 2-coboundary, but it certainly satisfies the cocycle equation (4.12). If we put

$$\sigma(g, g') = \exp(i \cdot \Im L(g, g')), \tag{4.13}$$

the resulting map $\sigma\colon G \times G \to \mathbb{S}$ is again a 2-cocycle: It satisfies the multiplicative analogue

$$\sigma(g_1, g_2)\sigma(g_1 g_2, g_3) = \sigma(g_1, g_2 g_3)\sigma(g_2, g_3) \tag{4.14}$$

of (4.12) for all $g_1, g_2, g_3 \in G$.

In order to construct the putative positive definite function ϕ in (4.9) and its associated unitary representation V on a Hilbert space $\mathcal{H}$ in (4.1) we introduce the *symmetric Fock space* $\operatorname{Exp}\mathcal{K}$ over $\mathcal{K}$, given by

$$\operatorname{Exp}\mathcal{K} = \mathbb{C} \oplus \mathcal{K} \oplus \mathcal{K} \otimes_s \mathcal{K} \oplus \mathcal{K} \otimes_s \mathcal{K} \otimes_s \mathcal{K} \oplus \cdots, \tag{4.15}$$

where $\otimes_s$ denotes the *symmetric* tensor product of Hilbert spaces (cf. [13] pp. 29-30). For every $v \in \mathcal{K}$, put

$$\operatorname{Exp} v = 1 \oplus v \oplus \tfrac{1}{\sqrt{2}}(v \otimes v) \oplus \tfrac{1}{\sqrt{3}}(v \otimes v \otimes v) \cdots \in \operatorname{Exp}\mathcal{K}.$$

If $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ denotes the obvious inner product on $\operatorname{Exp}\mathcal{K}$, then

$$\langle\!\langle \operatorname{Exp} v, \operatorname{Exp} v' \rangle\!\rangle = \exp\langle v, v' \rangle$$

for all $v, v' \in \mathcal{K}$. We denote by $\mathcal{H} \subset \operatorname{Exp}\mathcal{K}$ the closed linear span of the set $\{\operatorname{Exp}\delta(g) \mid g \in G\}$. For every $g, h \in G$, let

$$V_g \operatorname{Exp}\delta(h) = \exp(L(g, h) - \|\delta(g)\|^2/2) \operatorname{Exp}\delta(gh), \tag{4.16}$$

where L is defined in (4.11). Since

$$\langle\!\langle V_g \operatorname{Exp}\delta(h_1), V_g \operatorname{Exp}\delta(h_2) \rangle\!\rangle = \langle\!\langle \operatorname{Exp}\delta(h_1), \operatorname{Exp}\delta(h_2) \rangle\!\rangle$$

for every $h_1, h_2 \in G$, V_g extends by linearity to a unitary operator on $\mathcal{H}$. The map $g \mapsto V_g$ from G into the unitary group $\mathcal{U}(\mathcal{H})$ is continuous, but will in general not be a unitary representation of G: It is a *projective representation* satisfying

$$\sigma(g, g') V_{gg'} = V_g V_{g'} \tag{4.17}$$

for all $g, g' \in G$, where the *multiplier* σ of V is the 2-cocycle appearing in (4.13). By definition, the vector $v = \operatorname{Exp}\delta(1_G) = \operatorname{Exp} 0 \in \mathcal{H}$ is *cyclic* for this projective representation, and

$$\Phi(g) := \langle\!\langle V_g v, v\rangle\!\rangle = \exp(-\|\delta(g)\|^2/2) \tag{4.18}$$

for every $g \in G$. Hence

$$\Phi(gg')\Phi(g)^{-1}\Phi(g')^{-1} = \exp(\Re\langle\delta(g'), \delta(g^{-1})\rangle).$$

This isn't quite what we wanted in (4.9). However, if σ is a coboundary, i.e., if

$$\sigma(g, gg') = \beta(gg')b(g)^{-1}b(g')^{-1} \tag{4.19}$$

for some Borel (and, in fact, automatically continuous) map $\beta\colon G \to \mathbb{S}$, then the map

$$\tilde{V}\colon g \mapsto \overline{\beta(g)}V_g$$

is a continuous unitary representation of G, and $\tilde{\Phi}(g) = \langle\!\langle \tilde{V}_g v, v\rangle\!\rangle$ satisfies that

$$\tilde{\Phi}(gg')\tilde{\Phi}(g)^{-1}\tilde{\Phi}(g')^{-1} = \langle\!\langle \tilde{V}_g v, v\rangle\!\rangle$$

for every $g \in G$. It is easy to check that the map $\tilde{\Phi}\colon G \to \mathbb{C}$ has all the good properties one could hope for: It is continuous, normalized, positive definite, infinitely divisible, and it is a limit of a uniformly infinitesimal array of continuous positive definite functions. Since this is obviously a very nice situation, let me introduce some ad-hoc (and completely non-standard) terminology:

Definition 4. *Let U be a continuous unitary representation of a Polish group G on a complex separable Hilbert space $\mathcal{K}$. A continuous 1-cocycle $\delta\colon G \to \mathcal{K}$ is* rectifiable *if the 2-cocycle $\sigma\colon G \times G \to \mathbb{S}$ in (4.13) is a coboundary in the sense of (4.19).*

Which conditions on the pair (V, δ) will guarantee that δ is rectifiable?

Proposition 1. (1) *A cocycle $\delta\colon G \to \mathcal{K}$ is a coboundary if and only if it is norm-bounded.*
(2) *If δ is a coboundary, then it is rectifiable.*

Proof. (1) Suppose that δ is bounded in the sense that $K = \sup_{g\in G}\|\delta(g)\|_\infty$. For every $g \in G$ and $v \in \mathcal{K}$, let

$$A_g x = V_g x - \delta(g).$$

Since δ is bounded, the set $S = \{A_g 0 \mid g \in G\} \subset \mathcal{K}$ is bounded in norm, and so is the closed convex hull $C(S)$ of S in $\mathcal{K}$. By the Ryll-Nardzewski fixed point theorem [7] p. 444, there exists a point $z \in C(S)$ such that $A_g z = z$ for every $g \in G$, so that δ is indeed trivial. Conversely, if δ is a coboundary, then it is obviously bounded.

(2) If $\delta(g) = V_g z - z$ for all g, then

$$\begin{aligned} L(g,h) = \langle\delta(h), \delta(g^{-1})\rangle &= \langle U_h z - z, U_{g^{-1}}z - z\rangle - \|z\|^2 \\ &= (\langle U_{gh}z - z, z\rangle - \|z\|^2) - (\langle U_g z, z\rangle - \|z\|^2) - (\langle U_h z, z\rangle - \|z\|^2). \end{aligned}$$

Hence the cocycle σ in (4.13) satisfies (4.19) with $\beta(g) = \exp(i \cdot \Im(\langle U_g z, z\rangle)$, and is thus a 2-coboundary. □

If the 2-cocycle σ in (4.13) is nontrivial (i.e., not a 2-coboundary), the projective representation V in (4.16) – (4.17) is again infinitely divisible in the sense that there exists, for every $n \geq 1$, a cyclic projective representation $V^{(n)}$ of G on a Hilbert space $\mathcal{H}^{(n)}$ with multiplier $\sigma^{(n)} = \exp(i \cdot \Im L/n)$ and cyclic vector $v^{(n)}$ such that V is unitarily equivalent to the projective representation $V^{(n)} \otimes \cdots \otimes V^{(n)}$

of G on the n-fold tensor product $\mathcal{H}^{(n)} \otimes \cdots \otimes \mathcal{H}^{(n)}$, restricted to the cyclic subspace generated by $v^{(n)} \otimes \cdots \otimes v^{(n)}$.

Since there is such a close correspondence between continuous 1-cocycles of continuous unitary representations of G and continuous infinitely divisible projective representations of G, any attempt to classify all infinitely divisible projective representations of G (and, in particular, all infinitely divisible positive definite functions on G) would require a classification of all 1-cocycles of continuous unitary representations of our Polish group G.

If G is locally compact and second countable, the usual decomposition techniques allow one to restrict one's attention to *irreducible* representations of G: If we can write a unitary representation U of G as a direct integral $U = \int_X^{\oplus} U^\omega \, d\mu(\omega)$ of irreducible unitary representations over some finite measure space (Ω, μ), then [13] Theorem 13.2 shows that every continuous cocycle δ for U is of the form $\delta(g) = \int_X^{\oplus} \delta^\omega(g) \, d\mu(\omega)$, where each δ^ω is a continuous cocycle of the representation U^ω.

For example, if G is abelian, every irreducible unitary representation U of G is one-dimensional and thus a continuous homomorphism from G into $\mathbb{S}$. If $U_h \neq 1$ for some $h \in G$ then $U_h - 1$ is invertible, so that every $v \in \mathcal{H} = \mathbb{C}$ is of the form $v = U_h b - b$ for some $b \in \mathbb{C}$. In particular, any cocycle δ for U satisfies that

$$\delta(h) = U_h b - b$$

for some $b \in \mathcal{H}$. If $h' \in G$ is a second element with $U_{h'} \neq 1$, then

$$\delta(h') = U_{h'} b' - b'$$

for some $b' \in \mathbb{C}$, and

$$\begin{aligned} \delta(hh') &= U_h \delta(h') + \delta(h) = U_{hh'} b' - U_h b' + U_h b - b \\ &= U_{h'} \delta(h) + \delta(h') = U_{hh'} b - U_h b + U_h b' - b', \end{aligned}$$

so that

$$(U_h - 1)(U_{h'} - 1)(b - b') = 0$$

and $b = b'$. It follows that $\delta(g) = U_g b - b$ for every $g \in G$, i.e. that δ is trivial.

Proposition 2. *Let G be an abelian Polish group, U a continuous irreducible representation of G on $\mathcal{H} = \mathbb{C}$, and $\delta\colon G \to \mathbb{C}$ a cocycle of V. If U is nontrivial, then δ is a coboundary (cf. (4.10)). If U is trivial (i.e., if $U_h = 1$ for all h), then δ is a homomorphism from G into $\mathbb{C}$.*

Proposition 2 might give the impression that, for abelian groups, only trivial representations could have nontrivial cocycles (i.e. cocycles which are not coboundaries). However, the Lévy-Khinchine formula (4.6) is evidence that non-compact locally compact abelian groups can have unitary representations with unbounded cocycles, but without nonzero invariant vectors. For obvious reasons such cocycles are sometimes called *generalised coboundaries*. These generalised coboundaries give rise to the "Poisson parts" of the Lévy-Khinchine formulas (cf. [13] Theorem 16.3).

Furthermore, if $\delta\colon G \to \mathcal{K}$ is a nontrivial homomorphism arising from the trivial representation of G on a Hilbert space $\mathcal{K}$, then δ is not only nontrivial (since it is unbounded), but it can also give rise to a nontrivial 2-cocycle $\sigma\colon G \times G \to \mathbb{S}$ in (4.13) – (4.14). A simple example of this is described in [13] §8: if $G = \mathcal{K} = \mathbb{C}$ and U is the trivial representation of G on $\mathcal{K}$, then the identity map $\delta\colon z \mapsto z$ from G to $\mathcal{K}$ defines a cocycle for U on $\mathcal{K}$ which gives rise to the projective representation V in (4.16) with multiplier $\sigma(z_1, z_2) = \exp(-i \cdot \Im(z_2 \bar{z}_1))$ in (4.17).

In [13] Part III there are a few further results about cocycles for irreducible representations of other classes of groups (e.g., connected nilpotent or semi-simple Lie groups), but a more complete picture arises from the papers [4] by P. Delorme and [22] by A. M. Veršhik and S. I. Karpushev. The latter paper explains in particular the apparent scarcity of nontrivial 1-cocycles for irreducible unitary representations of locally compact groups (for terminology we refer to [22]).

Theorem 6. *([22], Theorem 2) Let G be a locally compact second countable group, and let V be an irreducible unitary representation of G on a complex separable Hilbert space $\mathcal{K}$ which has a nontrivial continuous 1-cocycle. Then V is infinitely small (i.e., Hausdorff inseparable from the trivial representation).*

For nontrivial irreducible unitary representations, non-rectifiability of 1-cocycles is, of course, even "scarcer" than non-triviality. The following proposition is a corollary of [9] p. 40, Corollary 1.

Proposition 3. *If G is a connected and simply connected semi-simple Lie group, U a continuous irreducible unitary representation of G on a complex Hilbert space $\mathcal{K}$, and $\delta\colon G \to \mathcal{K}$ a continuous cocycle for V, then δ is rectifiable in the sense of Definition 4.*

Factorisable projective representations

Before moving on to representations of current groups in the spirit of [1] and [19], let me point out an important property of the projective representation V of G on the Hilbert space $\mathcal{H} \subset \operatorname{Exp}\mathcal{K}$ arising from a unitary representation U of G on $\mathcal{K}$ and a cocycle $\delta\colon G \to \mathcal{K}$ in (4.16) – (4.17): if $\mathcal{K}$ can be written as a direct sum $\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2$ of two closed U-invariant subspaces we can decompose the cocycle $\delta\colon G \to \mathcal{K}$ as a direct sum $\delta_1 \oplus \delta_2$ of two cocycles $\delta_i\colon G \to \mathcal{K}_i$ and construct the corresponding infinitely divisible projective representations V_i of G on $\mathcal{H}_i \subset \operatorname{Exp}\mathcal{K}_i$ as above. Then the projective representation V of G on $\mathcal{H}$ is unitarily equivalent to the tensor product representation $V_1 \otimes V_2$ on the tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$ of the spaces $\mathcal{H}_i$.

This elementary observation can be developed to construct *factorisable representations* of so-called *current groups*. The following exposition is based on [12] – [13].

Assume for convenience that X is a compact metrizable space with a nonatomic Borel probability measure μ, G a Polish group, U a continuous unitary representation of G on a complex Hilbert space $\mathcal{K}$ with inner product $\langle\cdot,\cdot\rangle$, and $\delta\colon G \to \mathcal{K}$ a continuous 1-cocycle of U. We denote by $\boldsymbol{\mathcal{K}} = \int_X^{\oplus} \mathcal{K}\, d\mu = L^2_\mu(X, \mathcal{K})$ the Hilbert space of square-integrable maps $f\colon X \to \mathcal{K}$ with inner product $[f, f'] = \int_X \langle f(x), f'(x)\rangle\, d\mu(x)$.

Let $\Gamma = C(X, G)$ be the group of continuous maps $\gamma\colon X \to G$, furnished with pointwise multiplication and the topology of uniform convergence, and define a continuous unitary representation $\mathbf{U}$ of Γ on $\boldsymbol{\mathcal{K}}$ setting

$$(\mathbf{U}_\gamma f)(x) = U_{\gamma(x)} f(x)$$

for every $\gamma \in \Gamma$, $x \in X$, and $f \in \boldsymbol{\mathcal{K}}$. The cocycle δ gives rise to a continuous 1-cocycle $\boldsymbol{\delta}\colon \Gamma \to \boldsymbol{\mathcal{K}}$ for $\mathbf{U}$ with

$$\boldsymbol{\delta}(\gamma)(x) = \delta(\gamma(x)),$$

and

$$\mathbf{U}_{\gamma_1}\boldsymbol{\delta}(\gamma_2) = \boldsymbol{\delta}(\gamma_1\gamma_2) - \boldsymbol{\delta}(\gamma_1)$$

for all $\gamma, \gamma_1, \gamma_2 \in \Gamma$ and $x \in X$.

