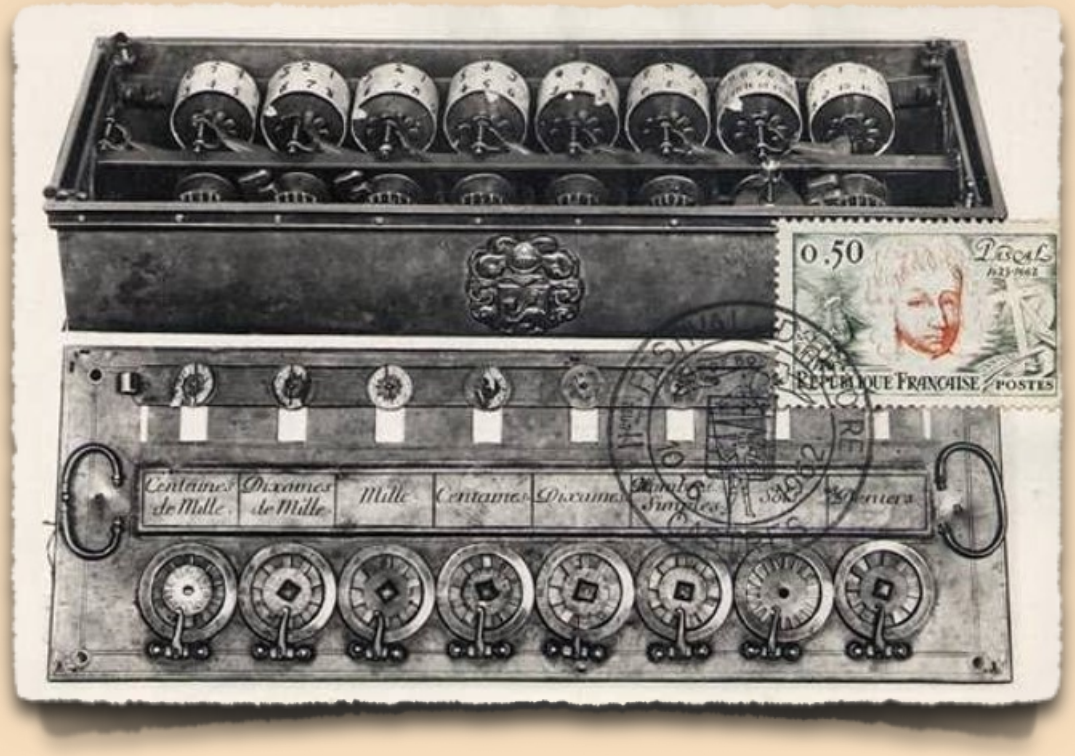
The Mathematics Consortium

BULLETIN

July 2023 A TMC Publication Vol. 5, Issue 1

Remembering Legendary Mathematician Blaise Pascal On His 400th Birth Anniversary (June 19, 1623 - August 19, 1662)



Pascaline - Pascal's Calculator

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Bulletin

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About the Cover Page: The figure on the cover page is an image of Pascal's adding machine which is called Pascaline. The stamp inscribed in the figure, issued in France, shows Blaise Pascal (1623-1662). Source: *https://www.123rf.com/photo_150132627_france-circa-1962- a-stamp-printed-in-france-shows-blaise-pascal-1623-1662- mathematician-physicist-ci.html*

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From the Editors' Desk

The Mathematics world is celebrating 400th birth anniversary of a legendary French mathematician Blaise Pascal (19 June, 1623 - 19 August, 1662). In a short span of life, extending only 39 years, he made amazing contribution to diverse fields of Mathematics.

Without any formal training in Mathematics, he went on to experiment with geometrical figures, and wrote an important short work on projective geometry, "Essay on Conics" at the age of just 16. Also, he formulated a theorem named after him, proving a mystic property of hexagons.

Between 1642 and 1644, Pascal conceived and constructed the first digital calculator, the Pascaline, to help his father in his tax computations, which can be marked as a beginning of Computing technology.

He reproduced and amplified experiments on atmospheric pressure by constructing mercury barometers and measuring air pressure. These tests paved the way for further studies in hydrodynamics and hydrostatics. While experimenting, Pascal invented the syringe and created the hydraulic press, an instrument based upon the principle that became known as Pascal's principle.

In his "Treatise on the Arithmetical Triangle" which was published in 1653, he studied the Pascal Triangle, making many new mathematical observations and proved a useful theorem of combinatorics dealing with binomial coefficients. Pascal's triangle in fact corresponds to Meruprastara, introduced by Pingala around 3^{rd} century BCE, in his book Chhandahshastra on prosody.

In 1654, Pascal began corresponding with mathematician Pierre de Fermat. He conducted experiments with dice and discovered that there was a fixed likelihood of a particular outcome. Thus, along with Fermat, he laid the foundation of the probability theory.

Pascal's scientific journey was guided partly by his curiosity and partly by his concern regarding real world problems and hence his discoveries were so relevant and innovative and marked the beginning of quite a few areas of mathematics research. In Article 7, Prof. V. O. Thomas and Dr. D. V. Shah throw some light on his life and works.

Article 1 is the third and last part of multi-article series by Prof. Ambat Vijayakumar started in the July, 2022 issue of TMCB. In part 3, he discusses in general, about real world networks and more on brain networks, and also introduces an upcoming branch of graph theory called evolutionary graph theory and epidemiological networks.

In the second article, Prof. S. A. Katre establishes some interesting results on the degree of an extension field of a field k, obtained by adjoining square roots of n elements of k. Some interesting properties of Ramanujan's Birthday Magic Square is the subject of Article 3.

In Article 4, Dr. D. V. Shah gives an account of significant developments in the Mathematical world during recent past, including brief write-ups on the winners of the 2023 Abel Prize, and 2023 Shaw prize, and on a recently elected Fellow of the Royal society.

In Article 5, Kalyan Sinha pays tributes to India's one of the most renowned mathematician, Prof. K. R. Parthasarathy who passed away on 14^{th} June 2023, by highlighting significance of his contributions to Mathematics. We also intend to publish a special issue of TMCB in the honour of Prof. Parthasarathy in near future. We also mourn the sad demise of Dr. Mangala Narlikar who passed away in July 2023 and Prof. S. A. Katre pays tributes to her in article 6.

In the Problem Corner, Dr. Udayan Prajapati presents two solutions to the problem posed in the April 2023 issue. These solutions are given by a student Shrestha Suraiya, and by Prof. Rajendra Pawale, who posed the problem. A problem on Geometry is also posed for our readers. Dr. Ramesh Kasilingam gives a calendar of Mathematics Events, planned during August, 2023 to October, 2023, in Article 9.

We are happy to bring out the first issue of Volume 5 in July 2023. We thank all the authors, all the editors, our designers Mrs. Prajakta Holkar and Dr. R. D. Holkar, and all those who have directly or indirectly helped us in bringing out this issue on time.

Chief Editor, TMC Bulletin

1. Graph Theory 1736 -1936 and Beyond - PART 3

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1.1 Introduction

In this concluding article we shall discuss some aspects of graph theory in the 'information age'. The prediction of the renowned popular science writer Arthur C. Clarke in the Popular Science Magazine (May,1970), that satellites would some day "bring the accumulated knowledge of the world to your fingertips" became a reality in 1989 through the invention of **World Wide Web (WWW)** by the British computer scientist Tim Berners-Lee while working at **CERN**(Conseil Européen pour la Recherche Nucléaire).

WWW the "universal linked information system" is a global collection of documents and other resources, linked by hyperlinks and Uniform Resource Identifiers (URIs) and serves as a startling example of a Real World Network (RWN) of very huge order, approximately 1.5 billion. WWW is also called an "infinite imaginary library". RWNs are the main topic of discussion in this part. The basic research problem will be to find the various parameters to 'understand' such networks, which are abundant in nature.