As in (4.15) we define the symmetric Fock space $\operatorname{Exp}\boldsymbol{\mathcal{K}}$ with inner product $\langle\!\langle\cdot,\cdot\rangle\!\rangle$ and set, for every $\gamma, \gamma' \in \Gamma$,

$$\mathbf{V}_\gamma \operatorname{Exp}\boldsymbol{\delta}(\gamma') = \exp\left(\mathbf{L}(\gamma, \gamma') - \tfrac{1}{2}\|\boldsymbol{\delta}(\gamma)\|^2\right) \operatorname{Exp}\boldsymbol{\delta}(\gamma'), \tag{4.20}$$

where

$$\mathbf{L}(\gamma, \gamma') = [\boldsymbol{\delta}(\gamma'), \boldsymbol{\delta}(\gamma^{-1})].$$

We denote by $\boldsymbol{\mathcal{H}} \subset \operatorname{Exp}\boldsymbol{\mathcal{K}}$ the closed linear span of the set $\{\operatorname{Exp}\boldsymbol{\delta}(\gamma) \mid \gamma \in \Gamma\}$. As in (4.12), (4.14) and (4.16) we observe that the maps $(\gamma_1, \gamma_2) \mapsto \mathbf{L}(\gamma_1, \gamma_2) \in \mathbb{C}$ and $(\gamma_1, \gamma_2) \mapsto \sigma(\gamma_1, \gamma_2) \in \mathbb{S}$ are 2-cocycles. The maps $\mathbf{V}_\gamma$ in (4.20) extend by linearity to unitary operators on $\boldsymbol{\mathcal{H}}$, and

$$\boldsymbol{\sigma}(\gamma, \gamma')\mathbf{V}_{\gamma\gamma'} = \mathbf{V}_\gamma \mathbf{V}_{\gamma'},$$

for all $\gamma, \gamma' \in \Gamma$, where

$$\boldsymbol{\sigma}(\gamma, \gamma') = \exp(i \cdot \Im(\mathbf{L}(\gamma, \gamma')).$$

As in (4.18) we note that the vector $\operatorname{Exp}(1_\Gamma)$ is cyclic for the projective representation $\mathbf{V}$ on $\mathcal{H}$.

In order to explain the factorisability property of the cyclic projective representation $\mathbf{V}$ of Γ on $\mathcal{H}$ we set, for any nonempty, open subset $\mathcal{O} \subset X$,

$$\Gamma_{\mathcal{O}} = \{\gamma \in \Gamma \mid \gamma(x) = 1_G \text{ for every } x \in X \setminus \mathcal{O}\}$$

and denote by $\mathcal{H}_{\mathcal{O}}$ the closed linear span of $\{\operatorname{Exp} \boldsymbol{\delta}(\gamma) \mid \gamma \in \Gamma_{\mathcal{O}}\} \subset \mathcal{H}$. If $\mathcal{O}_1, \mathcal{O}_2$ are two disjoint nonempty open subsets of X, then

$$\Gamma_{\mathcal{O}_1 \cup \mathcal{O}_2} \cong \Gamma_{\mathcal{O}_1} \times \Gamma_{\mathcal{O}_2},$$
$$\mathcal{H}_{\mathcal{O}_1 \cup \mathcal{O}_2} \cong \mathcal{H}_{\mathcal{O}_1} \otimes \mathcal{H}_{\mathcal{O}_2},$$

and the restriction to $\Gamma_{\mathcal{O}_1 \cup \mathcal{O}_2}$ of $\mathbf{V}$ is unitarily equivalent to the representation $(\gamma_1, \gamma_2) \mapsto \mathbf{V}_{\gamma_1} \otimes \mathbf{V}_{\gamma_2}$ of $\Gamma_{\mathcal{O}_1} \times \Gamma_{\mathcal{O}_2}$. For further results and details concerning factorisable representations of current groups arising from cocycles of unitary representations we refer to the papers [20], [21] and [6].

The factorisability of $\mathbf{V}$ means that the operator algebras generated by the unitary operators $\{\mathbf{V}_\gamma \mid \gamma \in \Gamma_{\mathcal{O}_i}\}$, $i = 1, 2$, are *independent*. For physical considerations this local independence condition should really be weakened to hold only if the closures of the sets $\mathcal{O}_i$, $i = 1, 2$, are disjoint. In [18] and [15], KRP and I made attempts in this direction, where we considered the group $\Gamma = C_C^\infty(\mathbb{R}, G)$ of all C^∞-maps with compact supports from $\mathbb{R}$ to a connected and simply connected Lie group G. By viewing the elements of Γ as smooth maps from $\mathbb{R}$ into a semidirect product $G^{(n)}$ of G with the exponential group of a Lie algebra involving the first n derivatives of the maps $\gamma\colon \mathbb{R} \to G$ (the n-th *Leibnitz extension* of the Lie algebra of G) and constructing factorisable representations of this latter group along the lines of [13] we obtained representations of current groups with slightly more restricted factorisability properties.

After 1975, KRP and I still collaborated on a couple of occasions (cf. [14], [16]), but by that time our research interests had begun to diverge, and KRP's focus had moved on to *quantum stochastic calculus*, *quantum probability* and related problems. However, my memories of KRP's enthusiasm for mathematics, his intellectual generosity, and his – and Shyama's – personal kindness, remain undimmed. Some years ago I was mistakenly listed as one of KRP's former PhD students in the *Mathematics Genealogy Project*; although this is factually incorrect, there may nevertheless be a grain of truth in this mis-attribution.

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5. KRP and His Imprimitivity Theorem

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Dedicated to the memory of K. R. Parthasarathy

I first met Professor Parthasarathy when he was visiting Sambalpur university where I was a postgraduate student. He then stopped over at Bhubaneswar on his way to Delhi. This provided me an opportunity to talk to him in a somewhat informal atmosphere. After several years, I joined the Indian Statistical Institute Kolkata in 1986. Soon after, Professor Parthasarathy invited me for a short visit to the Delhi Center of the Indian Statistical Institute. He was, of course, well known for his very meticulously prepared lectures that he delivered with great clarity. So, when I was asked to give a lecture with Professor Parthasarathy in the audience, I was very nervous. I remember to this date how during my lecture he was making several suggestions for picking better notation among a myriad of other things. I am sure this and many other friendly tips over the years has made me rethink my own approach to both teaching and research.

In the early nineties, together with my colleague Bhaskar Bagchi, I was trying to understand the Wallach set: Let $K : X \times X \to \mathbb{C}$ be a positive definite kernel defined on a set X, that is, the $n \times n$ matrix $\big(\!\big(K(x_j, x_k)\big)\!\big)_{j,k=1}^{n}$ is positive definite for all subsets $\{x_1, \ldots, x_n\}$ of X and all $n \in \mathbb{N}$. The Wallach set of the pair (X, K) for any bounded domain X in $\mathbb{C}^n$ is the set

$$\{\lambda > 0 \mid K^\lambda \text{ is positive definite}\},$$

where K is assumed to be holomorphic in the first variable and anti-holomorphic in the second. Moreover, K^λ is defined by first defining $K(w, w)^\lambda$ for any $\lambda > 0$ and then defining $K(z, w)^\lambda$ by polarizing the power series of the real analytic function $K(w, w)^\lambda$ in a neighbourhood of the set $\{(w, \bar{w}) \mid w \in X\}$. In [3], topics closely related to the Wallach set are discussed. Therefore, I thought it would be great if KRP (by now, like everybody else, I have switched to addressing Professor Parthasarathy by the more familiar name of KRP) can visit us at ISI Bangalore and give a few lectures on positive definite kernels. To my delight, when I checked with him, he happily agreed and delivered a series of mesmerizing lectures on positive definite kernels. He left his very detailed and complete lecture notes with me. Although, he never said it, I think, the idea was for me to convert his carefully prepared handwritten notes to a more formal set of lecture notes or a book. It is entirely my misfortune that I never got around to actually doing it.

A week long conference, "Mathematical Foundations of Quantum Mechanics" at IISER Kolkata in the year 2010 provided another opportunity for me to talk to KRP at length. After my lecture on imprimitivity in this conference, he said that I should learn Quantum Mechanics. We used to take long walks in the evening around the campus. During these long walks, he made it a point to patiently explain some of the basic principles of Quantum Mechanics to someone who had absolutely no idea about the subject. Among other things, he recommended that I get hold of a copy of "PCT, Spin and Statistics, and all that" and read it. Following his advice, of course, I bought the book promptly but I can't say I have been able to read much of it. Nevertheless let me attempt to describe a version of the imprimitivity theorem due to KRP [1] that is both deep, like many of his other theorems, and is at the confluence of the broad themes of Representation theory and Quantum mechanics.

5.1 States

We assume all Hilbert spaces are complex and separable and all operators are bounded. Replace a Borel σ- algebra by the lattice $\mathcal{P}(\mathcal{H})$ of projections on a Hilbert space $\mathcal{H}$ and a Borel measure by a function,

$$\mu : \mathcal{P}(\mathcal{H}) \to [0,1], \text{ satisfying } \mu(0) = 0 \text{ and } \mu(I) = 1;\ \mu\Big(\bigvee_{i=1}^{\infty} P_i\Big) = \sum_{i=1}^{\infty} \mu(P_i)$$

whenever $P_i P_j = 0$ for every $i \neq j$. The map μ is called a *state* on $\mathcal{P}(\mathcal{H})$. Examples are easy to construct: Given a unit vector u in $\mathcal{H}$ define $\mu_u : \mathcal{P}(\mathcal{H}) \to [0,1]$ by setting $\mu_u(P) = \langle Pu, u \rangle$. Are there other states? In general, we have the following theorem due to Gleason.

Theorem 7 (Gleason)**.** *Any state μ must be of the form $\mu(P) = \mathrm{tr}(PT)$ for some non-negative operator T on a Hilbert space $\mathcal{H}$, $\dim(\mathcal{H}) \geqslant 3$, with $\mathrm{tr}(T) = 1$.*

Since T is a non-negative operator with trace 1 there exists an orthonormal set of eigenvectors $\{u_j : j = 1, 2, \cdots\}$ of T with $Tu_j = \lambda_j u_j, \lambda_j \geq 0, \sum_{j=1}^{\infty} \lambda_j = 1$, such that

$$\mathrm{Tr}\,(PT) = \sum_{j=1}^{\infty} \lambda_j \left\langle Pu_j, u_j \right\rangle.$$

Consequently, $\{\mu_u \mid \|u\| = 1, u \in \mathcal{H}\}$ are the extreme points of the convex set consisting of all states. These are called *pure* states. For a pure state μ_u, we have $\mu_u(u) = \mu_u(cu)$ for any c in the unit circle $\mathbb{T}$. We can therefore identify pure states with elements of the Projective Hilbert space $P(\mathcal{H})$ obtained by identifying any two unit vectors u and v in $\mathcal{H}$ if $u = \alpha v$ for some $\alpha \in \mathbb{T}$.

Suppose that $\Gamma : \mathcal{P}(\mathcal{H}) \to \mathcal{P}(\mathcal{H})$ is a one to one onto map satisfying

(i) $\Gamma(0) = 0, \Gamma(I) = I$;

(ii) $\Gamma\left(\bigvee_j P_j\right) = \bigvee_j \Gamma\left(P_j\right), \Gamma\left(\bigwedge_j P_j\right) = \bigwedge_j \Gamma\left(P_j\right)$ for every sequence $\{P_j\}$ in $\mathcal{P}(\mathcal{H})$,

(iii) $\Gamma(I - P) = I - \Gamma(P)$.

Then Γ is called an automorphism of $\mathcal{P}(\mathcal{H})$. All such automorphisms constitute a group under composition. Let $\mathrm{Aut}\,\mathcal{P}(\mathcal{H})$ denote this group. Evidently, if U is a unitary operator on $\mathcal{H}$, then the map $\Gamma_U : \mathcal{P}(\mathcal{H}) \to \mathcal{P}(\mathcal{H})$ defined by

$$\Gamma_U(P) = UPU^{-1}, P \in \mathcal{P}(\mathcal{H})$$

is an automorphism. Are there other automorphisms? Wigner's theorem says that every automorphism Γ of $\mathcal{P}(\mathcal{H})$ is induced in this manner by a unitary or antiunitary operator (UA operator in short), namely, a map $V : \mathcal{H} \to \mathcal{H}$ that is onto, $V(u+v) = Vu + Vv$ for all $u, v \in \mathcal{H}, V(cu) = \bar{c}Vu$, $c \in \mathbb{C}$ and $u \in \mathcal{H}$ and $\langle Vu, Vv \rangle = \langle v, u \rangle$ for all $u, v \in \mathcal{H}$.

Theorem 8 (Wigner)**.** *Let $\mathcal{H}$ be a Hilbert space with $\dim(\mathcal{H}) \geqslant 3$. Then to every automorphism Γ of $\mathcal{P}(\mathcal{H})$ there corresponds a unitary or antiunitary operator U satisfying*

$$\Gamma(P) = UPU^{-1} \qquad \textit{for all } P \in \mathcal{P}(\mathcal{H}).$$

If V is another unitary or antiunitary operator satisfying the identity $\Gamma(P) = VPV^{-1}$ for all P then there exists $c \in \mathbb{T}$ such that $V = cU$.

Complete self-contained proof of Gleason's theorem as well as the theorem of Wigner is in [4].

5.2 Projective Unitary Antiunitary representations

All unitary and antiunitary operators on $\mathcal{H}$ form a group $\mathcal{UA}(\mathcal{H})$. The product of two antiunitary operators is unitary. The product of a unitary and an antiunitary operator is antiunitary. The group $\mathcal{U}(\mathcal{H})$ of all unitary operators is a normal open subgroup of $\mathcal{UA}(\mathcal{H})$ and the quotient $\mathcal{UA}(\mathcal{H})/\mathcal{U}(\mathcal{H})$ consists of two elements. Let $\pi(\mathcal{H})$ denote the compact subgroup $\{cI, |c| = 1\}$. Then $\pi(\mathcal{H})$ is the centre of $\mathcal{UA}(\mathcal{H})$. Wigner's theorem implies that there is a group isomorphism between $\operatorname{Aut}\mathcal{P}(\mathcal{H})$ and the quotient group $\tilde{\mathcal{U}}(\mathcal{H}) := \mathcal{UA}(\mathcal{H})/\pi(\mathcal{H})$. The group $\mathcal{UA}(\mathcal{H})$ with the weak topology (equivalently, the strong topology) inherited by it, is shown in [1, page. 308] to be a complete and separable metric group. When endowed with the quotient topology, $\tilde{\mathcal{U}}(\mathcal{H})$ becomes a separable metric group. Moreover, [1, Lemma 2.3] implies that it is actually a complete and separable metric group. Let

$$\sim : \mathcal{UA}(\mathcal{H}) \to \tilde{\mathcal{U}}(\mathcal{H})$$

be the canonical quotient homomorphism. Thus we may topologise $\operatorname{Aut}\mathcal{P}(\mathcal{H})$ by giving it the quotient topology of $\tilde{\mathcal{U}}(\mathcal{H})$ through Wigner's isomorphism. This makes $\operatorname{Aut}\mathcal{P}(\mathcal{H})$ a complete and separable metric group. A sequence $\{\Gamma_n\}$ in $\operatorname{Aut}\mathcal{P}(\mathcal{H})$ converges to an automorphism Γ if the weak limit, as $n \to \infty$, of $\Gamma_n(P)$ is $\Gamma(P)$ for every $P \in \mathcal{P}(\mathcal{H})$. Moreover, there exists a Borel cross-section for $\sim$, namely, a one to one Borel map $\eta : \tilde{\mathcal{U}}(\mathcal{H}) \to \mathcal{UA}(\mathcal{H})$ such that $\eta(U^\sim)^\sim = U^\sim$, see [1, Corollary 2.2].