In an earlier part of this article we had discussed in detail the frivolous origin of graph theory and most of the examples mentioned were static, in the sense that the order, size and other parameters of the network were constants. Later on researchers started the study of probability distributions over graphs. One of the first of its kind, the Erdös -Rényi random graph models were studied in 1959 by Paul Erdös and Alfred Rényi in their celebrated paper entitled '**Random Graphs**'. A random graph is obtained by starting with a set of n isolated vertices and adding successive edges between them at random. We have to determine at what stage a particular property of the graph is likely to arise. Béla Bollobás initiated an exciting discussion on Random Graphs in [4].

This article will discuss in general, about real world networks and more on **brain networks** [8, 16], introduce an upcoming branch of graph theory called **evolutionary graph theory** [15] and **epidemiological networks** [14] which gained a great leap due to the Covid-19 pandemic.

1.2 Real World Networks (RWN)

The first systematic study of real world networks was done by Derek John de Solla Price (1922 - 1983), a British Physicist and a historian of science, in his quantitative studies on scientific publications. This gave birth to 'Scientometrics', the study of measuring and analysing scholarly articles. He showed in 1965 that the number of citations that a paper receive had a heavy-tailed distribution following **a Pareto distribution** or **power law**.

He also proposed that the power law behaviour is due to the notion of '**Preferential Attachment**'. A preferential attachment process is any of a class of processes in which some quantity, typically some form of wealth or credit, is distributed among a number of individuals or objects according to how much they already have, so that those who are already wealthy receive more than those who are not. This phenomenon is now popularly known as '**Mathew Effect**' due to Robert Merton [12] an American sociologist. He in the article, '**Mathew Effect in Science**' which appeared in the prestigious journal '**Science**' [12] says that scientific achievements exhibit a Mathew Effect described in the Gospell of Matthew, the first book of the New Testament of the Holy Bible.

Matthew 25:29 - "For to every one who has will more be given, and he will have abundance; but from him who has not, even what he has will be taken away".

1.3 Scale-Free Networks

Real world networks are basically huge and 'dynamic' in the sense that they are generated by a web application, but still follow the '**Small World Phenomena**' that two persons having a common friend are likely to be friends [7]. This is the basic principle by which the 'Facebook Network' also functions. Metabolic Networks, food webs, semantic networks, biological networks and the protein-protein interaction networks are some other examples. In such massive networks, traditional graph invariants are unlikely to be of any significance.

Recent interest in **Scale-Free Networks** started in 1999 with the work by Albert-László Barabási and Réka Albert [2] who mapped the topology of a portion of the WWW, and found that some vertices, which they called "hubs", had many more connections than others and that the network as a whole had a **Power Law Distribution** of the number of links connecting to a vertex. Hubs have a significant impact on the network topology. After finding that a few other networks, including some social and biological networks, also had heavy-tailed degree distributions, they coined the term "scale-free network" to describe the class of networks that exhibit a power-law degree distribution.

Real-world networks are often claimed to be scale free, meaning that the fraction of vertices with degree k follows a power law $k^{-\alpha}$, a pattern with broad implications for the structure and dynamics of complex systems [13]. However, there are conflicting views on the universality of scale-free networks.

1.4 Small World Networks - Some Parameters

Everyone meets with the small-world phenomenon, though many of us may be not familiar with the term. As a typical case, in a get together most of us will be strangers to begin with. But soon after meeting a stranger, we are surprised to discover that we have a mutual friend, or we are connected through a short chain of acquaintances.

In a small-world network, most vertices are not neighbours of one another, but the neighbours of any given vertex are likely to be neighbours of each other. Due to this, most neighbouring vertices can be reached from every other vertex by a small number of steps. In other words, a small-world network is defined to be a network where the distance between two randomly chosen vertices (i.e. the number of steps required) grows proportionally to the logarithm of the number of vertices in the network [18].

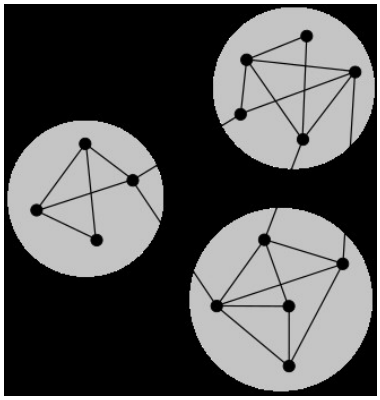


Figure 1

An interesting property of any RWN is the 'Community structure'. A **community**, with respect to a network, is a subset of vertices that are densely connected to each other, and loosely connected to the vertices in the other communities in the same networks (see Figure 1). As an example, in social media platforms such as Facebook, Instagram, or Twitter, where we try to connect with other people, we end up being connected with people belonging to different social circles. These social circles can be a group of relatives, school mates, colleagues, etc. which are communities. Detecting communities in a network is one of the most important and difficult tasks in network analysis. However, the **Girvan-Newman Algorithm** (named after Michelle Girvan, a Physics Professor in the University of Maryland, and Mark Newman FRS, a Professor of Physics in the University of Michigan)) is a hierarchical method used to detect communities in complex systems [9]. It relies on the notion of edge betweenness and iterative elimination of edges that have the highest number of shortest paths between nodes passing through them.

We shall now discuss in detail some parameters [5, 11] of real world networks.

a) Clustering Coefficient: is a measure of the degree to which vertices in a graph tend to

cluster together. In most real-world networks, and in particular social networks, vertices tend to create tightly knit groups characterised by a relatively high density of ties. Duncan J. Watts and Steven Strogatz [17] introduced this measure in 1998 to determine whether a graph is a small-world network. It locally quantifies how close its neighbours are to being a clique (complete graph) and hence the clustering coefficient of such networks will not be small. There are some efficient clustering algorithms - density based, distribution based etc. using Python.

b) Average Path Length: is defined as the average number of steps along the shortest paths for all possible pairs of network vertices. It is a measure of the efficiency of information or mass transport on a network. Some examples are, the average number of clicks which will lead you from one website to another, or the number of people you will have to communicate through, on an average, to contact a complete stranger. Obviously in a small world network the average path length will be very small. Biological networks display "small-world" properties, whereby any two genes are separated by only a small number of links [18]. In fact, the average path length in biological networks varies between roughly two to four interactions. Thus, nearly all genes are linked within a short number of steps to all other genes. For detailed discussion on these topics, see [3].

c) Centrality: In the context of the classical facility location problems, the notions like centre, centroid, median etc are well known. Depending on the nature of the problem, we choose the appropriate measure of centrality. In the case of real world networks, we have to identify, the most important person(s) in a social network, most cited researcher in a field of study, super spreaders in the epidemiological networks etc. Consequently, we have different measures of centrality such as degree centrality, betweenness centrality, eigen vector centrality, closeness centrality etc. Each of these measures assigns numbers to the vertices of the graph so that they can be ranked.

In the **degree centrality**, we assign degrees to each vertex and choose those with highest degrees.

American sociologist Linton Freeman introduced the notion of **betweenness centrality** in 1977, a notion which depends on the number of shortest paths through a particular vertex. It is the ratio of the total number of $u-v$ paths and the total number of $u-v$ paths through a particular vertex. As an example, in a telecommunications network, a vertex with higher betweenness centrality would have more control over the network, because more information will pass through that vertex.

The **closeness centrality** is calculated as the reciprocal of the sum of the length of the shortest paths between the vertex and all other vertices in the graph (also known as transmission index in communication theory, the Wiener index in mathematical chemistry). Thus, the higher central a vertex is, the closer it is to all other vertices.

The **eigenvector centrality** is a measure of the influence a vertex has on a network. If a vertex is pointed to by many vertices (which will also have high eigenvector centrality) then that vertex will have high eigenvector centrality. If λ is the largest eigen value of the network, the eigenvector centrality of a vertex v_i is defined as the i^{th} entry in $\bar{x}$ where $\bar{x} = \frac{1}{\lambda}A\bar{x}$. Since the adjacency matrix of a connected graph is irreducible, the existence of λ is assured by the Perron-Frobenius Theorem [11].

1.5 Biological Networks

Different types of information can be represented in the shape of networks in order to model the cell. The meaning of the nodes and edges used in a network representation depends on the type of data used to build the network. Some of the most common types of biological networks [5] are: Protein-Protein Interaction networks, Brain Networks, Metabolic Networks, Gene regulatory networks etc. We shall now discuss some of these.