Let G denote a locally compact second countable group equipped with the natural Borel structure compatible with the topology. Also, for the sake of brevity, we write $\tilde{\mathcal{U}}$ instead of $\tilde{\mathcal{U}}(\mathcal{H})$. As before, it is equipped with the quotient topology.

A Borel homomorphism from G into $\tilde{\mathcal{U}}$ is called a *projective unitary antiunitary representation* or simply a PUA representation of G in $\mathcal{H}$.

A well-known theorem due to Mackey (cf.[3, Theorem 2.2]) states that if G is a locally compact second countable group and H is a separable metric group, and $\pi : G \to H$ is a Borel homomorphism from G into H, then π is continuous. Since $\tilde{\mathcal{U}}$ is a separable metric group, it follows that the map $g \mapsto \pi(U_g)$ is continuous. Thus, any PUA representation of G is continuous, see [1, Lemma 3.1].

5.3 Multipliers

The lifting of a projective unitary representation to a multiplier representation is well-known. In the paper [1], first, how to lift PUA representations to multiplier representations (see below) is discussed. This is necessarily more complicated since both unitary and antiunitary representations are involved. Secondly, the imprimitivity theorem due to Mackey, originally proved only for *projective unitary representations* is now proved for PUA representations. Let me conclude by providing some details briefly of the imprimitivity theorem of KRP following [1].

Suppose that $g \to U_g{}^\sim$ is a PUA representation of G. Making use of the cross section η, construct a Borel map $g \to \eta\left(U_g{}^\sim\right)$ from G into $\mathcal{UA}(\mathcal{H})$. Since $\eta\left(U_g{}^\sim\right)^\sim = U_g{}^\sim$, it follows that $U_g = \eta\left(U_g{}^\sim\right)$ without loss of generality. Then $g \to U_g$ is a Borel map and for any two elements $g_1, g_2 \in G$, $\left(U_{g_1}U_{g_2}\right)^\sim = U^\sim_{g_1g_2}$. Hence there exists a complex number $\sigma(g_1, g_2) \in \mathbb{T}$ such that

$$U_{g_1}U_{g_2} = \sigma(g_1, g_2)\, U_{g_1g_2} \text{ for all } g_1, g_2 \in G. \tag{5.1}$$

Assume that $U_e = I$, where e is the identity element of G. Then

$$\sigma(e, g) = \sigma(g, e) = 1 \quad \text{for all } g \in G. \tag{5.2}$$

Computing $U_{g_1}U_{g_2}U_{g_3}$ in two different ways, as $U_{g_1}\left(U_{g_2}U_{g_3}\right)$ and $\left(U_{g_1}U_{g_2}\right)U_{g_3}$, it is shown (see [1, Equation (3.3)]) that

$$\sigma(g_1, g_2)\,\sigma(g_1g_2, g_3) = \begin{cases} \sigma(g_1, g_2g_3)\,\sigma(g_2, g_3) & \text{if } g_1 \in G^+ \\ \sigma(g_1, g_2g_3)\,\bar{\sigma}(g_2, g_3) & \text{if } g_1 \in G^-, \end{cases} \tag{5.3}$$

where the set G^+ is the open and closed normal subgroup $\{g : U_g^\sim$ is unitary modulo $\pi(\mathcal{H})\}$, see [1, Lemma 3.1], and $G^- := \{g : U_g^\sim$ is antiunitary modulo $\pi(\mathcal{H})\}$.

A Borel function σ defined on $G \times G$ and taking values in $\mathbb{T}$ is called a multiplier if it satisfies Equations (5.2) and (5.3). A Borel map $g \to U_g$ from G into $\mathcal{UA}(\mathcal{H})$ is called a multiplier representation if there exists a multiplier σ such that Equation (5.3) is satisfied. When G^- is the empty set, that is, $U_g{}^\sim$ is a projective unitary representation, Equations (5.2) and (5.3) coincide with the usual multiplier identities, see [3, page 2].

Theorem 9. (Theorem 3.1, [1]). *Let G be a locally compact second countable group and $g \to U_g^\sim$ be a PUA representation of G. Then there exists a multiplier representation $g \to V_g$ of G such that $V_g^\sim = U_g^\sim$ for all $g \in G$. Conversely every multiplier representation $g \to V_g$ of G determines a PUA representation $g \to V_g^\sim$ of G.*

5.4 Imprimitivity

We first recall Mackey's imprimitivity theorem and then describe the non-trivial generalization of this theorem obtained by KRP.

Let G be a locally compact second countable group and X be a locally compact G - space, that is, there is a map $\alpha : G \times X \to X$, such that for a fixed $g \in G$, the map $x \to \alpha_g(x)$, $\alpha_g(x) := \alpha(g, x)$ is bijective and continuous on X, moreover, $g \to \alpha_g$ is a homomorphism. The action of G on X is said to be *transitive* if for every pair x_1, x_2 in X, there is a $g \in G$ such that $g \cdot x_1 = x_2$, $g \cdot x := \alpha(g, x)$. Let $H \subseteq G$ be a closed subgroup and let $X := G/H$ be the space of cosets: $\{gH \mid g \in G\}$. Equipped with the action of G by left multiplication: $g'(gH) := (g'g)\, H, g', g \in G$, the coset space X is a transitive G- space.

Let $(X, \mathcal{B})$ be the Borel measurable space, and note that each $g \in G$ defines a continuous map on X by our assumption. Given a σ-finite measure μ on X, define the *push-forward* $g_*\mu$ of the measure μ by the requirement

$$(g_*\mu)(A) := \mu(g \cdot A), \;\; g \cdot A := \{g^{-1} \cdot s \mid s \in A\}, A \in \mathcal{B}.$$

The measure μ on X is said to be invariant if $g_*\mu = \mu$ and *quasi-invariant* if $g_*\mu$ is equivalent (mutually absolutely continuous) to μ for all $g \in G$. There is a quasi-invariant measure uniquely determined modulo equivalence on X, see page 313 of [1].

If G is second countable, then there is a Borel cross-section $p : G/H \to G$, that is, a Borel subset $B \subset G$ that meets each coset of H in exactly one point. Thus, each $g \in G$ can be written uniquely as $g = g_1 g_0$ with $g_0 \in H$ and $g_1 \in B$, see page 315 of [1].

A spectral measure, or a projection valued measure, defined on X is a projection valued map $P : \mathcal{B} \to \mathcal{P}(\mathcal{H})$ such that $P(X) = I$ and $P(\cup E_k) = \sum_{k=1}^{\infty} P(E_k)$ for any disjoint collection of sets E_k, $k = 1, 2, \dots$, in $\mathcal{B}$, where the convergence is in the strong operator topology.

A system of imprimitivity $(\mathcal{H}, U_g, P(E))$ introduced by Mackey consists of a *projective unitary representation* U of a second countable locally compact group G on a Hilbert space $\mathcal{H}$ and a regular $\mathcal{H}$-projection-valued measure P on X such that

$$U(g)P(E)U(g)^{-1} = P(g \cdot E) \tag{5.4}$$

for all $g \in G$ and every Borel subset E of X.

The imprimitivity theorem of Mackey (*involving only projective unitary representations*) has two parts: Firstly, any transitive imprimitivity $(\mathcal{H}, U_g, P(E))$ is equivalent to a canonical imprimitivity, where $\mathcal{H} = L^2(X, \mu, \mathcal{H}_n)$, U is a projective unitary representation on $L^2(X, \mu, \mathcal{H}_n)$, that is,

$$(U(g)h)(x) = c(g, x)(g \cdot h)(x), \;\; h \in L^2(X, \mu, \mathcal{H}_n), \;\; g \in G, (g \cdot h)(x) = h(g \cdot x),$$

where $c : G \times X \to \mathcal{U}(\mathcal{H}_n)$ is a Borel map taking values in the group of unitary operators acting on the Hilbert space $\mathcal{H}_n$ of dimension n. For U to be a homomorphism, the function c must be a cocycle. The spectral measure P is defined, via the functional calculus, by setting $P(E) = M_{\mathbb{1}_E}$, $E \in \mathcal{B}$ and $\mathbb{1}_E$ is the characteristic function of E. Here M_f denotes the multiplication by f, $f \in L^\infty(X, \mu, \mathcal{H}_n)$ on $L^2(X, \mu, \mathcal{H}_n)$. Secondly, the imprimitivity theorem asserts that such a multiplier representation is induced from a unitary representation of the subgroup H acting on the Hilbert space $\mathcal{H}_n$.

In the generalization of Mackey's imprimitivity theorem obtained by KRP, the projective unitary representation is replaced by a PUA (projective unitary antiunitary) representation. An automorphism of the lattice of projections induces a map on the state space and that along with Wigner's theorem discussed in the beginning not only justifies such a generalization but makes it indispensable. However, there are new complications arising from the decomposition of the group $G = G^+ \biguplus G^-$. As we have seen, the multiplier identities are a lot more complicated.

Obtaining a canonical form of the imprimitivity when both projective unitary and antiunitary (PUA) representations are involved is the first non-trivial step in the generalization of Mackey's imprimitivity theorem to the case of PUA representations. Let me reproduce below how KRP achieves this in [1], where his $L_2(\mu, n)$ stands for what we have called $L^2(X, \mu, \mathcal{H}_n)$.

"In the space $L_2(\mu, n)$, the complex conjugation which maps f to $\overline{f}$ is a canonical antiunitary operator. By the discussion in §2, it follows that every antiunitary operator is the product of a unitary operator and this conjugation. Making use of this fact and following the arguments of Mackey [2] one can prove the following lemma.

Lemma 4.2. Let $\left\{L_2(\mu, n), V_g, P^0(E)\right\}$ be an imprimitivity system for G on X. Let $G = G^+ \cup G^-$be the UA decomposition of G associated with the PUA representation $g \to V_g{}^\sim$. Then there exist functions $C(g, x)$ and $D(g, x)$ defined respectively on $G^+ \times X$ and $G^- \times X$ and taking values in the space of unitary operators in $\mathbb{C}^n$ such that

$$
\begin{aligned}
\left(V_g f\right)(x) &= \left[\frac{d\mu}{d\mu^g}\left(g^{-1}x\right)\right]^{\frac{1}{2}} C\left(g, g^{-1}x\right) f\left(g^{-1}x\right) \quad \text{if} \quad g \in G^+ \\
&= \left[\frac{d\mu}{d\mu^g}\left(g^{-1}x\right)\right]^{\frac{1}{2}} D\left(g, g^{-1}x\right) \overline{f}\left(g^{-1}x\right) \quad \text{if} \quad g \in G^-
\end{aligned}
$$

where μ^g is the quasi invariant measure defined by the equation $\mu^g(E) = \mu(gE), E \in \mathcal{B}_X$."

The second part of the Imprimitivity theorem is to show that the unitary representation U in any system of imprimitivity based on $X = G/H$ is a representation induced from a representation κ of the subgroup H, that is, the representation U is of the form

$$
(U(g)f)(x) = \sqrt{\frac{d\mu(g \cdot x)}{d\mu(x)}}\kappa(h) f(g^{-1} \cdot x).
$$

Here $h \in H$ is determined from the relation $g\, p(g^{-1} \cdot x) = p(x)h$, $x \in X$, where $p : G/H \to G$ is a Borel cross-section.

It is impossible to go through all the intricacies of the powerful generalization obtained by KRP of Mackey's imprimitivity theorem in a short article like this one. Therefore, I have decided to conclude by reproducing one of the main theorems of KRP from [1]. I hope that the reader will not have any difficulty with what is being said and in appreciating the depth of what is involved. I am sure this will be motivation enough to read the original work of KRP.

Theorem 10. (Theorem 4.1, [1]). *Let G be a locally compact second countable group, $H \subset G$ a closed subgroup and $X = G/H$ the homogeneous space of left cosets. Let $\left\{\mathcal{H}, U_g, P(E)\right\}$ be an imprimitivity system for G on X. Let $G = G^+ \cup G^-$be the UA decomposition of G with respect to PUA representation $g \to U_g, g \in G$. Suppose that G^+acts transitively on X and σ is the multiplier of the representation $g \to U_g$. Let γ be a one one Borel map from X into G^+such that $\pi\gamma$ is the identity map of X onto itself. Then there exists an equivalent imprimitivity system*

$\{L_2(\mu,n), V_g, P^0(E)\}$ *where*

$$(V_g f)(x) = \frac{\sigma\left(g, \gamma\left(g^{-1}x\right)\right)}{\sigma\left(\gamma(x), \gamma(x)^{-1} g \gamma\left(g^{-1}x\right)\right)} \left\{ \left(\frac{d\mu}{d\mu^g}\right)\left(g^{-1}x\right) \right\}^{\frac{1}{2}} \cdot M_{\gamma(x)^{-1}\gamma(g^{-1}x)} f\left(g^{-1}x\right),$$

μ is a quasi invariant measure, n is a finite or countable cardinal, $h \to M_h$ is a σ- representation of H and P^0 is the canonical projection valued measure on $\mathcal{B}_X$. This imprimitivity system is irreducible if and only if the σ representation $h \to M_h$ of H is irreducible.

In the statement of the theorem reproduced above from [1], (i) P^0 is the canonical projection valued measure on $L^2(\mu, n)$ as described at the bottom of page 313, [1], and (ii) a "σ- representation" is a multiplier representation with multiplier σ, Definition 3.3, [1].

The slight familiarity that I have with the terminology from mathematical physics is mostly from my conversations with KRP. This article, based on one of the papers of KRP that I have always admired, is dedicated to his fond memory.

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At a conference at the University of Columbia (1995)
L. Accardi and T. Hida are seen on the two sides of KRP and
S. R. S. Varadhan is seen behind Hida.
(Photo courtesy of Mrs. Shyamala Parthasarathy)

6. K. R. Parthasarathy: A Great Mathematician and a Master

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In the paper [9] we have tried to give an idea of some of the main achievements in the huge scientific production of K. R. Parthasarathy (KRP or Partha in the following) highlighting the conceptual structures behind them and some, among the many, possible developments. KRP's production, in more than 60 years of uninterrupted scientific activity, would have been sufficient to create the reputation, not for one, but for three or four mathematicians of the highest international level. The fact that his main contribution to science has been the creation, with Robin Hudson, of quantum stochastic calculus, risks to let people forget about his contributions to the theory of infinitely divisible laws of classical probability and its quantum development into the theory of factorizable structures, to the theory of group representations, to classical and quantum information as well as to classical statistics.

His books are masterpieces of clarity of exposition and completeness of information: some of them will remain as classics of mathematics. He had many students and many collaborators, but they constitute a tiny fraction with respect to the generations of mathematicians he has influenced with his books and papers.

This note is dedicated to some memories of my personal interactions with KRP in a scientific dialogue that has continued uninterrupted for fifty years and which for me continues even after his passing. My emphasis will be on the many things I learned from him directly, from long discussions over the years, or indirectly, by studying his papers and books or listening to his talks or lectures. I will quickly describe how, some of the seeds planted in my brain through these interactions blossomed, sometimes years, sometimes decades later. The message I would like to convey is that the study and meditation on his writings has great inspirational potential for those who intend to participate in the construction of the new historical level of probability theory that begun to emerge in the early 1970s with the development of quantum probability.