1.6 Protein-Protein Interaction Networks

In this network, vertices represent proteins and by edges, their interactions. Proteins are the main agents of biological function. Proteins control molecular and cellular mechanisms and determine healthy and diseased states of organisms. However, they are not functional in isolated forms but they interact with each other and with other molecules (e.g., DNA and RNA) to perform their functions.

1.7 Brain Networks

In the human brain, there is a complex network of neurons that encompass brain network systems. Neurons don't exist in isolation. These neurons need to interact with each other to create a brain network system. The interaction between different neurons in the brain is needed to communicate and process information according to what we see, hear, think, and move. This network is undoubtedly the most complex network known to humanity. A human brain has about 100 million (10^{11}) neurons linked by 100 trillion (10^{14}) synapses. One of the goals of neuroscience is to study the 'brain connectivity'. Olaf Sporns(Psychological and Brain Sciences, Indiana University, USA), Giulio Tononi (Consciousness Science, University of Wisconsin) and Rolf Kötter (1961-2010) coined the term, **Connectome**, which reduced the whole study in terms of a '**Connection Matrix**' representing all possible pairwise anatomical connections between neural elements in the brain. The possibility of visualisation of the brain due to technological advances such as the open source project called 'Open Worm', has made the study all the more exciting and challenging. The book, Fundamentals of Brain Network Analysis, by Alex Fornito etal. [8] discusses in detail all these aspects. In the case of brain networks, a new kind of centrality- **Leverage Centrality** - has been studied [10], which considers the extent of connectivity of a vertex relative to the connectivity of its neighbours.

C. elegans: Caenorhabditis elegans with just 302 neurons in its nervous system is a free-living transparent nematode about 1 mm in length [11] that lives in temperate soil environments and is frequently chosen as a model organism to study human diseases. C. elegans lacks respiratory or circulatory systems. In 1963, Sydney Brenner (1927 - 2019), an African biologist who shared the Nobel prize in physiology in 2002 proposed research into C. elegans, primarily in the area of neuronal development. It was the first multicellular organism to have its whole genome sequenced, and in 2019 it was the first organism to have its connectome completed and it turned out to be a small world network. There are some recent attempts in studying the connectome of C. elegans using graph theory. S. R. Kingan explains in her exciting blog (http://graphsandnetworks.blogspot.com) that the picture (see Figure 2) of C. elegans which are featured in the front cover page of [11] was drawn using a program called neuroConstruct. The data is taken from the Worm Atlas Project - a data base featuring behavioural and structural anatomy of C. elegans and other nematodes and drawn in Gephi 0.9.2 using the Fruchterman - Reingold layout. An open source project 'Openworm'(https://openworm.org) will also help us in creating virtual organisms. The Centre for Neuroscience in our university also is conducting research in C. elegans and I was quite excited to see this worm through stereo microscope recently.

1.8 World Wide Web - Structure and Diameter

It is interesting to ask, what is the structure of the WWW and viewed as a network, what is its diameter?

It is interesting to note that, the structure of WWW is that of a 'Bow-Tie' as proposed by Broder et al. in [6]. It models the websites and pages in the World Wide Web as a bow-tie structure with most pages in either IN, OUT, or the SCC component. SCC represents a large strongly connected component where most websites are directed to each other; IN represents

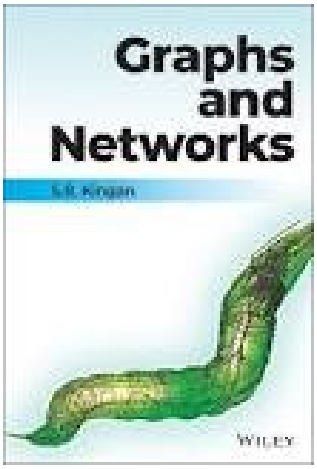


Figure 2

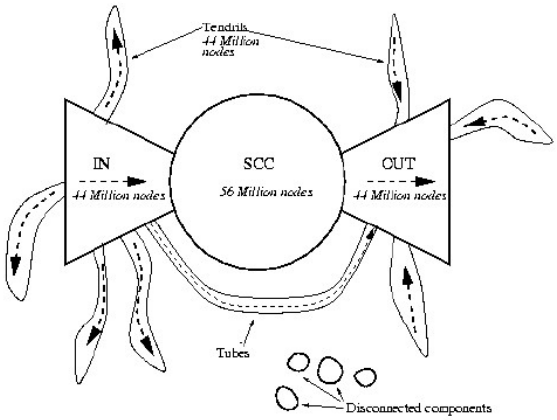


Figure 3

pages that direct to pages in the SCC; OUT represents pages that are directed by pages in the SCC (see Figure 3).

As of January 2023, it is estimated that there are 202 million active websites and 920 million inactive websites in the world. The topology of this graph determines the web's connectivity and consequently our effectiveness in locating information on the WWW. However, due to its large size, and the continuously changing documents and links, it is impossible to catalogue all vertices and edges. Despite its huge size, the WWW is a highly connected graph of average diameter of only 19 links. Due to the surprisingly small diameter of the web, all the information is just a few clicks away - that is, WWW is also is a small world phenomenon.

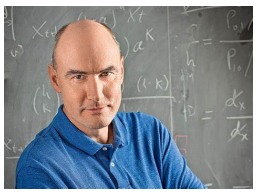
1.9 Power Law in Networks

Though there are billions of webpages, how is that some are more popular? How is that some pages become the center of the web activity? Though the WWW is dynamic, its growth is governed by a principle- Pareto principle, named after Vilfredo Pareto (1848-1923) an Italian polymath, which states that for many outcomes, roughly 80% of consequences come from 20% of causes. A power law distribution (Pareto distribution) is a characteristic of the 80/20 rule. One attribute of power laws is their scale invariance.

The power law can be precisely explained as follows.

Let G be a network of order n and $N(k,G) = \{v \text{ in G such that degree}(v) = k\}$. The degree distribution of G is the sequence $\{N(k,G) : 0 \leq k \leq n\}$. The degree distribution of G follows a power law if for each degree k, $N(k,G)/n$ is asymptotically proportional to $k^{-\beta}$, for some constant $\beta > 1$, called **the exponent of the power law**. Taking logarithm, $\log(N(k,G) = \log(n) - \beta\log(k)$. Hence, the $\log - \log$ plot will be a straight line with slope $-\beta$.

Evolutionary graph theory is an approach to study how topology affects evolution of a population as detailed by Martin Andreas Nowak [15] and using the techniques of graph theory, probability theory, and mathematical biology.



Martin Andreas Nowak (1965 -), a pioneer in this emerging field is an Austrian-born professor of mathematical biology, at Harvard University, heading the Program for Evolutionary Dynamics (PED) since 2003. Nowak realised that he could apply graph theoretic tools to explain in a better way, the theory of origin of species and the principle of natural selection propagated by the British naturalist and biologist, Charles Darwin (1809 - 1882). It is quite exciting to note that Martin Nowak has over 300 scientific publications, of which 40 are in *Nature* and 15 in *Science*.

We conclude this article with a brief note on epidemiological networks, which are of current

interest due to the recent Covid-19 pandemic.

1.10 Epidemiological Networks

In this network the vertices are living organisms like humans, animals or plants and edges indicate the transmission of the disease. Average degree is called the **reproduction number** because when a disease is transmitted to another person in a susceptible population, it reproduces itself. The most important use of this parameter is to determine if an emerging infectious disease can spread in a population and find what proportion of the population should be immunized through vaccination to eradicate a disease. When this parameter becomes less than 1, we interpret that the pandemic shows signs of ending. In the case of Sars-Cov-2 it had varied from 2 to 6. One person infected with COVID-19 will usually infect around three other people. A **super-spreader** is someone who infects more than this number. Super-spreaders are highly infectious. In any pandemic, 20% of the population spread 80% of the infections. That is, the spread follows Pareto distribution. This is true for the COVID-19 pandemic also. For a discussion on epidemiological networks, see [14].