The remembrance of our first meeting is still so vivid in my memory that I can see it in my mind like a movie. It was in the summer 1982, during the $QP1$–conference in Villa Mondragone. It was the second public occasion in which he and Robin Hudson were presenting at a conference the just born quantum stochastic calculus **(QSC)**, not yet completed with the preservation process: no published paper about it had yet appeared.

In the first one, a few months earlier, they had expounded the contents of the two papers: [4], with emphasis on the classical formulation of the basic quantum martingales, and [5] dealing with the purely quantum case.

Their contribution to the $QP1$–conference [3] was a development of [5], because it was addressed to a public of quantum probabilists and the classical language was abandoned except as a pure analogy.

I was sitting next to KRP, who had not yet given his talk, and Robin was expounding the new stochastic calculus to a public who, like me never heard about such a thing. I knew, from Robin's previous work on the quantum central limit theorem, that he was realizing, in two ways, i.e. through the position and momentum fields, the classical brownian motion process inside the boson Fock space over $L^2(0, +\infty)$, but I was missing the main point because I had the feeling that the techniques used were of classical probability. I remember expressing under my breath, during Robin's talk, this doubt to KRP by saying: *But these are two classical processes!* His immediate reply, *Yes, but they do not commute!*, made me think. I looked more carefully at Robin's transparencies on the screen(in 1982 they were still the main tool used in conferences!), where the precise commutation relation between these two classical brownian motions was written, and for the first time I realized that QSC is a stochastic calculus in which two classical brownian motions, satisfying a very special and non–trivial commutation relation are simultaneously considered. I was not able to understand much more than this vague intuition, because at those times my knowledge even of classical stochastic calculus was very superficial, since I had never used it in my research. I could not imagine that, in the next twenty years, my research would have rotated around this barycenter, like the planetary system around the sun. Yet this vague intuition was enough to convince me that I was in front of something deep and I am at a loss for words to describe my happiness when Partha, towards the end of the conference, approached me asking if I wanted to visit him at the Indian Statistical Institute (ISI) in New Delhi.

The two months I spent at ISI in Delhi in the early 1983, were probably the most intensive period of my scientific life, in the sense that I think that never before or after in my life I learned so many things in such a short period. Partha's knowledge of mathematics was deep and wide and he loved to transmit it to younger people. On my side, I perceived this depth and width and was eager to treasure even the smallest piece of information I received from him. I dedicated my evenings to improve and expand the notes I had been taking during our conversations in the day, trying to fill my comprehension gaps and to complete some of the proofs, that were often sketchy. However the material kept on accumulating at such a rate that the task I had set for myself became impossible to accomplish. These notes continued to accompany me throughout my scientific life up to today and I periodically take up and elaborate some parts of them in various directions, both out of pure curiosity and for concrete applications to my research work. These notes turned out to be a treasure for me in later years but, during my 1983 stay in ISI in Delhi, I have not had time to delve into all the numerous scientific stimuli received from Partha because the main part of my efforts were dedicated to learn from him quantum stochastic calculus (and classical as well: I think I have been one of the first mathematicians to learn classical SC as a corollary of quantum SC, rather than learning QSC as a generalization of classical SC). It was clear for me I was experiencing a unique opportunity in my scientific life and I was determined not to waste it. Among the many things I learned from Partha in that period one played a particular role. He told me that Lévy had proven that any martingale with continuous trajectories is, up to a random time change, a classical brownian motion (BM). Since quantum stochastic calculus was based on quantum brownian motion, it was natural to wonder if the above mentioned Lévy's characterization could be extended from classical to quantum BM. I made a formal calculation that indicated that such a characterization should be possible: it was far from being a proof, just an intuitive suggestion. So, when I explained this idea to Partha, I was afraid he would have dismissed it as nonsense. To my surprise, that wasn't the case: he showed interest and we spent long hours over the next few days discussing the possible ways to make this intuition rigorous.

The problem was fraught with difficulties, both technical and conceptual: the essential assumption in Lévy's theorem was the continuity of trajectories, but trajectories do not exist in the quantum case; moreover, nobody up to that moment had considered the quantum analogue of a random time change in a QSDE (simply because such equations had not yet been invented); but the crucial issue was that the boson commutation relations (CR), which are the starting point to define quantum BM, were not available. On the contrary, not only the CR, but even the gaussianity of the state, had to be deduced starting from two purely probabilistic assumptions (martingale property and continuity of the trajectories) of which only one had a quantum analogue at that time. Therefore, to extend the proof technique developed by P. Lévy to the quantum case, we could not rely on the Hudson–Parthasarathy stochastic calculus because the central aim of our research was to deduce the quantum BM, not to start from it.

Evidently such an undertaking required time while, precisely in those days, my stay at ISI in Delhi was coming to an end. We therefore continued to work in this direction by communicating through letters and, after just over a year, we were able to present a first version of our results [8] at the QP2 conference held in Heidelberg from 1 to 5 October, 1984. The paper [7], that was accepted in 1986, but appeared in print only in 1988, summed up the minimal analytical conditions sufficient to guarantee the emergence of the boson commutation relations and of the gaussian nature of the quantum brownian process from quantum probabilisic assumptions of Lévy type. The assumptions in [7] were slightly stronger than in [8] with the exception of the fermion case that, differently from [8], was deduced from the unification result, obtained one year before by Hudson and KRP [2]. The most difficult conceptual problem, i.e. how to express continuity of trajectories in a quantum context, was eventually overcome with the observation that a condition on the 4th moments of a classical stochastic process that, according to a theorem due to Kolmogorov, guaranteed the continuity of trajectories of the process, made sense also in the quantum case. So we simply took this sufficient condition as a quantum analogue of the continuity of trajectories and it turned out that this was sufficient to prove the coincidence of the mutual quadratic variation with the conditional quadratic variation, which is the crucial distinction between diffusions and processes with jumps. The history of quantum Lévy theorems had many sequels, in different ramifications of quantum probability, but for reasons of space, these will be discussed elsewhere.

I limit myself to mention that *the fourth moment condition*, introduced in [7] was used in classical probability, and became very fashionable in that field, about twenty years later to prove results similar to those considered by us in the quantum case (which includes the classical).

The above mentioned two papers are the only joint papers I have with Partha, but several of my own articles after them have been affected by his influence. I would like to conclude this small note giving a few examples of how the influence of the notes I took on his Delhi teachings in 1983 has materialized concretely in my own research activity.
One of the first things KRP taught me was the connection between the classical theory of random processes with independent and stationary increments, the theory of projective representations of groups and the process of second quantization. He explained to me how all these connections were born in a paper of Kolmogorov on *helices in Hilbert spaces*, evolved twenty years later in a series of papers by Araki and Araki–Woods who introduced the notion of *factorizable structures* and were further extended by himself and Schmidt in their Springer LNM volume [6] and later by Guichardet (see [9] for more details on this issue). Curiosity for these structures is one of the themes that has accompanied me since 1983 till the present time, but about twenty years after my first visit to Delhi, I noticed that some factorizable structures naturally emerged in the programs of non–linear quantization and of C^*–quantization (which for reasons of space cannot be described here), so I decided that the time had come for me to go deeper into this issue and I begun to look at the original papers. In this historical–scientific research, I discovered that, in Kolmogorov's original papers [16], [17], where he first introduced Hilbert space techniques in probability theory, using them to determine the structure of stationary processes with independent increments, the author does not use the term *helix* but generically speaks of *curves in Hilbert space*. In [9] it is described how this approach led to the notions of positive definite and conditionally positive definite kernels, to the identification of Kolmogorov's *curves in Hilbert space* with the increments of a real valued stationary increment stochastic process indexed by the intervals of $\mathbb{R}$ and with what today we call 1–cocycles (for the action of $\mathbb{R}$ on itself by translations), to the fact that any such process is canonically associated to a Hilbert space uniquely determined by its Lévy-Khinchine function and finally to the proof that this Hilbert space is naturally identified to a boson Fock space, … .

In addition to probability theory, Kolmogorov's papers had a tremendous influence in many branches of mathematics. One year after the first Kolmogorov paper, which was set in complex Hilbert spaces, Schönberg and von Neumann [21] discussed the real Hilbert space case. Shortly after, Aronszajn [11], describing a part of Kolmogorov's construction, introduced the name *reproducing kernels* and this name caught up in the western literature, giving rise to an intensive line of research on various aspects of these kernels which continues to this day. To my knowledge, the first use of the term *helix in Hilbert space* is due to Masani [19], [18]. A. M. Yaglom extended Kolmogorov's approach from stationary processes to translation invariant random fields and he also considered kernels given not by functions, but by positive definite distributions (a direction pursued later also by W. von Waldenfels). Quite interesting is another, relatively recent, extension of Kolmogorov's theory due to Fuglede, where helices are generalized to *spirals*.

Definition 5. ([14]) A (logarithmic) spiral of order $\alpha \in \mathbb{R}$ is defined as a continuous path $t \mapsto x(t)$ in a real Hilbert space such that

$$\|x(t_1 + t) - x(t_2 + t)\| = e^{\alpha t}\|x(t_1) - x(t_2)\| \quad , \quad t, t_1, t_2 \in \mathbb{R}. \tag{6.1}$$

For $\alpha = 0$ the spiral becomes a helix.

This definition highlights the fact that, in the early developments of the theory, the natural environment for the notion of helix, generalizing that of *increments of a stochastic process*, was that of a normed (or at least metric) space. As testified by the above mentioned Fuglede's paper, this terminology continues to be used nowadays.

However, in the extensions of Kolmogorov approach to the theory of stationary independent increment processes that begun with Araki's work [10] and culminated with the KRP–Schmidt monograph [6], the various generalizations of the notion of increments of a process (to groups, or spaces acted upon by a group, or Lie algebras or co–algebras, ...) are described in terms of the more algebraic notion of 1–*cocycle* (for the action of a given group). The last part of the monograph [6], dedicated to the determination of the structure of the 1–cocycles for various types of group actions, suggests that the use of the cohomological terminology derives from the work of Bargmann who determined the structure of the 2–cocycles of the additive group $\mathbb{R}^d$. In the theory of group representations, the term 2–*cocycle* is used as a synonymous of *multiplier* (or of logarithm of multiplier) and it would be historically interesting to know if the theory of independent increment stationary processes played a role in the development of this terminology. In any case it would be interesting to understand if, in any application, the metric notions of helix or spiral can be deduced from an underlying notion of 1–cocycle or a modification of it.

For example, denoting $\mathbb{R}^* := \mathbb{R} \setminus \{0\}$ and

(i) $t \in \mathbb{R} \mapsto T_t$ the translation action of the additive group $\mathbb{R}$ on an increment process $(x_{(s,t]})$, indexed by the family of intervals of the form $(r, s] \subset \mathbb{R}$,

$$T_t x_{(r,s]} := x_{(r+t,s+t]}.$$

(ii) $M : (y, x) \in \mathbb{R}^* \times \mathbb{R} \to M_y x := y \cdot x \in \mathbb{R}$ the multiplication action of the multiplicative group $\mathbb{R}^*$ on $\mathbb{R}$ (considered now as the state space of the process $(x_{(s,t]})$), we see that, for each $\alpha \in \mathbb{R}$, $t \in \mathbb{R} \mapsto e^{\alpha t} \in \mathbb{R}^*$ is a parametrization of $\mathbb{R}^*$. So the group representations $(T_{\cdot})$ and $(M_{\cdot})$ are examples respectively of the **kinematical** and **state space symmetries** discussed in Section 3 of [9].

With these notations, the spiral condition (6.1) is implied by the algebraic condition

$$T_t x_{(t_1,t_2]} = M_{e^{\alpha t}} x_{(t_1,t_2]} := e^{\alpha t} \cdot x_{(t_1,t_2]} := e^{\alpha t} \cdot (x_{(0,t_2]} - x_{(0,t_1]}). \tag{6.2}$$

Now notice that (6.2) can be equivalently written in the form

$$T_t x_{(t_1,t_2]} = T_t(x_{(0,t_2]} - x_{(0,t_1]}) = x_{(t,t_2+t]} - x_{(t,t_1+t]} = M_{e^{\alpha t}} x_{(t_1,t_2]}. \tag{6.3}$$

Therefore, putting $t_1 = 0$ and using $x_\emptyset = 0$, (6.2) and (6.3) imply that (6.2) can be equivalently written in the form

$$T_t x_{(0,t_2]} = M_{e^{\alpha t}}(x_{(0,t_2+t]} - x_{(0,t]}) \tag{6.4}$$

which, for $\alpha = 0$ expresses the fact that the map $t \mapsto x_{(0,t]}$ is a 1–$(T_{\cdot})$–cocycle. In this sense we say that **the metric condition** (6.1) can be **deduced** from the natural modification of **the algebraic notion** of 1–cocycle given by (6.4). It is likely that this deduction can be extended to more general groups.

Additional notes on the historical development of the subject can be found in [19], [18] and in the more recent [13].

Another example of non–trivial implications for my research of an input received by Partha in 1983 is also related to factorizable structures, but this time the input concerns a very technical result. Among the many things Partha taught me about factorizable structures in my 1983 stay in Delhi, one remained in a corner of my mind for several years. Namely, when explaining to me the proof of the totality of the exponential vectors in the boson Fock space, he told me that, while it is easy to prove that exponential vectors with test functions restricted to step–functions are total, there was a non–trivial result, proved in the unpublished thesis of the French mathematician Delorme, according to which, to prove totality it is sufficient to further restrict the test functions to finite sums of characteristic functions of disjoint intervals (see [12], in fact Partha mentioned to me Delorme's thesis, but I was not able to find a reference for it).

As I already said, my main concern in that period was to learn QSC and to make some progress in the proof of the quantum Lévy martingale characterization of BM, so I had no much time to devote to other issues. Nevertheless, I made some unsuccessful attempts to prove this result by myself and I arrived to the conclusion that the result was indeed non–trivial. Some years later KRP and Sunder [1] published a proof of this result, but still rather complicated. Eventually Michael Skeide, using some ideas of Arveson, was able to produce a short proof of this result [22].

In the early 2000s, I was studying with S. Kozyrev the quantum theory of interacting particle systems and some difficulties with the infinite volume limit lead us to consider the following question: *It is known, and intuitively obvious, that a stochastic differential equation contains more information than the semigroup equation deduced by it. Can one quantify how much more is this information?* The question applies both to classical and quantum SDE. I don't now why but, thinking about this problem, the idea came to my mind that Delorme's result, that Partha described to me about twenty years earlier, could be of help in this issue. In fact it turned out to be the essential technical tool for the proof that a stochastic flow (i.e. the solution of a, classical or quantum, SDE) is equivalent to four semigroups (the 4–semigroup theorem). I reported this result at the conference in honour of Prof. Kalyan B. Sinha held at the ISI in Kolkata (20—23 December 2003) and still remember Partha's reaction because this has been the only time in my life in which he explicitly told me that he liked a result I had obtained.

Between the last decades of 20th century and the first decades of the 21th, probability theory has undergone a revolution that can only be compared to what happened in geometry starting from the beginning of 1800. In this revolution, the role of KRP has been outstanding. Those who had the privilege of having scientific interactions with him will remember him with gratitude and admiration. Those who had not, can rest assured that their scientific personality will grow enormously through meditation on his writings several of which can already be considered classics of contemporary mathematics.

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7. Contributions of KRP to the Mathematics of Quantum Theory

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I became aware of KRP's work in the late 1960's from his two papers: "On the derivation of the Schrödinger equation in a Riemannian manifold" [20], and "Projective unitary, antiunitary representations of locally compact groups" [21] when I was a student in the U.S.A. completing my PhD thesis work on Mathematical theory of Quantum Fields. These two, written while KRP was working in the U.K., alerted me to his style of writing and thinking. They were both related to the mathematical concepts of Quantum Mechanics - the first one establishes Schrödinger equation in Riemannian manifold, while the second one about the consequences of Wigner's theorem in Quantum Mechanics to representation theory of locally compact groups, which often appear as groups of symmetry of a quantum system. At this stage, KRP seemed to be interested in the structural aspects of the foundations of Quantum Theory and not so much the details of the solutions of the Schrödinger equation or of the (unbounded) generators of the various symmetry groups involved, which gives possibilities of describing different physical models and details of the Schrödinger evolutions.

KRP, as his early training and background shows, is first and foremost a probabilist with a wide perception of the subject. However, in the mid 1960's, in ISI Kolkata, the lectures of V S Varadarajan on the structures (in algebro-geometric settings rather than probabilistic) of Quantum theory had a deep influence on him. Later in his professional life, the synthesis of Probability theory and concepts of Quantum Mechanics seemed to have played a bigger role.

At some point in the late 1970's after his return to India from the UK, KRP developed interest in the theory of perturbation of linear operators in an infinite dimensional Hilbert space and gave a lecture course in the Bangalore campus of TIFR, mostly based on the work of T. Kato on the subject. This first change of direction in his mathematical attention led to two papers: the first on the eigenvalues of analytic matrix-valued maps [23] and the second on the exact bound of the Coulomb potential with respect to the free Dirac operator [22]. These researches possibly grew out of his lecture course on perturbation theory earlier.

It may be important to emphasize that during this decade of the 70's, KRP's research were in various areas of mathematics of the Quantum theory. Some of these were directly related, some not so much, to Probability theory, e.g. on positive definite functions, infinitely divisible distributions, current algebras, etc. These will be written about in this collection elsewhere. Here it is mentioned to bring forth a feeling of the breadth of the mathematical canvas of KRP and also to prepare for the next level of major research activity that erupts in this canvas of KRP.

The next major direction of KRP's research possibly begins with his work with R. L. Hudson and P. D. F. Ion in "The Feynman-Kac Formula for Boson Wiener Processes" [7], and in which probably the first attempt to model Wiener process as processes associated with Bosonic Field was made. It must be mentioned that approximately around this time (1980-), in the UK, R. L. Hudson and R. F. Streater along with their collaborators, had already started related mathematical constructions, Hudson in the CCR Fock space over $L^2(\mathbb{R}_+)$ and Streater in the CAR-Fock space (e.g. [4]). Having completed a PhD thesis work on the representation of the CCR (with infinite degrees of freedom) and mathematical theory of quantum fields, I went to University of Geneva and switched fields completely to the mathematical theory of scattering in the Quantum Mechanics (of finite degrees of freedom). The major developments in the Quantum Field Theory (QFT) in the 1950's and 60's were very exciting, but in my views, did not go far into solving physically important problems. The real reason in my view is that may be it was very difficult and demanding. On the other hand, the mathematics needed for a wide range of physical problems with finite degrees of freedom were either already developed or were emerging rapidly. Thus I had spent the whole of the decade of 1970, working in this area of quantum theory, ending with writing the first book on

Mathematical Theory of Scattering, with Jauch and Amrein in Geneva. However the subject of the theory of Quantum Fields remained alive, though in the back of my mind. It is possible that listening to the lectures (mostly of KRP) in the ISI Delhi centre on this newly emerging subject of Quantum Stochastic Calculus at first ignited my interest - may be this will provide an interesting approach to some of the unsolved problems of QFT. But as it turned out, that's not to be; rather the new theory of Quantum Stochastic Processes, probably did more to model the dissipative processes in quantum theory than for the QFT, aside from expanding the mathematical horizon to bring in different theories of probability, some of which do not satisfy the axioms of Kolmogorov for the classical probability theory. Having had a background in the theory of unbounded operators in a Hilbert space and after learning some probability theory from KRP's lectures, I was ready to venture into a marriage of Stochastics and Quantum theory.

The initial thrust of Hudson and KRP seemed to be in the direction of a kind of "multiplicative and non-commutative integral" (e.g. with Hudson and Ion [8] and the earlier references) as infinite product of "unitary infinitesimals" driven by the "increments of the creation and annihilation operators" in a Fock space. Formally this made good sense but computations are very awkward in this formulation and hence good results were hard to come by. In the same period, KRP and I started looking at a Trotter-Kato product of evolutions with unbounded operator-coefficients driven by classical Brownian motion [29] and we could show that having a limit of these random products is the same as having solutions of classical stochastic differential equations with operator coefficients in a Hilbert space. These were constructive evolutions, constructed out of the independent increments of classical Brownian process. Further generalisations of this idea were carried out by J. M. Lindsay and this author [18]driven by all the three basic quantum martingales with bounded operator coefficients and by D. Goswami, B. Das, and this author [6] on quantum stochastic flows on $*$-algebras with unbounded coefficients. In 1981-82, one of the two important ingredients, viz. stochastic differential equations to replace "infinite product integrals" has lodged itself in the minds of Hudson and Parthasarathy. The second, viz. that of continuous tensor product, possibly came from his much earlier work on "infinitely divisible distributions" and his collaborations with K. Schmidt [28]. These led Hudson and KRP to realise that the Fock space representation of the Weyl-Segal commutation relation (often used by physicists) on $L^2(\mathbb{R}_+)$ provides an ideal setup to implement the continuous tensor product structure. Another way of expressing this would be to say that the "exponential functoriality" of the map

$$\Gamma : h \ni \mathcal{H} \mapsto \text{Fock (over } \mathcal{H}),$$

(this is called the "second quantisation" by the physicists) stating that $\Gamma(h_1 \oplus h_2)$ is unitarily equivalent to $\Gamma(h_1) \otimes \Gamma(h_2)$, where $h_j \in \mathcal{H}_j$, $j = 1, 2$, weaves neatly into the factorisation

$$\begin{aligned} \Gamma(L^2(\mathbb{R}_+)) &= \Gamma\left(L^2([0,s]) \oplus L^2([s,t]) \oplus L^2([t,\infty))\right) \\ &\simeq \Gamma\left(L^2([0,s])\right) \otimes \Gamma\left(L^2([s,t])\right) \otimes \Gamma\left(L^2\left([t,\infty)\right)\right) \qquad \text{for } 0 \le s \le t < \infty. \end{aligned}$$

Formally speaking this can accommodate a "product integral" as "infinitesimal factorised product" in

$$\Gamma(L^2(\mathbb{R}_+)) \simeq \Gamma(L^2([0,t])) \otimes \Gamma(L^2([t, t+\delta t])) \otimes \Gamma\left(L^2\left([t+\delta t, \infty)\right)\right),$$

leading to writing of an integral with respect to a stochastic (yet possibly by non-commutating or quantum) "integrator". The first factor would represent the "past", the last the "future", while the infinitesimal operator-integrator (the second factor) commutes with "what is in the past space", though the final integrated solution may not be commutating.

The stage is now set for the emergence of the seminal paper of Hudson and Parthasarathy [10] giving:

(i) mathematically satisfactory structure in Bosonic Fock space $\Gamma(L^2(\mathbb{R}_+))$ of operator valued processes,

(ii) the operator-integrators (operator-martingales),

(iii) a theory of integration in this background (somewhat in the spirit of Ito integrals), and

(iv) quantum Ito multiplication table of products of operator increments.

Though Hudson in an earlier single authored paper, did give Ito-table with only creation and annihilation increments, this is the first time the full table with conservation process as well appeared. Then they went on to consider quantum stochastic differential equation (qsde) with bounded operator coefficients and the conditions under which the solution will be an unitary operator valued process in $h \otimes L^2(\mathbb{R}_+)$, where h is the space in which the quantum physical system is described.

This paper also observed a remarkable feature of the theory - viz. in the same structure, both classical processes - Brownian motion and Poisson process - are mathematically describable which is impossible in the classical theory of stochastic process. This impossibility is due to the fact that the σ-algebra constructed out of the increments of Brownian motion and that of the Poisson process are incompatible: "In a world of knowledge made up of experiments on the Brownian motion, one cannot answer questions on the Poisson process". But this is the magic of quantum theory - in a sense, one can create a large algebra (in this case $\mathcal{B}(\Gamma(L^2(\mathbb{R}_+)))$, the $*$-algebra of bounded linear operators on Fock space) which contains as commutative sub-$*$-algebras, both the Brownian motion and the Poisson algebras. In fact, $\mathcal{B}(\Gamma(L^2(\mathbb{R}_+)))$ is generated by these two commutative (but *mutually non-commutative*) sub-algebras.

Possibly for sometimes in 1985 onwards, KRP started regular courses on this newly emerging subject which was regularly attended by me and a few graduate students. There was of course the KRP-hallmark in each of his lectures - the clarity of statements and proofs, even the depth of casual remarks and above all, the masterly organisation of material. These lectures metamorphosed to first ISI lecture notes and then to his famed monograph "An introduction to Quantum Stochastic Calculus", Birkhauser, 1992 [27].

This book also contains some of other very interesting observations of Hudson and KRP - viz. that the Fermionic process $\{F(t)\}_{t\geq 0}$ can be realised in the Bosonic Fock space as a quantum stochastic integral (with respect to the Bosonic operator-integrators) satisfying the canonical anti-commutator relation (CAR);

$$\begin{aligned} \{F(t), F(s)\} &\equiv F(t)F(s) + F(s)F(t) = 0, \\ \{F(t), F(s)^*\} &= \min\{t, s\}, \quad 0 \leq s, t \leq \infty. \end{aligned}$$

The book describes in details the construction of the processes, driven by a countable-infinite member of mutually independent basic quantum process, i.e. creation, annihilation, and conservation process, originally by Mohari and the present author (another presentation appears in the book [19]). This later book by the very well known classical probabilist Paul-André Meyer gave, in part a classical probabilist's point of view to the new subject at that time and introduced the concept of a toy Fock space, which often turned out to be quite useful in understanding the basic probabilistic ideas. Furthermore, the central ideas behind the construction of quantum stochastic evolutions via a family of $*$-homomorphic flows on a $*$-algebra of observables are also studied in both these books. This is constructed by a conjugation by unitary solutions of appropriate qsde's just as one computes the Heisenberg evolution from the unitary Schrodinger operators [11]. Some of the central theme of these topics appeared slightly earlier in the paper by L. A. Accardi, A. Frigerio, and J. T. Lewis [1]. These evolutions have more recently been called *quantum diffusions* or *quantum flows*.

During the period of 1985-1992 before these books made their appearance and created an impact in the domains of quantum theory and of probability, KRP had a flurry of academic activities in many different directions, with the central theme of quantum stochastic processes, of which I'll mention a few. The first of these on "The passage from random walk to diffusion in quantum probability" [15], in which he continued successfully his efforts to model quantum flows on a $*$-sub-algebra of $\mathcal{B}(\mathcal{H})$ by approximating the stochastic (noise)-part by countably infinite tensor products of $\mathcal{B}(\mathbb{C}^2)$-pieces. Each step of the random walk produces a $\mathcal{B}(\mathbb{C}^2)$-element associated with the interval $(jh, \overline{j+1}h)$ as a new tensor element. With bounded operator coefficients, it was demonstrated that the quantum random walk (suitably scaled with respect to h, the walk-length) converges strongly to a quantum stochastic diffusion in $\mathcal{B}(\mathcal{H})$, obtained by conjugation by the unique unitary solutions of a stochastic Hudson-Parthasarathy (H-P) equation. Some improvements on these results were noted in the Article of this author "Quantum random walk revisited"[36].

The concept and use of the (random) stop times (in relation to a stochastic process) is a beautiful and powerful tool in the hands of classical probabilists. Hudson had already started an investigation into the possibility of incorporating this concept into the realm of quantum probability as an adapted family of spectral measures in Fock spaces $\Gamma(L^2(\mathbb{R}_+))$. KRP and the present author ([31], [32]) undertook a systematic analysis of this in the Fock space calculus, defined the random stopped-past and -future Fock spaces, and proved that for each stop-time operator S (unbounded non-negative self-adjoint operator in Fock space), the *strong Markov property* holds, i.e.

$$\Gamma(L^2(\mathbb{R}_+)) = \text{ Fock space for } S\text{-stopped past} \otimes \text{Fock space for } S\text{-stopped future.}$$

This work led to further researches into interesting questions on the anomalies in the consequential definitions of random stopped past- and future- quantum observables (algebras) ([3],[5]).There are still many unanswered questions in this sub-area.

Slightly earlier there was the fundamental work of KRP with the present author on integral representation of martingales [30], emulating the famous representation theory of square integrable martingales of Kunita and Watanabe. Hudson and Lindsay had also looked at this problem, in a limited context, in [9]. The 1987 paper mentioned above, introduces a concept of bounded regular martingales and proves that a quantum martingale in Fock space satisfies a H-P qsde with bounded operator coefficients if and only if it is bounded regular. This led to many interesting consequences, e.g. a note on "A martingale of bounded operators that is not representable as a stochastic integral", by J.-L Journé. and P.-A. Meyer, in [14] and a rejoinder by KRP on the note in the same volume [24]. The *bounded regularity* is too strong a constraint and some progress was made by going over to *white noise theory* (or looking at second quantisation over a *somewhat restricted class of functions* in constructing the Fock-like space (but not Fock), see e.g. the article by Un Cig Ji and this author in [13].

A mention needs to be made of several significant papers in this period: "A martingale characterization of canonical commutation and anticommutation relations," ([2]), "Azéma martingales and quantum stochastic calculus", in [26], and "Cohomology of power sets with applications in quantum probability" in [16]. The first of the above (with Accardi) emulates classical work of Lévy in characterising the Brownian motion from the property of it being a martingale and having a characteristic quadratic variation to that of the CCR and CAR in Fock space by their respective properties. The second of the three relates to a very special classical martingale (studied by Azéma) made out of the signature of the Brownian motion and its last zero before time t (it is a process without continuous trajectories). Following up on a suggestion of P.-A. Mayer, KRP showed that the Azéma martingale is one of a class of commutative quantum martingales satisfying a linear qsde. This allowed the use of considerable amount of resources of the development of the quantum stochastic calculus to the problem to study the properties of the whole class, parameterised by a real parameter $c \in [-1, 1]$. This article is one of the many instances of writing by KRP which displays beautifully his mastery of the classical and quantum probability theories. He uses the Fock space quantum stochastic calculus, Maassen's kernel calculus in Guichardet Fock space and the classical Ito calculus of standard Brownian motion in an interwoven fashion to prove a large number of interesting results. For example, for each $c \in [-1, 1]$, the solution $X_c(t)$ is a family of commutative self-adjoint processes, of which all but $X_1(t)$ are bounded operator families and furthermore he shows that while $X_1(t)$ is the classical Brownian motion, $X_{-1}(t)$ is closely related to the Fermionic process, and the process $(2t)^{-\frac{1}{2}}X_0(t)$ has the density function closely related to the arc-sine law. Thus this article establishes one kind of continuous interpolation among various quantum stochastic processes.