Conclusion: In this three-parts article I have tried to give an expository account of how the subject of graph theory/networks which had its origin in some simple looking puzzles has grown to explain very many interdisciplinary topics including brain networks and epidemiology. It also finds applications in some unexpected fields of knowledge and opens up very many challenging problems of interdisciplinary nature.

Acknowledgment: The author thanks the unknown referee for suggesting some changes, which improved the article and Dr. K. Pravas, Government Maharaja's College, Ernakulam, for helping in the typesetting.

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Result on page 31

2. On Field Extensions by Square Roots

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Abstract: In this article we obtain some necessary and sufficient conditions for an extension field of a field k, obtained by adjoining square roots of n elements of k, to have degree 2^n. The criterion helps in finding the degree of such an extension.

Key words: *Field extensions, square roots, degree of an extension.*

If K and F are fields and F is a subfield of K, then K is called an extension field of F. We say that K/F is a field extension. In this case K is a vector space over F and the dimension of K as a vector space over F is denoted by $[K : F]$. This dimension is called the degree of the extension K/F.

R. L. Roth [2] has given an interesting proof of the fact that if $p_1, \ldots, p_n$ are distinct primes, then $[\mathbb{Q}(\sqrt{p_1}, \ldots, \sqrt{p_n}) : \mathbb{Q}] = 2^n$. More generally, one would like to find the degree of the extension field over $\mathbb{Q}$ when the p_i are replaced by composite integers or even by arbitrary rational numbers. This question can also be considered for an arbitrary field k. An elementary treatment for $k = \mathbb{Q}$ is found in [1]. Our answer depends upon the characteristic of k. One can find the degree of the extension algorithmically by calculations over k as is illustrated in this article.

Characteristic of a field: If F is a field and $k \cdot 1$ is not zero for any positive integer k, then F is said to be of characteristic 0. In this case the subset of $\mathbb{Z}$ consisting of those $k \in \mathbb{Z}$ such that $k \cdot 1 = 0$ is (0). If for some $k \neq 0$, $k \cdot 1 = 0$, then the least such positive integer k is called as the characteristic of F. In both the cases, if m is the characteristic, then the set of all k such that $k \cdot 1 = 0$ is $m\mathbb{Z}$. $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ are fields of characteristic 0. If the characteristic of a field is not 0, then we can see that it must be a prime number. The field $\mathbb{Z}_p$, p prime, of integers mod p, is of characteristic p, so is the field of rational functions in one or more variables over $\mathbb{Z}_p$. A field is of characteristic 0 (resp. p) if and only if it contains an isomorphic copy of $\mathbb{Q}$ (resp. $\mathbb{Z}_p$).

If F is a field of characteristic 2, then $2 = 2 \cdot 1 = 0$. Hence $-1 = 1$. For any $a \in F, 2 \cdot a = 0$ and 2 is not invertible in F. Also for $a, b \in F$, $(a+b)^2 = a^2 + 2ab + b^2 = a^2 \pm b^2$. Thus a sum or difference of squares is a square. Clearly the product of two squares and the reciprocal of a nonzero square is a square. Thus, if char $(F) = 2$, the set of squares of elements of F is a subfield of F.

Notation: From now on k, F, K will denote fields. For $a \in k$, $\sqrt{a}$ denotes one of the square roots of a, possibly in an extension field of k. Square roots of elements of k under consideration will lie in a given extension field of k, e.g. a fixed algebraic closure of k. If char.$(k) = 2$, an element $a \in k$ has a unique square root, possibly in an extension of k. For, if $a = b^2 = c^2$, then $0 = b^2 - c^2 = (b-c)^2 = 0$, so $b - c = 0$, or $b = c$. Note also that for char$(k) = 2$ and $a, b \in k$, $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$.

In what follows we shall consider a field $K = k(\sqrt{a_1}, \ldots, \sqrt{a_n})$, where all a_i's are in k and are nonzero. Observe that $k(\sqrt{a_1}, \ldots, \sqrt{a_n}) = k[\sqrt{a_1}, \ldots, \sqrt{a_n}]$ and for any field F and $b \notin F$, $F(\sqrt{b}) = F[\sqrt{b}] = \{\alpha + \beta\sqrt{b} : \alpha, \beta \in F\}$. This can be seen by rationalising the denominator.

The case char$(\mathbf{k}) \neq 2$

Theorem 1. Let k be a field of characteristic $\neq 2$. Let $a_1, \ldots, a_n \in k$ and let $K = k(\sqrt{a_1}, \ldots, \sqrt{a_n})$. Then $[K : k] = 2^n$ if and only if
$(^*)$ all the products $a_{i_1} a_{i_2} \ldots a_{i_r}$ for $1 \leq i_1 < i_2 < \ldots < i_r \leq n, 1 \leq r \leq n$, are nonsquares in k.

Proof.$(\Leftarrow)$ The proof is by induction on n.
If $n = 1$, then a_1 is a nonsquare in k, so that $\sqrt{a_1} \notin k$, and $[K : k] = 2$.

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[1]Talk given on 3^{rd} August 2021 at Bhaskaracharya Pratishthana, Pune, in Symposium in Number Theory in honour of 60^{th} Birthday of Prof. Dinesh Thakur.

Assume the result for $n-1$, i.e. if $(^*)$ holds for any $b_1,\ldots,b_{n-1}\in k$, then $[k(\sqrt{b_1},\ldots,\sqrt{b_{n-1}}):k]=2^{n-1}$.

Assume that $(^*)$ holds for $a_1,\ldots,a_n$. Then it also holds for $a_1,\ldots,a_{n-1}$ so that for $k_0=k(\sqrt{a_1},\ldots,\sqrt{a_{n-1}})$, $[k_0:k]=2^{n-1}$. Note that if $F=k(\sqrt{a_1},\ldots,\sqrt{a_{n-2}})$, then $k_0=F(\sqrt{a_{n-1}})$ with $\sqrt{a_{n-1}}\notin F$. Now to show that $[K:k]=2^n$, it suffices to prove that $\sqrt{a_n}\notin k_0$. Assume the contrary. Then

$$\sqrt{a_n} \quad = \quad \alpha+\beta\sqrt{a_{n-1}} \tag{2.1}$$

where $\alpha,\beta\in F$. Squaring

$$a_n \quad = \quad \alpha^2+a_{n-1}\beta^2+2\alpha\beta\sqrt{a_{n-1}}. \tag{2.2}$$

If both α, β are nonzero, then as $\mathrm{char}(k)\neq 2$, from (2.2) we see that $\sqrt{a_{n-1}}\in F$, a contradiction.

If $\alpha=0$, then by (2.1), $\sqrt{a_na_{n-1}}\in F=k(\sqrt{a_1},\ldots,\sqrt{a_{n-2}})$. Hence by induction assumption, $(^*)$ does not hold for $a_1,\ldots,a_{n-2},a_na_{n-1}$. Hence some product of $a_1,\ldots,a_{n-2},a_na_{n-1}$ and thus also of $a_1,\ldots,a_{n-1},a_n$ is a nonsquare. This contradicts that $(^*)$ holds for $a_1,\ldots,a_n$.

If $\beta=0$, then $\sqrt{a_n}\in F=k(\sqrt{a_1},\ldots,\sqrt{a_{n-2}})$ and we get a similar contradiction.

$(\Rightarrow)$ Conversely, assume that $[K:k]=2^n$. Then $a_1,\ldots,a_n$ are nonzero.
If a product $a_{i_1}\ldots a_{i_r}$ is a square in k, then $\sqrt{a_{i_1}}\in k(\sqrt{a_{i_2}},\ldots,\sqrt{a_{i_r}})$, so that K is generated over k by elements from the set $\{\sqrt{a_1},\ldots,\sqrt{a_n}\}-\{\sqrt{a_{i_1}}\}$. This implies that $[K:k]\leq 2^{n-1}$, a contradiction.

This completes the proof of the theorem.

If $\mathrm{char}(k)=2$, then the condition $(^*)$ in the above theorem is necessary (the same proof works) but not sufficient. For example, let $\mathrm{char}(k)=2$ and let a be a nonsquare in k. Then $[k(\sqrt{a}):k]=2$ as expected; however, $\sqrt{a+1}=\sqrt{a}+1$, so that $[k(\sqrt{a},\sqrt{a+1}):k]=[k(\sqrt{a}):k]=2$, although $a,a+1$ and a^2+a are all nonsquares in k.

To facilitate writing the corresponding condition when $\mathrm{char}(k)=2$, define

$$T_n \quad = \quad \phi\cup\{(i_1,\ldots,i_r)\,|\,1\leq i_1<i_2<\ldots<i_r\leq n,\,1\leq r\leq n\}, \tag{2.3}$$

n being a nonnegative integer. Thus T_n is the set of subsets of $\{1,2,\cdots,n\}$, including the empty set. T_n has 2^n elements. The elements s of T_n are subsets or ordered tuples written in increasing order. Thus e.g. $s=(1,3,4,6)$ or $\{1,3,4,6\}$ where 1,3,4,6 are elements of s and we write $1,3,4,6\in s$.

We define $A_\phi=1$ and for $s=(i_1,\ldots,i_r)\in T_n$, let A_s denote the product $a_{i_1}\ldots a_{i_r}$. If $K=k(\sqrt{a_1},\ldots,\sqrt{a_n})$, by rationalising denominators, $K=k[\sqrt{a_1},\ldots,\sqrt{a_n}]$. We can then write every element of K as $\sum_{s\in T_n}\sqrt{A_s}b_s$ for some $b_s\in k$.
For each $n\geq 0$, consider the 2^n variables X_s for $s\in T_n$. Consider the quadratic form $f_n=\sum_{s\in T_n}A_sX_s^2$. We may write, for convenience, $X_\phi=X_0$ and for $s=(i_1,\ldots,i_r)\in T_n, X_s=X_{i_1,\ldots,i_r}$. Thus we have $f_0=X_\phi^2=X_0^2$ and $f_1=X_\phi^2+a_1X_{\{a_1\}}^2=X_0^2+a_1X_1^2$. Similarly, $f_2=X_0^2+a_1X_1^2+a_2X_2^2+a_1a_2X_{1,2}^2$.

Then for $\mathrm{char}(k)\neq 2$, we can put Theorem 1 in the following form. This is done just for comparison with the case $\mathrm{char}(k)=2$.

Theorem $1'$. Let $\mathrm{char}(k)\neq 2$. Let $a_1,\ldots,a_n\in k$ and $K=k(\sqrt{a_1},\ldots,\sqrt{a_n})$. Then $[K:k]=2^n$ if and only if for each m such that $1\leq m\leq n$, a_m is not representable by the quadratic form

$$f_{m-1}=\sum_{s\in T_{m-1}}A_sX_s^2, \tag{2.4}$$

with values of X_s in k and **with all X_s, except one, being zero**.

Proof: If for every m such that $1 \le m \le n$, a_m is not representable by f_{m-1} with all X_s except one being zero, then for every $s \in T_{m-1}$, $a_m \ne A_s b_s^2$ for every $b_s \ne 0$; so $a_m A_s \ne A_s^2 b_s^2$, for every $b_s \ne 0$, and so is a nonsquare. Thus $A_{s\cup\{m\}}$ is a nonsquare for every $s \in T_{m-1}$. Hence for every $s \in T_n - \{\phi\}$, A_s is a nonsquare. Hence by Theorem 1, $[K : k] = 2^n$. The converse is similar.

Note that for $\mathrm{char}(k) \ne 2$, for $[K : k] = 2^n$, it can happen that a_m is representable by f_{m-1} with two or more X_s nonzero, e.g. for $k = \mathbb{Q}$, $a_1 = 2$, $a_2 = 3$, we have $3 = 1^2 + 2 \cdot 1^2$, so that a_2 is represented by $f_1 = X_0^2 + a_1 X_1^2$ although $\mathbb{Q}(\sqrt{2}, \sqrt{3}) = 2^2$.

The case char(k) = 2

For $\mathrm{char}(k) = 2$, we have the following

Theorem 2. Let $\mathrm{char}(k) = 2$. The following are equivalent.

1. $[k(\sqrt{a_1}, \sqrt{a_2}, \ldots, \sqrt{a_n}) : k] = 2^n$.
2. For each $1 \le m \le n$, a_m is not representable by the quadratic form $f_{m-1} = \sum_{s \in T_{m-1}} A_s X_s^2$ with values of X_s in k.
3. The quadratic form f_n is anisotropic over k, i.e. $f_n = 0$ has no nonzero solution over k.
4. The 2^n elements $A_s, s \in T_n$, are linearly independent over k_0, where k_0, is the subfield of k consisting of the squares of elements of k.

Proof. (1) $\Rightarrow$ (2): If for some m, $a_m = \sum_{s \in T_{m-1}} A_s b_s^2$, with $b_s \in k$, then $\sqrt{a_m} = \sum_{s \in T_{m-1}} \sqrt{A_s} b_s \in k(\sqrt{a_1}, \ldots, \sqrt{a_{m-1}})$ so that $[k(\sqrt{a_1}, \ldots, \sqrt{a_m}) : k] \le 2^{m-1}$, a contradiction.

(2) $\Rightarrow$ (1): Conversely, if a_m is not represented by f_{m-1}, then $a_m \ne \sum_{s \in T_{m-1}} A_s b_s^2$, $b_s \in k$. So, $\sqrt{a_m} \ne \sum_{s \in T_{m-1}} \sqrt{A_s}\, b_s$, for any $b_s \in k$. Hence $\sqrt{a_m} \notin k(\sqrt{a_1}, \ldots, \sqrt{a_{m-1}})$ as $\{\sqrt{A_s} \mid s \in T_{m-1}\}$ generates $k(\sqrt{a_1}, \ldots, \sqrt{a_{m-1}})$ as a vector space over k. This is true for each $m, 1 \le m \le n$. Hence $[k(\sqrt{a_1}, \ldots, \sqrt{a_n}) : k] = 2^n$.

(3) $\Rightarrow$ (2). If (2) is false, then there is $m, 1 \le m \le n$ such that a_m is representable by f_{m-1} over k. This gives a nonzero solution for $f_n = 0$ over k, so (3) is false. Hence (3) $\Rightarrow$ (2).

(2) $\Rightarrow$ (3): The proof is by induction on n. If $n = 1$, suppose $f_1 = X_0^2 + a_1 X_1^2$ is isotropic. So for some $(b_0, b_1) \ne (0,0), b_0^2 + a_1 b_1^2 = 0$. If $b_1 = 0$, then $b_0 = 0$. Hence, $b_1 \ne 0$. Hence $a_1 = -b_0^2/b_1^2 = b_0^2/b_1^2$ is representable by $f_0 = X_0^2$, which contradicts (2).
Let $n \ge 2$ and suppose (2) $\Rightarrow$ (3) holds upto $n-1$. Suppose that the quadratic form f_n is isotropic over k. Thus there are $b_s \in k, s \in S$ such that some $b_s \ne 0$ and $\sum_{s \in T_n} A_s b_s^2 = 0$. Here if for every $s \ne \phi$, $b_s = 0$, then $b_\phi = 0$. Hence there is $s \ne \phi$ such that $b_s \ne 0$. Let m be the largest such that $m \in s$ and $b_s \ne 0$ for some $s \in T_m$. Thus $b_s = 0$ whenever $i \in s$ for some $i > m$.

If $m = 1$, then $b_0^2 + a_1 b_1^2 = 0$, with $b_1 \ne 0$. So a_1 is representable by f_0, contradicting (2). So $m > 1$. Again, $0 = \sum_{s \in T_m} A_s b_s^2 = (\sum_{s \in T_{m-1}} A_s b_s^2) + a_m (\sum_{s \in T_{m-1}} A_s b_{s\cup\{m\}}^2) = Q_0 + a_m Q_1$ say. Now $\sqrt{Q_0} = \sum_{s \in T_{m-1}} \sqrt{A_s} b_s$ and $\sqrt{Q_1} = \sum_{s \in T_{m-1}} \sqrt{A_s} b_{s\cup\{m\}}$ are both in $k[\sqrt{a_1}, \ldots, \sqrt{a_{m-1}}]$. Here, if $Q_1 \ne 0$, then $a_m = \dfrac{Q_0}{Q_1}$. Taking square roots, $\sqrt{a_m} \in k(\sqrt{a_1}, \ldots, \sqrt{a_{m-1}}) = k[\sqrt{a_1}, \ldots, \sqrt{a_{m-1}}]$.