Another article by KRP and this author [33] addressed one fundamental issue in the subject, taking a quantum stochastic flow $j_t : \mathcal{A} \mapsto \mathcal{A} \otimes \mathcal{B}(\Gamma(L^2(\mathbb{R}_+)))$ where $\mathcal{A} \subseteq \mathcal{B}(h)$ with h being the initial Hilbert space of the observed system, does the family $\{j_t(x) : a \in \mathcal{A}; t \geq 0\}$ constitute a commutative sub-algebra, if $\mathcal{A}$ is a commutative $*$-algebra to start with. Equivalently, one can ask if the initial observable set are all 'classical', does the flow remains 'classical'? In the said paper, the authors showed that if the map coefficients driving the flow $\hat{j}_t(\cdot)$ map $\mathcal{A}$ boundedly into $\mathcal{A}$, then the answer is yes. This structure accommodates a large class of classical Markov processes on a measurable set, and this aspect of the theory does need further exploration, which somehow has not happened.

The theme of unification of various noises (classical, Bosonic, and Fermionic) engaged KRP in his article in [25], and was also extended (with this author, [34]) where the free noise (coming from *free* commutation relations) also could be accommodated, however at the cost of the need to use *non-adapted integration theory*, or more precisely, the integration of the *future-adapted integrands*.

The structure of the generator of uniformly continuous dynamical semigroups, described by Lindblad, and later by Christensen and Evans in von Neumann algebras led KRP to two important articles. The first (with the present author: [35]) achieves the quantum stochastic dilation of the said semigroup with CE-parameters by

(i) constructing a compensated Poisson process in the Fock space by using the CP-map in the CE-parameters and then,

(ii) perturbing this process by a suitable unitary-cocycle to reach the stochastic dilation.

The second paper (with J. M. Lindsay, [17]) investigates the properties of CP-map-valued, uniformly contractive, though not necessarily $*$-homomorphic, stochastic flows on a von Neumann algebra, and also constructs a class of such flows for a given set of CE-parameters.

There were a few other areas of *chaotic/stochastic* motions in non-commutative sets, in which KRP made important contributions in collaboration with R. L. Hudson and others, e.g. [12] about which very little will be said here. Another important area that will be dealt with elsewhere in this volume is, his work with B. V. R. Bhat in constructing general quantum Markov process, not necessarily described by qsde's and his later systematic studies on finite dimensional quantum Gaussian distribution. And there will remain unsaid, unwritten areas/ideas of discussions that I have had the privilege of having with him (e.g. relations between non-commutative probability and non-commutative geometry) which shall have to wait for future to see their unfolding.

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KRP with Kalyan Sinha and Barry Simon
ISI Bangalore (2019)

KRP at ISI Bangalore (2015) - V. S. Sunder and Kalyan Sinha are seen in the photo

8. Weak Markov Flows and Extreme Points of Convex Sets

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8.1 Introduction

The goal here is to provide an overview of articles by K. R. Parthasarathy on two specific somewhat unrelated areas mentioned in the title. The papers on 'Weak Markov Flows' is limited to the decade 1990-1999, and is about a quantum analogue of classical Markov processes. The second topic 'Extreme points of Convex sets' is a lifelong favorite of Parthasarathy and the papers are spread over several decades and it includes problems and ideas from both classical and quantum probability. Typically the tools come from elementary linear algebra and the final answers are simple and elegant. For the convenience of reader we include the references of papers of Parthasarathy in the main text itself.

8.2 Weak Markov Flows

After the discovery of quantum Ito formula by Hudson and Parthasarathy in 1980's, the subject of Quantum Stochastic Calculus came into existence and quickly it became an area of hectic research activity. Apart from the founders several prominent names such as L. Accardi, V. Belavkin, F. Fagnola, A. Holevo, Un Cig Ji, J. M. Lindsay, K. B. Sinha, W. Waldenfels, and their collaborators and students also got engaged in developing this theory.

One of the early achievements of this theory was to describe a beautiful class of QSDE's with unitary processes as solutions. When their defining coefficients satisfy some structure equations, conjugation by these unitary processes solve the dilation problem for quantum dynamical semigroups in the bounded generator case. Here we elaborate a bit on this problem. One parameter semigroups of unital completely positive maps on C^*-algebras are the quantum analogues of semigroups of stochastic maps describing probabilities of Markov transitions in classical Markov process theory. They are known as Quantum Markov semigroups. They are used to model open systems or irreversible systems in quantum theory. Can we write them as expectation semigroups of 'quantum Markov processes' in some sense is the question. Now if the quantum Markov semigroup has a bounded generator then we can explicitly write down quantum stochastic differential equations and as mentioned above Hudson-Parthasarathy theory provides conjugations by unitary processes as solutions. In other words the theory is quite satisfactory in the bounded generator case. However, keeping interesting examples in mind, it is important to consider quantum Markov semigroups with unbounded generators. As demonstrated by Fagnola, Sinha and others, extending the Hudson-Parthasarathy theory to include such unbounded generators has been successful in some special cases with stringent domain conditions on the generator. However, the method is very analytic in nature and so it has its limitations.

On the other hand, in classical probability theory, thanks to Kolmogorov's existence theorem there is a Markov process arising from every Markov semigroup of stochastic maps. This raised the question as to whether we can have an analogous theory of constructing a suitable Markov process for quantum dynamical semigroups without dealing with the generator. This is what the theory of 'Weak Markov Flows' accomplishes. Now let us look at the papers of K. R. Parthasarathy in this area.

1. K. R. Parthasarathy, A continuous time version of Stinespring's theorem, Quantum Probability and Applications -V, Lecture Notes in Mathematics **1442** (1990) 296-300.

   Every unital completely positive (CP) map dilates to a unital $*$-homomorphism and this is known as Stinespring's theorem. Construction of a Markov process, starting with a quantum Markov semigroup required stitching together Stinespring dilations of individual completely positive maps of the semigroup. An early attempt in this direction can be seen in this paper. However, there is no explicit mention of quantum Markov processes in this article.

2. B. V. Rajarama Bhat and K. R. Parthasarathy, Markov dilations of nonconservative quantum dynamical semigroups and a quantum boundary theory, Annales de l'I.H.P. Probabilités et statistiques, **31**(1995) no. 4, pp. 601-651.

   This article was the main content of the author's Ph.D. thesis, written under the supervision of K. R. Parthasarathy. Here for the first time, the notion of 'Weak Markov Flows' has been defined and studied. In this article unital semigroups are called conservative, and contractive completely positive semigroups, which are not necessarily unital are called nonconservative semigroups. This is a terminology inspired from classical stochastic processes, and is not so common in recent papers.

   Let $\mathcal{H}_0$ be a complex separable Hilbert space and let $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H}_0)$ be a von Neumann algebra (for some of the constructions C^*-algebras also will do). Consider a non-conservative one parameter semigroup of completely positive maps (quantum dynamical semigroup) on $\mathcal{A}$:

   $$T = \{T_t : t \geq 0\}.$$

   The basic construction in this paper, allows us to construct a triple $(\mathcal{H}, F, j)$, where:

   (a) $\mathcal{H}$ is a Hilbert space containing $\mathcal{H}_0$ as a subspace and is called the dilation space. It is like the measure space constructed in the Kolmogorov construction of Markov processes;

   (b) $F = \{F_t : t \geq 0\}$ is an increasing family of projections on $\mathcal{H}$, where $F(0)$ is the projection onto $\mathcal{H}_0$. This family provides the analogue of filtration of classical theory. The von Neumann algebra $\{F(t)XF(t) : X \in \mathcal{B}(\mathcal{H})\}$ is like the algebra of bounded functions measurable with respect to the σ-field at time t. The maps $E_t : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ defined by

   $$E_t(X) = F(t)XF(t), \quad X \in \mathcal{B}(\mathcal{H})$$

   are the associated conditional expectation maps.

   (c) $j = \{j_t : t \geq 0\}$ is a family of (not necessarily unital) $*$-homomorphisms, $j_t : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ satisfying (i) $E_0(j_0(X)) = XF(0), \ \ j_t(X)F(t) = E(t)(j_t(X)), \ \ X \in \mathcal{A}$; (ii) $E_s(j_t(X)) = j_s(T_{t-s}(X))F_s, \ \ X \in \mathcal{A}, 0 \leq s \leq t < \infty$. For obvious reasons the property (i) is called adaptedness and (ii) is called Markov property.

   The triple $(\mathcal{H}, F, j_t)$ is called a *weak Markov flow* with T as its expectation semigroup. Under certain natural minimality conditions, up to unitary equivalence, there exists a unique weak Markov flow $(\mathcal{H}, F, j)$ associated with any non-conservative quantum dynamical semigroup.

   The naturalness and the power of this construction has been demonstrated in the article by developing a quantum version of Feller's boundary theory for nonconservative semigroups.

3. B. V. Rajarama Bhat and K. R. Parthasarathy, Kolmogorov's existence theorem for Markov processes in C* algebras, Proc. Indian Acad. Sci. (Math. Sci.), **104**(1), (1994) pp. 253-262.

   This extends the basic construction of the previous article, where now instead of semigroups we have a family of contractive completely positive maps, $\{T_{s,t} : 0 \leq s \leq t\}$ on a von Neumann algebra $\mathcal{A}$, satisfying $T_{r,t} = T_{s,t}T_{r,s}$ for $0 \leq r \leq s \leq t$. Now we quote from the abstract of the article: Given a family of transition probability functions between measure spaces and an initial distribution Kolmogorov's existence theorem associates a unique Markov process on the product space. Here a canonical non-commutative analogue of this result is established for families of completely positive maps between C* algebras satisfying the Chapman-Kolmogorov equations. This could be the starting point for a theory of quantum Markov processes.

4. K. R. Parthasarathy and K. B. Sinha, Quantum Markov processes with a Christensen-Evans generator in a von Neumann algebra, Bull. London. Math. Soc., **31**(5), (1999) 616-626.

   Suppose a unital quantum dynamical semigroup $T = \{T_t : t \geq 0\}$ on a von Neumann algebra has a bounded generator. Then thanks to Christensen and Evans [7], we know the structure of the generator. Now it is a natural problem as to whether we can construct a weak Markov flow concretely using the generator alone instead of the abstract algebraic approach described in [2] and [3] above, which need the knowledge of the semigroup. This article answers this affirmatively. The construction is quite involved and is inspired by compound Poisson process of classical stochastic process theory.

5. K. R. Parthasarathy and V. S. Sunder, Exponentials of indicator functions are total in the Boson Fock space, $\Gamma(L^2[0,1])$.

   This article proves a beautiful technical result on exponential vectors in Fock spaces. It was established by the authors in 1980's when Sunder was a faculty member in Indian Statistical Institute, Delhi. However, the result remained unpublished until some use for it was found in [4] to prove minimality of certain weak Markov flows. This is the connection of this article with the theory of weak Markov flows. The proof is measure theoretic and uses tools such as martingale convergence theorem. An indirect proof can be found in [6]. Now much simpler direct proofs are available [10]. The result has found interesting applications in the study of cocycles on Boson Fock spaces ([1] and [9]).

   In due course, the theory of weak Markov flows was used to obtain dilation of quantum Markov semigroups to E_0-semigroups (semigroups of unital endomorphisms) making the theory more useful in Operator Algebra theory [5].

8.3 Extreme points of convex sets

The well-known Krein-Milman theorem tells us that any compact convex subset of a Hausdorff locally convex topological vector space is the closed convex hull of its extreme points. This makes it important to determine the set of extreme points of naturally occurring convex sets. It is seen that often the set of extreme points has a nice description with various symmetry properties. However, in some situations there may not be any explicit description of extreme points and we may have to be satisfied with some abstract characterizations.

Here we look at contributions of K. R. Parthasarathy in this circle of ideas. He has considered various convex sets of primary interest in quantum theory. One of the recurring themes, is to look at coupled systems with fixed marginals. To begin with we see some papers which deal with elementary classical probability.

1. K. R. Parthasarathy, On extremal correlations, C R Rao 80th birthday felicitation volume, Journal of Statistical Planning and Inference, 103(2002) 173-180.

   By definition a correlation matrix of order n is an $n \times n$ positive semi-definite matrix whose diagonal entries are equal to 1. They are correlation matrices of random variables with unit variance. Clearly the set E_n of all correlation matrices of order n forms a convex compact set in the space of all $n \times n$ (real or complex) matrices. Determining its set of extreme points seems to be a hard problem. Earlier it was shown by Grone et al [8] that if A is an extreme point of E_n, then $k := \text{rank }(A)$, satisfies

   $$k(k+1) \leq 2n.$$

   Parthasarathy completely characterizes the set of extreme correlation matrices of rank k, as matrices having some special block decompositions. The result of Grone et al., is a simple corollary of this characterization. Some remarks connecting this result to Bell inequalities of quantum theory can also be found in this article.

2. K. R. Parthasarathy, A Remark on Spin Correlations, Sankhyā: The Indian Journal of Statistics, Series A (1961-2002), **51**(2), (1989), pp. 192-195.

   In quantum theory, self-adjoint operators are the quantum random variables and are known as observables. On any measurement, the values taken by observables are points in the spectrum. Spin observables are those which take only $\{+1, -1\}$ as their values. In other words, they are both self-adjoint and unitary.

   Given a non-empty set X, a function $K : X \times X \to \mathbb{C}$, is said to be a positive kernel if for any finite subset $\{x_1, x_2, \dots, x_n\}$ of X, the matrix:

   $$[K(x_i, x_j)]_{1 \le i,j \le n}$$

   is positive semi-definite. Observe that if $K(x, x) = 1$ for every $x \in X$, then these matrices are correlation matrices.

   Consider a positive kernel K on a set X, such that $K(x, x) = 1$ for all x. It is shown that there exists a Hilbert space $\mathcal{H}$ with a unit vector Ω, a family $\{U_x, x \in X\}$ of unitary operators and a spin observable S, such that on considering spin observables $S_x = U_x^{-1} S U_x, \quad x \in X$,:

   $$\langle \Omega, S_x \Omega \rangle = 0;$$

   $$\langle \Omega, S_x S_y \Omega \rangle = K(x, y), \quad \forall x, y \in X.$$

   In other words, any preassigned correlation structure can be achieved by a family of spin observables, which are all unitarily equivalent to each other.

3. K. Balasubramanian; J. C. Gupta and K. R. Parthasarathy, Remarks on Bell's inequality for spin correlations, Sankhyā, Ser. A **60**(1), (1998), 29-35.

   A random variable ξ is said to be a spin variable if the only values taken by it are $\{+1, -1\}$. This is the classical probability analogue of spin observables introduced before. In the current article, only 'symmetric' spin variables, i.e., those satisfying $P(\xi = +1) = P(\xi = -1) = \frac{1}{2}$ are considered.

   If $\{\xi_i : 1 \le i \le n\}$ is a family of symmetric spin variables. Clearly they have expectation $E(\xi_i)$ equal to zero. Consider the correlation matrix $\Sigma = [\sigma_{ij}]_{n \times n}$ defined by

   $$\sigma_{ij} = E(\xi_i \xi_j), \quad 1 \le i, j \le n.$$

   Note that $\sigma_{ii} = 1$ as $P(\xi_i^2 = 1) = 1$. Now the famous, Bell's inequalities ([3]) can be expressed in the form

   $$1 + \epsilon_i \epsilon_j \sigma_{ij} + \epsilon_j \epsilon_k \sigma_{jk} + \epsilon_k \epsilon_i \sigma_{ki} \ge 0, \quad \forall 1 < i < j < k \le n,$$

   where $\epsilon_1, \epsilon_2, \dots, \epsilon_n$ are ± 1.