Hence $\sqrt{a_m} = \sum_{s\in T_{m-1}} c_s\sqrt{A_s}$, for some $c_s \in k$. Hence $a_m = \sum_{s\in T_{m-1}} A_s c_s^2$, so that a_m is representable by the quadratic form f_{m-1}. This contradicts (2). Hence $Q_1 = 0$. Now, by the assumption on m, for some $s \in T_{m-1}$, $b_{s\cup\{m\}} \neq 0$, so the quadratic form $\sum_{s\in T_{m-1}} a_s X_s^2$ is isotropic. As $1 \leq m-1 < n$, by induction hypothesis, for some $r, 1 \leq r \leq m-1$, $a_r = \sum_{s\in T_{r-1}} A_s e_s^2$ for some $e_s \in k$. This again contradicts (2). This proves (2) $\Rightarrow$ (3).

(**3**) $\Leftrightarrow$ (**4**): Since $\mathrm{char}(k) = 2$, the set k_0 of squares in k forms a subfield of k. Suppose (4) is false. Then the 2^n elements $A_s, s \in T_n$, are linearly dependent over k_0. Thus there are elements b_s^2 in k_0, not all zero, with b_s in k, such that $\sum_{s\in T_n} b_s^2 A_s = 0$. Hence f_n is isotropic. i.e. (3) is false. Thus (3) $\Rightarrow$ (4). Similarly (4) $\Rightarrow$ (3). This proves the theorem.

Remark. Using Theorem 1 and Theorem 2, one can determine $[k(\sqrt{a_1},\ldots,\sqrt{a_n}) : k]$ if one has sufficient information about k. Thus if $\mathrm{char}(k) \neq 2$, then from the set $S = \{a_1,\ldots,a_n\}$ drop a_i where i is the least such that a certain product (of entries from S) of the form $a_{j_1}a_{j_2}\ldots a_{j_r}a_i$, $1 \leq j_1 < j_2 < \ldots < j_r \leq i-1$, $0 \leq r \leq i-1$ is a square. Continue the process with the remaining set. If finally t a_j's remain, then
$[k(\sqrt{a_1},\ldots,\sqrt{a_n}) : k] = 2^t$ and the extension field is generated by the square roots the remaining t a_j's.

Example 1. $K = \mathbb{Q}(\sqrt{-10}, \sqrt{\frac{2}{15}}, \sqrt{-6}, \sqrt{2}, \sqrt{\frac{1}{15}}, \sqrt{\frac{7}{5}})$.
Here we consider the set $S = \{-10, \frac{2}{15}, -6, 2, \frac{1}{15}, \frac{7}{5}\}$.
We list the elements and their products with all previous products of distinct elements. If an element or its product with a previous product is a square, then we drop that element from S and continue. Thus we have the sets $\{-10\}, \{\frac{2}{15}, -\frac{4}{3}\}, \{-6, 60, -\frac{4}{5}, 8\}$ consisting of nonsquares. So keep -10, $\frac{2}{15}$ and -6 in S. The next number 2 when multiplied with 8 gives 16, which is a square, so drop 2 from S. Next $60 \cdot \frac{1}{15}$ is a square. So drop $\frac{1}{15}$ from S. $\frac{7}{5}$ cannot be dropped as its product with previous products contains 7 with first power, so we get nonsquares. Hence

$$K = \mathbb{Q}(\sqrt{-10}, \sqrt{\tfrac{2}{15}}, \sqrt{-6}, \sqrt{\tfrac{7}{5}}) \text{ and } [K : \mathbb{Q}] = 2^4 = 16.$$

Example 2. If $a_1,\ldots,a_n$ are nonsquare integers relatively prime in pairs, then $[\mathbb{Q}(\sqrt{a_1},\ldots,\sqrt{a_n}) : \mathbb{Q}] = 2^n$.

Example 3. Let $f_1(x),\ldots,f_n(x)$ be nonsquare polynomials over $\mathbb{C}$, which are relatively prime in pairs. Let $K = \mathbb{C}(x)(\sqrt{f_1(x)},\ldots,\sqrt{f_n(x)})$. Then $[K : \mathbb{C}(x)] = 2^n$.

Example 4. Let R be a UFD (Unique Factorisation Domain) of characteristic $\neq 2$. Let K be the quotient field of R. Let $a_1,\ldots,a_n$ be nonunit nonsquares in R and relatively prime in pairs. Then $[K(\sqrt{a_1},\ldots,\sqrt{a_n}) : K] = 2^n$.

Example 5. Let F be of characteristic 2. Let a be a nonsquare in F, Then a^3+a+1 is representable as $X_0^2 + aX_1^2$ for $X_0 = 1$ and $X_1 = a+1$. Thus $[F(\sqrt{a}, \sqrt{a^3+1}) : F] \neq 2^2$ and this degree must be 2.

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3. Ramanujan Birthday Magic Squares

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Abstract: Ramanujan Birthday Magic square (RBMS) is one of the interesting contributions of the great Indian mathematician Srinivasa Ramanujan involving his birthday 22-12-1887. In this article we discuss these magic squares and their properties.

3.1 Introduction

Magic squares have fascinated mathematicians and mathematics lovers for centuries. The legendary mathematical genius Srinivasa Ramanujan has also contributed to the theme an ingenious construction of 4×4 magic square, with 4 parameters. In this article we present its construction and discuss some of its interesting properties. We begin with the definition of a magic square.

Definition 1. *A magic square of order n is an $n \times n$ square grid with n^2 distinct positive integer entries, such that the sums of the entries in each of the rows, each of the columns and also of the entries along each of the two diagonals, is the same number; the latter is called the magic constant or the magic sum, the entries in the cells of the magic square are called the magic numbers.*

Ramanujan introduced the remarkable magic square of order 4,

A	B	C	D
D + 1	C - 1	B - 3	A + 3
B - 2	A + 2	D + 2	C - 2
C + 1	D - 1	A + 1	B - 1

(1)

allowing four independent parameters A, B, C, D, which only need to be distinct positive integers for which that all the entrees in the square are distinct positive integers. We note that the entries of the square are made up of four quadruplets $(A, A+1, A+2, A+3)$, $(B-3, B-2, B-1, B)$, $(C-2, C-1, C, C+1)$ and $(D-1, D, D+1, D+2)$, and the condition on the parameters A, B, C and D is precisely that the four blocks consist of positive integers and are disjoint from each other. The condition holds in particular if for every pair of numbers picked from the set of values chosen for A, B, C and D differ by 7 or more; (this is however not a necessary condition). This shows that we have here an infinite collection of such magic squares. Moreover, we can also have several such magic squares for which the magic constant is the same.
In (1) if we choose $A = 22$, $B = 12$, $C = 18$, $D = 87$, we get the magic square

22	12	18	87
88	17	9	25
10	24	89	16
19	86	23	11

(2)

in which the first row is Ramanujan's birthdate (22-12-1887). In this context we shall call any magic square arising as in (1) from various admissible choices, a Ramanujan's Birthday Magic Square (RBMS for short).

An RBMS can be constructed analogously with other birthdays appearing as the first (or any other specific) row. However, not all birthdays are suited for this, as the conditions needed for all the entries to be positive integers and distinct may not be satisfied; in particular, for persons who are born in the months January, February or March we do not get an RBMS as above. Niels Henrik Abel, a Norwegian mathematician, who made pioneering contributions in a variety of fields, was born on 5^{th} August, 1802. It can be seen that for this birthdate we cannot construct an RBMS as above.

3.2 Identities satisfied by the entries of RBMS

Let S be the set of 16 pairs (i,j), with i and j between 1 and 4. We shall say that a subset T of S consisting of 4 elements is *cross-sectional* for RBMSs if the entries in the square (1) corresponding to the coordinate entries from T involve all of the parameters A, B, C, and D. For example, the subsets $(1,1),(1,2),(2,1),(2,2)$, and $(1,2),(2,2),(3,3),(4,3)$ are cross-sectional. As each of the parameters occurs in the square (1) at 4 places, and they can be chosen independently, we see that there are 256 cross-sectional subsets. We note that if T is a cross-sectional subset then for any RBMS with magic constant K the sum of the entries of the RBMS corresponding to T is $K+t$, where t is a number between -6 and 6, depending only on T; when T is cross-sectional, the entries in (1) corresponding to its elements are of the form $A+a, B+b, C+c$ and $D+d$, for some a, b, c and d between -3 and 3, and hence the sum is $A+B+C+D+a+b+c+d = K+t$, where $t = a+b+c+d$; since a, b, c and d depend on the positions of the elements of T (and not on the values of A, B, C, D), it follows that t depends only on T.

As there are 256 cross-sectional subsets and only 13 possible values of t, we see that there would be numerous blocks for which the corresponding sums are equal, not only for the given RBMS but also for all of them with the same value of the magic constant. The author has explored many of these identities further, and hopes to discuss the topic further elsewhere. It may be noted that the value t being 0 corresponds to the sum corresponding to the subset being K, the magic constant of the square. These include the rows, the columns, the two diagonals, and also subsets consisting of (i,j) with j from $1,2$ with i taking any two values between 1 and 4, and similarly with j from $3,4$ with i taking any two values between 1 and 4, ensuring that the subsets are cross-sectional.

3.3 Magic Squares with consecutive positive integers

One may ask whether one can construct an RBMS whose entries are 16 consecutive positive integers. We present here an affirmative answer to this, and also give a count of the number of ways in which it can be done. We note that without loss of generality we may assume the integers to be 1 to 16. Recall that the entries are made up of $(A, A+1, A+2, A+3)$, $(B-3, B-2, B-1, B)$, $(C-2, C-1, C, C+1)$ and $(D-1, D, D+1, D+2)$, which consist of consecutive integers. If they are to make up entries of an RBMS, their entries must be disjoint. Consequently, their entries should cover the set of integers from 1 to 16 if and only if the collection of the 4 subsets as above coincides with the collection of the 4-element subsets of the form $(k, k+1, k+2, k+3)$, with $k = 1, 5, 9$, and 13. Namely, to each one of the above subsets we need to associate one of these, in a one-one correspondence, and, conversely, setting up any such correspondence gives rise to a magic square.

For example, if we choose to follow for the choice the order in which the sets are written, viz. A, B, C, D, we should have $A = 1, B-3 = 5, C-2 = 9$, and $D-1 = 11$, it entails $A = 1, B = 8, C = 11$ and $D = 14$ and substituting the values we get the RBMS

1	8	11	14
15	10	5	4
6	3	16	9
12	13	2	7

(3)

Setting up a correspondence as required above is readily seen to correspond to choosing an order on the values to be assigned to A, B, C, D. Once we choose an order, say for example B, C, A, D as the order of the numbers (from 1 to 16) to be associated with them, then that forces the choice for A, B, C, D, which for this example will be $B-3 = 1, C-2 = 5, D-1 = 9$ and $A = 13$. Conversely, any RBMS with entries from 1 to 16 corresponds to a choice in the order for A, B, C, D (as the 4-element sets as above are disjoint and contain exactly one element from these). Since there are 24 ways in which we can choose an order between A, B, C, D it follows that there are 24 such RBMSs;

we note that they are all distinct since the order among A, B, C and D is reflected in the first row of the magic square.

Acknowledgement: I sincerely thank the Reviewer for meticulously reading the manuscript and rewriting it. I profusely thank Prof. V. G. Tikekar, Prof. S. A. Katre and Prof. J. N. Salunke for meticulously reading the manuscript and giving comments.

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Professor V. P. Saxena, Vice-President, TMC has been awarded Fellowship of ISMMACS - 2021

The Indian Society for Mathematical Modelling and Computer Simulation has conferred Fellowship of ISMMACS - 2021 upon PROF. DR. V. P. SAXENA, Former Vice-Chancellor and Professor of Mathematics at Jiwaji University, Gwalior, for his outstanding contributions in the field of Mathematical Modelling and Simulation, in August, 2022.

The Indian Society for Mathematical Modelling and Computer Simulation was formed in 1996 with Prof. Roddam Narasimha as its first President and Prof. J. B. Shukla as the first General Secretary. The society was formed with an objective to provide a forum where scientists from different disciplines can meet and discuss various relevant problems that human society faces today.

The earlier recipients of the Fellowship include Professors, J. M. Cushing (U.S.A.), H. Hethcote (Canada), G. Herbert (Germany), T. J. Pedley (England), Roddam Narasimha (India).

4. What is Happening in the Mathematical World?

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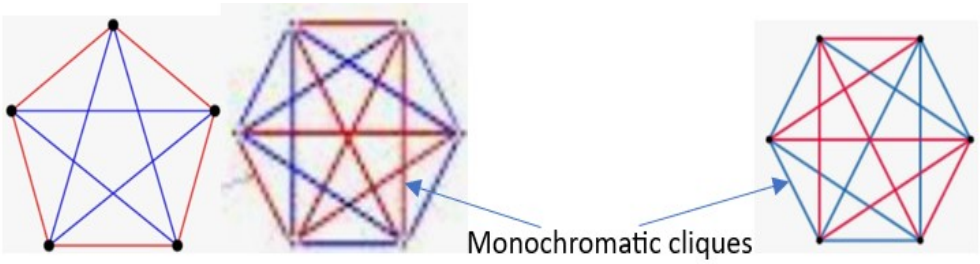
4.1 An Upper Bound On Ramsey Number Improved

Four mathematicians (from left to right) *Julian Sahasrabudhe, Rob Morris, Simon Griffiths* and *Marcelo Campos* have found a new upper limit to the "Ramsey number", a crucial property describing unavoidable structure in graphs.

In the March 16 seminars, and in a paper posted later that evening, the researchers announced an improvement in the upper bound on the Ramsey number by an exponential factor.

The Ramsey number describes how many nodes a complete graph must contain, to be forced to have a certain structure, say the edges of a complete graph are assigned one of two colors: red or blue. And say you try to color the edges in a way that avoids connecting a group of k nodes with edges of the same color. In 1930, Frank Ramsey proved that when a graph is big enough, it becomes impossible to avoid creating what is called a monochromatic clique, namely a group of nodes whose common edges are either all red, or all blue.

The question is: How big, exactly, must a graph be before a monochromatic clique is forced to emerge?



The answer depends on the size (number of nodes) of the clique. Ramsey showed that there exists a number $R(k)$, now called the diagonal Ramsey number, representing the minimum number of nodes for which a monochromatic clique of a given size k must exist, no matter how the edges are colored. Figure 2 shows that if we take a complete graph K_5 then it is possible to color each edge red or blue without creating a monochromatic clique of size 3 but for K_6 there is no way to avoid creating a monochromatic clique of size 3 and hence the Ramsey number $R(3) = 6$. It wasn't until 1955 that $R(4)$ was pinned down at 18. $R(5)$ remains unknown - it is at least 43 and no bigger than 48, and $R(10)$ has been estimated to lie between 102 and 23,556. Mathematicians are seeking a formula that will give a good estimate of $R(k)$, especially when k is extremely large.

The paper in which Ramsey introduced the number named after him was published in the Proceedings of the London Mathematical Society just weeks before he died, at the age of 26.

Five years later, Erdös and Szekeres showed that the Ramsey number $R(k)$ is less than 4^k. And 12 years after that, Erdös showed that it is bigger than about $\sqrt{2}^k$. This was achieved by calculating the probability that a graph with randomly colored edges contains a monochromatic clique. This "probabilistic" technique became very influential in Graph theory. It also trapped $R(k)$ between two exponentially growing numbers: $\sqrt{2}^k$ and 4^k.