   It is not hard to construct real correlation matrices which violate Bell's inequalities. In other words, there are correlation matrices which can't be realized as correlation matrices of classical symmetric spin random variables. In the present article it is shown that for $n \le 4$, Bell's inequalities are also sufficient to ensure that the given correlation matrix Σ is the covariance matrix of a set of symmetric spin random variables. However, for $n \ge 5$, this is not the case.

   Interestingly, a correlation matrix Σ where all off-diagonal entries are equal (say, $\sigma_{ij} \equiv \sigma$, $i \ne j$, is the correlation matrix of a family of symmetric spin random variables iff $1 \ge \sigma \ge -\frac{1}{n-1}$ if n is even and $1 \ge \sigma \ge -\frac{1}{n}$ if n is odd.

   In the quantum setting Bell's inequalities can be violated by spin observables. In fact it is shown that any correlation matrix can be realized as correlation matrix of spin observables.

4. K. R. Parthasarathy, Extreme Points of the Convex Set of Joint Probability Distributions with Fixed Marginals, Proc. Indian Acad. Sci. (Math. Sci.) **117**(4), (2007) 505-515.

   From the title of this article the convex set under consideration is clear. The surprise is in the proof technique. The article uses Stinespring's theorem, which characterizes completely positive maps on C^*-algebras. Applying this theorem to commutative algebras various characterizations are obtained for extreme points of the joint distributions with fixed marginals. As a special case, a new proof of the classical Birkhoff- Von Neumann theorem identifying extreme points of the set of doubly stochastic matrices as permutation matrices has been obtained. The following open problem has been mentioned: Characterize the possible support sets of extreme joint distributions in terms of the given marginal distributions.

5. K. R. Parthasarathy, Comparison of completely positive maps on a C*-algebra and a Lebesgue decomposition theorem, In Athens Conference on Applied Probability and Time Series Analysis, Lecture Notes in Statistics **-114** (1996), 34-54.

   The convex set of completely positive maps dominated by a given completely positive map can be described using positive contractions in the commutant of the range of the $*$-homomorphism of Stinespring's representation (W. Arveson [2] had observed this). The positive contraction which gives the dominated map can be considered as an analogue of Radon-Nikodym derivative. In the article we see a quantum version of the result from classical probability that the Radon-Nikodym derivatives of a measure Q dominated by a measure P, on an increasing filtration of sub-σ-fields forms a convergent martingale with respect to P. Further, a quantum analogue of Lebesgue decomposition theorem for two completely positive maps has been obtained.

6. K. R. Parthasarathy, Extreme Points of the Convex set of Stochastic Maps on a C^*-Algebra, Infinite Dimensional Analysis, Quantum Probability and Related Topics, **1**(4) (1998), pp. 599-609.

   This article characterizes the set of extreme points of the convex set of unital completely positive maps on a C^*-algebra. The main tool is the Stinespring's representation theorem. In case of maps on full matrix algebras, the extreme points can be characterized using the corresponding Choi-Kraus coefficients. The result with suitable modifications can be extended to unital quantum channels.

7. K. R. Parthasarathy, Extremal decision rules in quantum hypothesis setting, Infinite Dimensional Analysis, Quantum Probability and Related Topics, Vol. 02, No. 04, pp. 557-568 (1999),

   It is assumed that a quantum system can be in any of the n, known states (density matrices) $\{\rho_1, \rho_2, \dots, \rho_n\}$. The problem is to determine as to which is the right state. This requires a *'decision rule'*, which basically means choosing a positive operator valued measure (POVM) on $\{1, 2, \dots, n\}$ taking values in the cone of positive operators on a finite dimensional Hilbert space $\mathcal{H}$:
   $$\{(X_1, X_2, \dots, X_n) : X_j \in \mathcal{B}(\mathcal{H}), X_j \geq 0, \forall j, \sum_j X_j = I\}.$$
   Clearly the set of POVMs is a compact convex set. In fact, this collection is in a natural one to one correspondence with the set of unital CP maps on the commutative algebra of continuous functions on $\{1, 2, \dots, n\}$ with values in $\mathcal{B}(\mathcal{H})$. Consequently the extreme points of the set of decision rules can be determined using the results of previous paper. The optimal decision rules are those which minimize a given cost functional and Holevo had certain equations characterizing them. In this article Holevo's equations for an optimal decision are derived through elementary means and a simple example is given in order to illustrate the non-uniqueness of optimal decision rules.

8. K. R. Parthasarathy, Extremal quantum states in coupled systems, Ann. I. H. Poincaré, PR 41 (2005) 257-268.

   The problem here is very similar to the problem discussed above of joint measures with given marginals. Let $\mathcal{H}_1, \mathcal{H}_2$ be finite dimensional complex Hilbert spaces describing the states of two finite level quantum systems. Suppose ρ_i is a state in $\mathcal{H}_i, i = 1, 2$. Consider the convex set of all states in $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ whose marginal states in $\mathcal{H}_1$ and $\mathcal{H}_2$ are ρ_1 and ρ_2 respectively. The article provides a necessary and sufficient criterion for an element in this set to be an extreme point. Such a condition implies, in particular, that for a state ρ to be an extreme point, it is necessary that the rank of ρ does not exceed $(d_1^2 + d_2^2 - 1)^{\frac{1}{2}}$ where $d_i = \dim \mathcal{H}_i, i = 1, 2$.

   Further when $\mathcal{H}_1$ and $\mathcal{H}_2$ coincide with the 1-qubit Hilbert space $\mathbb{C}^2$ with its standard orthonormal basis $\{|0\rangle, |1\rangle\}$ and marginal states ρ_1, ρ_2 are given by $\rho_1 = \rho_2 = \frac{I}{2}$, it turns out that a state is extremal if and only if it is of the form $|\Omega\rangle\langle\Omega|$, where

   $$\Omega = |0\rangle|\psi_0\rangle + |1\rangle|\psi_1\rangle,$$

   with any arbitrary ortho-normal basis $\{|\psi_0\rangle, |\psi_1\rangle\}$ of $\mathbb{C}^2$. In particular, the extremal states are the maximally entangled states. Using the Weyl commutation relations in the space $L_2(A)$ of a finite Abelian group A, a mixed extremal state with marginals $\frac{1}{n}I_n, \frac{1}{n^2}I_{n^2}$ has been obtained for $n \geq 2$. This paper has garnered a large number of citations, showing the interest in this sort of problems in quantum information theory community.

Acknowledgements: I thank the editors of this volume for giving me this opportunity to contribute this article. Professor K. R. Parthasarathy (KRP) was my Ph.D. supervisor and it is impossible to put down in words what I owe to him in my life. My reminiscences of interactions with KRP are likely to appear in the *Indian Journal of Pure and Applied Mathematics* and in *Infinite Dimensional Analysis, Quantum Probability and Related Topics*.

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9. K. R. Parthasarathy's Contributions to Quantum Gaussian Distributions and Applications

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In the "short pedagogical essay" [1], published in 2010, K. R. Parthasarathy illustrated the notion of a quantum Gaussian state as a natural extension of the idea of Gaussian or normal distribution in classical probability. This presentation led to some interesting open problems on symmetry transformation and other properties of quantum Gaussian states, calling for further investigation which he continued in the following years.

This was just one, perhaps the last, topic he had considered in his long and productive life: Gaussian stochastic analysis within Quantum Probability. In the last decade, despite declining health, he continued his studies in this field along the lines that I will try to summarize later. However, I cannot resist saying a few words about my contacts with him and his influence on my research in Quantum Probability (QP).

I first met Partha, as his colleagues and friends familiarly called him, at the conference Quantum Probability and Applications III (QP3) held in Oberwolfach, 25-31 January 1987. I had started to be interested in QP after some seminars by Luigi Accardi and, having a background in the theory of classical stochastic processes, quantum stochastic calculus seemed to me to be a very promising and interesting development. I remember that the first words I exchanged with him concerned the notion of stopping time in QP. At a certain point, I asked him why some mathematical physicists present at the conference seemed skeptical about the usefulness of the notion. Briefly, he told me that he did not care too much with the applications of his discoveries, he was doing mathematics for fun. Today, I believe this was a distinctive feature of his mathematical research. Throughout his life Partha constantly enjoyed solving mathematical problems, in this way he left so many influential ideas and contributions that the community of mathematicians in general, and quantum probabilists in particular, owe him a great debt for the legacy of his work.

Over the years, in my visits to ISI and in the recurring meetings of the Quantum Probability conference series, I had the pleasure to follow his enjoyable brilliant talks and to discuss scientific problems. Moreover, I've been amused by several post-seminar walks; "after each talk, take a walk" was one of his favourite maxims. These discussions, his speeches, his scientific works (not only in content, but also in the way of organizing the presentation) and, more generally, his approach to mathematics were of such great inspiration to me that I consider him as a guru.

Returning to the thread of the discussion on Partha's contributions to quantum gaussian distributions and applications, I will briefly describe the problems he studied in the papers listed in the references.

In his second paper [2] he continued investigation on the structure of quantum Gaussian states in an n-mode Fock space. First he found in a direct way the explicit canonical form of the density as the product of an m-mode Gibbs state and an $(n-m)$ mode vacuum state conjugated with a unitary Weyl operator and another unitary operator implementing a Bogoliubov transformation. Then he showed that any Gaussian state in an n-mode Fock space can be viewed as the marginal of a pure Gaussian state on a $2n$-mode Fock space. Finally he classified Gaussian symmetries, namely unitary operators U such that, for any Gaussian state ρ, $U\rho U^*$ is still a Gaussian state, by proving that they are the product of a Weyl operator and another unitary implementing a Bogoliubov transformation up to a phase factor.

In [3] he first proved that every real $2n \times 2n$ matrix admits a dilation to (roughly speaking, can be realized as a corner of) a $2(n+m) \times 2(n+m)$ *symplectic* matrix, then he mentioned some open problems on quasi-free Gaussian channels (i.e. channels that transform Gaussian states to Gaussian states and are characterized by two matrices). This was the first of three papers in which he tried to use the concept of dilation in the analysis of Gaussian states and quantum Gaussian maps. I have the feeling that he wanted, on the one hand, to attack open problems on quantum Gaussian channels by using dilation techniques, as not yet explored in the literature on the topic, in a framework that was familiar to him and, on the other hand, to highlight how Hudson-Parthasarathy's quantum stochastic calculus is a natural tool for quantum stochastic Gaussian analysis. This aim is clear in [4] where he shows that quantum Markov (or quantum dynamical, in the physics terminology) semigroups of quasi-free completely positive maps admit a dilation by means of a family of unitary operators solving a quantum stochastic differential equation driven by the fundamental noises of Hudson-Parthasarathy's quantum stochastic calculus. These semigroups are not strongly continuous but their predual semigroups map quantum Gaussian states to quantum Gaussian states (see [10] for an account with Partha's notation also filling some gaps in the old mathematical physics literature). They were introduced in the seventies and their generators were shown to be of the Gorini-Kossakowski-Sudharsan-Lindblad (GKSL) type with unbounded operators in their representation in this form. They were recently investigated in the context of quantum information theory in some papers where some of the open problems stated in [3] were also considered.

His studies on Gaussian quantum channels also led him to the introduction of the non-trivial extension considered in [9]. There, using the tool of quantum characteristic functions, he constructed new concrete semigroups with unbounded generators that are also quantum Markov semigroups, but the form of their generators involves additional features which do not appear in the standard GKSL form. In the wake of this analysis many open problems arise naturally which are still waiting to be investigated.

Always taking inspiration from the relationship with classical probability, of which he was as well a master, Partha also studied other properties of quantum Gaussian states which had not been considered with due attention before such as exchangeability. However, he was also concerned with a typical concept regarding quantum states, namely entanglement that has no classical counterpart since all classical states are not entangled (and they are called separable). In the paper [5], in collaboration with Ritabrata Sengupta, a characterization of exchangeability of a chain of quantum Gaussian states, with a stationarity property, was given in terms of their covariance matrices. Developing this analysis, examples of entangled stationary states were also given. In the paper [6], in collaboration with Rajarama Bhat and Ritabrata Sengupta, the relationship between notions of extendability and separability was investigated extending and completing previous results in the mathematical-physics literature. The classical probabilistic viewpoint, always present, emerges also in his work [8] where various relative entropies of two n-mode quantum Gaussian states are computed and shown to be equal to the sum of a classical part (relative entropies of classical Gaussian distributions) and another term which is due to non-commutativity.

In all investigation on n-mode quantum Gaussian states and their properties the main difficulties are non-commutativity, which adds to the complexity due to the high dimensionality of the problems (as in the analysis of classical multidimensional Gaussian distributions), and the lack of a simple explicit formula for the density of a quantum Gaussian in terms of the covariance matrix.

When I last met Partha in late January 2020 at ISI Delhi in his office at the first floor, that he still visited once or twice a week, he had just published the work [7] in collaboration with Tiju J. Cherian in which, starting from Klauder-Bargmann integral representation of Gaussian symmetries in terms of coherent states, they developed a new parametrization of Gaussian states, as an alternative to the customary parametrization by position-momentum mean vectors and covariance matrices. He showed me with his usual enthusiasm and clarity how to write the density of a state ρ of the form $Z_1^\dagger Z_1$ where Z_1 is the product of a normalization constant, of the second quantization of a positive contraction operator and an operator of the form $\exp\left(\sum_{i,j=1}^n \alpha_{i,j}\, a_i a_j + \sum_{j=1}^n \lambda_j\, a_j\right)$, where $(a_j)_{1\leq j\leq n}$ are the annihilation operators corresponding to the n different modes, $[\alpha_{ij}]$ is a symmetric complex matrix and $\lambda_j \in \mathbb{C}$. This allowed them to get an explicit particle basis expansion of an arbitrary mean zero pure Gaussian state vector along with a density matrix formula for a general Gaussian state, a class of examples of pure n-mode Gaussian states that are completely entangled and applications to the tomography of an unknown Gaussian state and other interesting consequences.

These results clarify many properties of quantum Gaussian states, often hidden in complex formulas used by physicists. Furthermore, they reveal the profound relationship of classical Gaussian distributions of which quantum Gaussian states are the natural extension when one considers incompatible observables in quantum theory. I think that, in the future, they will receive more and more attention even if it will be a slow process because, as Partha said in the 90s (and today it is even more true), "people are so busy writing that they have no time to read".

Regardless of the immediate resonance of his work, he continued to produce mathematics "for fun", supported by Shyama, his faithful, devoted and helpful wife of a lifetime, who in recent years accompanied him on his walks "for exercise" throughout the day. For all those who had the invaluable privilege to know Partha and exchange ideas with him, his deep knowledge and far reaching insight on many areas of mathematics have always been inspiring.

Partha has left an indelible influence on all the community of Quantum Probability of which he was a founding father, his students, colleagues and many researchers that just came across his work. All his discoveries and new knowledge will keep his memory alive.

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10. Remembering KRP

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I met KRP when I arrived in Calcutta as a research scholar in August 1959. He shared an office with Ranga Rao and we ran into each other regularly at tea.

I was not sure what I wanted to do. I thought I was going to work on applications of Statistics. C. S. Ramakrishnan who was around at that time tried to convince me that Operation Research was the most important thing and that I should read Bellman. For a few months I was confused about my goals. Varadarajan and Bahadur gave courses on measure theory and point set topology. I attended them. KRP got hold of me and suggested we work on a real problem. Varadarajan had left for US by then. Ranga Rao, KRP and I started working on a problem. We worked hard and accomplished quite a bit in two years. We studied convolution properties of probability distributions on topological groups.

KRP was very disciplined. I would go frequently to the city to see a movie or eat at a restaurant. KRP would frown at it. I wanted to learn how to swim. The institute had a pond that one can swim in, and KRP offered to teach me. But only at 6 am. It was too cold and I never took up on it. For about eighteen months we worked on probability distributions on groups. It was a lot of fun. We would gather at 7 a.m. and work till early afternoon with a short lunch break. Kolmogorov visited ISI and some of us traveled with him to Bangalore. KRP who knew some Russian acted as the interpreter. Dr C.R. Rao asked Kolmogorov to be a reader for my thesis.

Then KRP left for Moscow. While there he was helpful in getting Kolmogorov to send a report on my thesis.

I graduated from ISI and moved to the US. I met KRP and Shyama many times in England, Mumbai, Delhi and in New York when he visited us at NYU for a month.

My years of working with him in the sixties was an experience that I will always cherish.

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KRP and the author are seen in the photos, with
Prof. Ranga Rao (top left) and Mrs. Parathasarathy (bottom left)
(Photos courtesy of Mrs. Shyamala Parathasarathy)

11. K. R. Parthasarathy as a Teacher of Mathematics

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K. R. Pathasarathy's life and personality were marked by seriousness, sincerity and a sense of purpose. These qualities suffused all his professional activities: learning and reading, research, teaching, editing and committee work. Over five decades of association with him, I had the good fortune of attending several of his courses, seminars, talks and public lectures. In this article I share some of my reminiscences.

The first course I took from KRP (as he was commonly known) was a course on Measure Theory given to masters students at IIT Delhi in 1973. Three weeks into the course I was struck by the realisation that he had already covered more material than what had been done in most of my one semester courses earlier. Further, he had managed to do so effortlessly. This was a hallmark of all his courses that I took subsequently.

Over the years I came to understand that KRP had a deep and thorough knowledge of his subject which he upgraded constantly. He had a genuine desire to communicate it clearly. In his first lecture he would give several examples of the objects that were going to be studied and make a case for the importance of the subject. He was always well-prepared and never casual. He delivered his lectures with great enthusiasm, energy and passion. He structured his course with great care. I remember his advice that one major theorem should be proved in each lecture.

To give an idea of the place of teaching in KRP's life I recall two conversations. Once he explained to his students that there are four stages in a mathematician's evolution. First, he should be able to understand mathematics. Second, he should be able to explain it clearly to others. Third, he should be able to solve problems, and finally the fourth, he should be able to create problems. The second stage was important in his thinking. On another occasion someone reported that a certain mathematician was very sick and his diet had been restricted to boiled rice and boiled banana. KRP looked sad and responded "Then you cannot lecture with energy." That was a very revealing reaction. He took teaching as a central purpose of his life.

Apart from his several masters courses, KRP regularly organised and lectured in research seminars. Whenever he travelled to attend committee meetings he would volunteer to give a talk at the host institution and often say that was more important than the meeting. This was especially valued in an era when travel funds for lectures and meetings were scarce.

At most Indian universities professors keep a distance from students. Informal discussions and common-room meetings are rare. KRP created both time and space for such meetings. He also invited students to his home. Because of his stature, his knowledge, his passion and his wit he was often the main light of these meetings. From these meetings a student gained a glimpse of the culture of mathematics. On one occasion early in my association with him he said that Harish-Chandra's work was so deep and difficult that if he decided to spend all his life just trying to understand it, he would not be able to do it. His life was just too short for it. Such a comment was an eye-opener for a young student. A few months later, in another context, he said Harish-Chandra is a problem solver whereas Gelfand is a creator of new fields (in mathematics). These conversations played a very valuable role in our education.

Incidentally, KRP had an unreserved and unbounded admiration for A. N. Kolmogorov with whom he worked soon after his PhD. The qualities he often mentioned were his greatness as a mathematician, his attempts at teaching students at all levels including high schools, his support to mathematicians in difficult situations including those from Jewish backgrounds facing discrimination from the State. KRP's stay in the Soviet Union in the early 1960's had influenced and affected him in several ways. In 1974 when Solzenitsyn's Gulag Archipelago appeared he obtained a copy of the French edition at some effort and often talked of it. On the lighter side he often regaled his students with an imitation of Nikita Khrushchev's speech in Russian that he had heard on the radio during the Cuban crisis.

KRP was fastidious about neatness in thinking, writing and talking. When a student would try to tell him about some idea he had and excitedly began writing on the board, KRP would sternly admonish "Start from the top left-hand corner of the board." He also discouraged obsequious behaviour. In India it has been a common practice for a speaker to begin a lecture by making a special reference to the senior persons present. For example, "Professor X, Professor Y, Professor Z and dear friends...." I remember an occasion when a speaker began his mathematics colloquium in this way. KRP sternly spoke from the first row "The first word in a mathematics talk should be 'Let'."

I recall with special admiration an event from 2011 when KRP was 75 years old. Despite my long association with KRP and with ISI my knowledge of statistics is somewhat meagre. Some work in matrix analysis that I did around 2005 seems to be of interest in areas like machine learning, image processing, brain-computer interface etc. The name "matrix information geometry" has been used to encompass some of these ideas. There are links with the classical differential geometric ideas in statistics going with the Rao-Fisher metric. (This was almost the first major work by C. R. Rao done in Calcutta of which KRP was a great admirer.) In 2011 one of my students and I were invited to an international conference whose theme was to be matrix information geometry. I requested KRP to explain to us the work of Rao and Fisher. As was his nature, he immediately agreed. Then he gave us two beautiful lectures on the topic and at the end handed me a few pages of notes. These were written like a very serious student's assignment in bold neat hand. To think that he did so when he was 75, and in response to a small request, is truly inspiring.

This characterised KRP's several conference talks at meetings and at instructional conferences. The audience was always treated to a well-prepared substantial talk, sonorously delivered with passion. Many people all over India heard and enjoyed his lectures. If one person has to be thought of as a "national teacher" of mathematics, it would be him.

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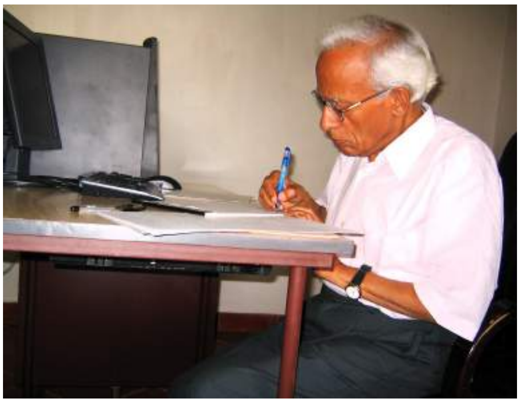

Glimpses of KRP as a scholar (Images courtesy of Mrs. Shyamala Parthasarathy)

12. Prof. K. R. Parthasarathy - a Profile

Kalyanapuram Rangachari Parthasarathy (K. R. Parthasarathy, Partha or KRP as he was variously known) was born on 25th June 1936 in Madras (now Chennai). He had his early schooling in Thanjavur and continued his studies at P. S. High School, in the Mylapore area of Madras. He then joined the Ramakrishna Mission Vivekananda College, Madras, from where he obtained his B. A. (Hons) in 1956 of the University of Madras, and the M. A. followed in 1957, by eflux of time. Following graduation, in 1956 he proceeded to join a "three-year advanced professional statisticians' course" at the Indian Statistical Institute, Calcutta (now Kolkata), in an environment in which there was a deep sense of a need for a large army of well-trained statisticians to monitor and formulate new economic programmes. Though the programme was focused on statistical practice, near the end of the programme the students were to do a project, which is when "after spending a few days in our well-equipped library" KRP made his choice to go along a theoretical line - and that was to stay - and did a project on the moments problem. Along with the certificate for the course he earned a scholarship to do research for the Ph.D. degree under the supervision of Prof. C. R. Rao. Two significant results that he contributed, one on the density of ergodic probability measures and another on integral representations in terms of ergodic components, earned him his Ph.D. from ISI, in 1962. lSI was declared an institute of national importance, by an act of parliament, in 1959, authorizing it to confer its own degrees, and KRP got his Ph.D. degree, along with another student J. Sethuraman, in the first convocation held on 12 February 1962, in a grand ceremony which also featured conferment of the degree of Doctor of Science (honoris causa) upon various luminaries, S. N. Bose, R. A. Fisher, Jawaharlal Nehru, A. N. Kolmogorov and W. A. Shewhart!

The following year, during 1962-63, he worked at the Steklov Institute of the USSR Academy of Sciences, in Moscow, as a lecturer, and participated in the Seminars of E. B. Dynkin and Ya. G. Sinai, and the famed Monday evening programmes of I. M. Gelfand.

Returning from Moscow, he served as lecturer at ISI, Calcutta, for two years, following which in 1965 he joined the University of Sheffield as a lecturer in Probability and Statistics. His outstanding research record earned him rapid promotions, to Senior lectureship in 1966 and Readership in 1967, and in 1968, he was appointed as Professor in the University of Manchester Statistical Laboratory, within the Sheffield-Manchester Joint School of Probability and Statistics. He however held the post only for two years, 1968 – 70, deciding to return to India. During his five years of stay in UK, he also visited and gave lectures, and lecture courses, at various institutions in Germany, France, Denmark, Norway as well as Australia, and also within UK.

While he may have been sceptical about good opportunities in India where he could pursue his work fruitfully, he received an offer of Professorship, in 1970, at the Centre for Advanced Study in Mathematics, set up by the University Grants Commission (UGC) at the University of Bombay (now Mumbai), which he accepted. He moved to Delhi in 1973, as Professor and Chairman of the Department of Mathematics, at the Indian Institute of Technology (IIT) Delhi.

On 31 December 1974 a new campus of the Indian Statistical Institute was inaugurated in New Delhi, and soon after that C. R. Rao suggested to KRP that when the buildings were completed he should think of moving over to lSI. He soon began to participate in the academic activities there and near the end of 1976 he joined the new centre of ISI, to build up its Mathematics and Statistics Unit. He continued there, as a regular member until the mandatory superannuation in 1996, as C. V. Raman Research Professor of INSA from 1996 to 2001, and later as Emeritus Professor. During the years he undertook numerous visits for conferences, research collaborations and lectures in Europe, USA, and China, and also spent one year, 2001, at the Institute of Mathematical Sciences, Chennai.

He authored over 150 research papers and 10 books, covering a wide range of areas, which have had a huge impact. In particular he pioneered the area of Quantum Stochastic Calculus, jointly with Robin L. Hudson. He had numerous collaborators around the world and guided many Ph.D. students. He also served on numerous committees influencing the course of Mathematics in India.

He was awarded the Shanti Swarup Bhatnagar Prize for Mathematical Sciences in 1976, and The World Academy of Sciences Prize in 1996.

He received also the Mahalanobis medal of the Indian Science Congress (2002) and the Srinivasa

Ramanujan Medal of the Indian National Science Academy (2013). He was awarded Honorary Doctorate of Nottingham Trent University (2000), Doctorate of Sciences of the Chennai Mathematical Institute (2009), and Honorary Doctorate of the Indian Statistical Institute (2013).

He was an invited speaker at the International Congress of Mathematicians at Zurich (1994), and was invited in 1995 for the Hardy Lecture Tour, covering several universities in UK, starting with Cambridge, on 21 May 1995, and continuing at Oxford, Swansea in Wales, Manchester, Edinburgh in scotland, Dublin in Ireland, and Nottingham, London, Sheffield, Liverpool and Warwick.

He served as a member of the Executive Committee of the International Mathematical Union during 1995 – 98. He was elected Fellow of The World Academy of Science (TWAS), the Indian Academy of Sciences (IASc) and the Indian National Science Academy (INSA), and served on the Council of INSA during 1983 – 85.

His lively and inspiring presence amidst us will be acutely missed by a large section of the mathematical community.

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KRP receiving the award of Honorary Doctorate of Nottingham Trent University (2000)

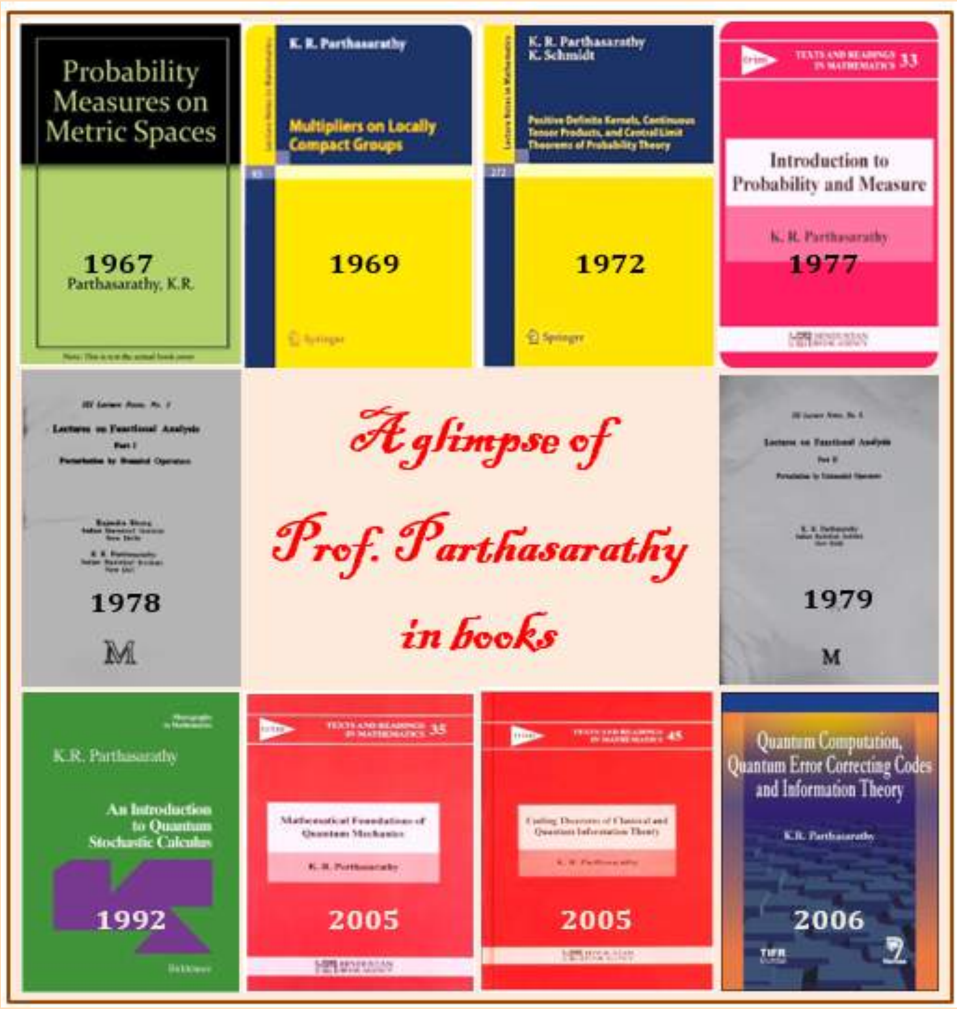
KRP receiving the Honorary Doctorate of the Indian Statistical Institute
from Dr. C. Rangarajan (2013)
(Images courtesy of Mrs. Shyamala Parthasarathy)

With Prof. Ruslan Stratonovich

With Prof. Huzihiro Araki

With Professors M. S. Narasimhan and David Mumford

KRP with the younger generation - a family photo
(Photos courtesy of Mrs. Shyamala Parthasarathy)



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