This January, Campos, Griffiths, Morris and Sahasrabudhe, decided to switch to another version of the problem, namely finding bounds on the "off-diagonal" Ramsey number $R(k,l)$, which measures how big a graph must be before it contains either a red clique with k nodes, or a blue clique with l nodes. They found a way to delete a bunch of blue edges at once following a particular protocol,

which increased the density of red edges, and led to an improved bound on the off-diagonal Ramsey number. This method, called a "density increment" argument then used to improve the upper bound on the Ramsey number by an exponential factor. They have claimed to have shown that $R(k) < 3.993^k$.

Source: *https://www.quantamagazine.org/after-nearly-a-century-a- new-limit-for-patterns- in - graphs -20230502/*

4.2 Ninth Dedekind Number Computed

Scientists at Paderborn University in Germany and KU Leuven University in Belgium have discovered the ninth Dedekind number, which experts have been looking for since 1991.

The Dedekind numbers were first defined by Richard Dedekind in 1897, and have interested mathematicians ever since. The Dedekind number $D(n)$ counts the number of monotone Boolean functions of n variables. A Boolean function is a function that takes as input n Boolean variables (that is, values that can be either false or true, or equivalently binary values that can be either 0 or 1), and produces as output another Boolean variable. It is monotonic if, for every combination of inputs, switching one of the inputs from false to true can only cause the output to switch from false to true (and not from true to false).

The first eight Dedekind numbers have been known and their exact values are 2, 3, 6, 20, 168, 7581, 7828354, 2414682040998, 56130437228687557907788. The first six of these numbers were given by Church (1940). $D(6)$ was calculated by Ward (1946), $D(7)$ was calculated by Church (1965) and Berman & Köhler (1976), and $D(8)$ was calculated by Wiedemann (1991). The ninth one $D(9)$ has remained elusive until now. Now, in 2023 Christian Jäkel from Technische Universität Dresden, Germany and Lennart Van Hirtum, Master student at KU Leuven and currently a research associate at the University of Paderborn, simultaneously determined $D(9)$.

The eighth Dedekind number was found using the most powerful computer of that time - a Cray 2. Lennart Van Hirtum and his team found $D(9)$ by performing computations on the Noctua 2 supercomputer at the Paderborn Center for Parallel Computing. The team ran the computation on this supercomputer for approximately five months. On March 8, 2023, the team found the ninth Dedekind number: 286386577668298411128469151667598498812366, having 42 digits.

Sources:

1. *https://interestingengineering.com/innovation/supercomputer-ninth-dedekind-number- problem*
2. *https://www.sciencealert.com/mathematicians-discover-the-ninth-dedekind-number-after- 32 years-of-searching*
3. *https://en.wikipedia.org/wiki/Dedekind_number*

4.3 Computer Scientists Prove Decades Old Problem Of Size Of A Set Of Integers

In 1936, Erdös and Turán published a paper that sparked nearly a century of research into the size of integer sets that avoid arithmetic progressions.

Zander Kelley (left), a graduate student at the University of Illinois, Urbana-Champaign and *Raghu Meka* (right) of the University of California, Los Angeles, both computer scientists, have found a new - and dramatically lower limit on the size of a set of integers bounded by a given number in which no three of them are evenly spaced (ruling out combinations like 3, 8 and 13 or 101, 201 and 301). Their claim smashed the previous record from 2020.

Kelley and Meka have focused on arithmetic progressions made up of just three numbers, following a line of research often traced to a 1936 paper by Paul Erdös and Paul Turán. Erdös

and Turán wanted to know how many numbers smaller than some ceiling N can be put into a set bounded without creating any three-term arithmetic progressions. N might be 1,000, 1 million, or some unimaginably huge number. They conjectured that as N grew larger, a set without three-term progressions would have to become incredibly sparse. There is some structure that is going to pop into the set, no matter how one chose the set. How large a set does one really need to guarantee that this structure in there?

In 1946, Felix Behrend found a way to construct sets of numbers between 1 and N not containing any three-term progressions. His method resulted in sets that got bigger as N did, but achingly slowly. When N is 100,000, Behrend's set contains just 171 elements. When N is 1 million, his set has 586 numbers - less than 0.06% of the numbers between 1 and 1 million. Until Kelley and Meka's paper arrived, the maximum size of a progression-free set was penned in from below by Behrend's formula and from above by Bloom and Sisask. In their paper from July 2020 they showed that a progression-free set must have substantially fewer than $N/(logN)$ elements. Together with Kelley, Meka showed that for some constant $k > 0$, the size of progression-free subset of integers $\{1, 2, \ldots, N\}$ of size at least $N \cdot 2^{-O((logN)^k)}$ as compared to previously known $N/(logN)^{1+c}$ for a constant $c > 0$.

Sources:

1. *https://www.quantamagazine.org/surprise-computer-science-proof-stuns-mathematicians - 20230321/*
2. *https://gilkalai.wordpress.com/2023/02/14/absolutely-sensational-morning-news-zander- kelley -and-raghua meka-proved-behrend-type-bounds-for-3aps/*

4.4 A Shape With A Pattern that Never Repeats

Experts have searched for decades for a polygon that only makes non-repeating patterns. But no one knew it was possible until now.

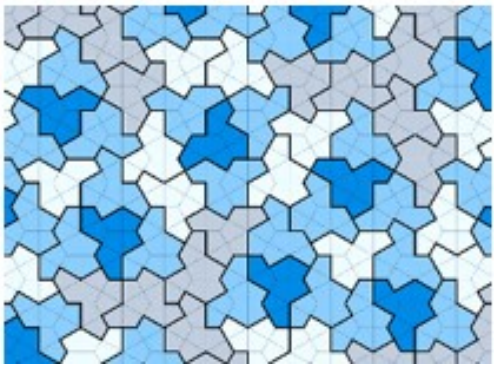


Figure 1

From bathroom floors to honeycombs or even groups of cells, tilings surround us. These patterns cover a space without overlapping or leaving any gaps. Like a rug filled with diamond shapes, where each section looks the same as the one next to it. Until now every tiling ever recorded has eventually repeated itself. After decades of searching for mathematicians call an "einstein tile (named for the German words for "one stone", or one tile)" - an elusive shape producing a tiling which does not repeat - researchers say they have finally identified one. The 13-sided figure is the first that can fill an infinite surface with a pattern that is always original.

Repeating patterns have translational symmetry, meaning you can shift one part of the pattern and it will overlap perfectly with another part, without being rotated or reflected. The new shape described does not have translational symmetry - each section of its tiling looks different from every part that comes before it.

Now, (from left to right) *David Smith*, a retired printing technician and nonprofessional mathematician of the East Riding of Yorkshire, *Craig S. Kaplan*, a computer scientist at the University of Waterloo, Canada, *Chaim Goodman-Strauss* of University of Arkansas and *Joseph Myers*, a software developer in Cambridge, England came up with the shape that could be a solution to the long-standing "einstein problem". Kaplan and Smith had

gotten about halfway through the problem, and Goodman-Strauss and Myers were able to sort of fill in the rest of the puzzle and complete the proof. The team published a preprint paper detailing the findings on the site arXiv. The "einstein" tile is made up of eight kites, or four-sided polygons with two pairs of adjacent, equal-length sides (see Figure 1). Researchers call it "the hat" because of its resemblance to a cap. The shape is simpler than some experts expected it to be.

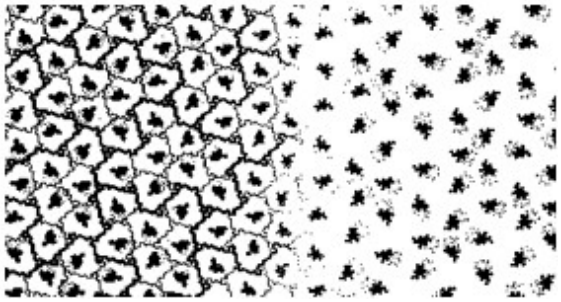


Figure 2

Less than a week after this paper came out, Smith emailed Kaplan a new shape. Kaplan refused to believe it at first. But analysis confirmed that the new Tile was a "non-reflective Einstein". Something still bothered them - while this tile could go on forever without repeating a pattern, this required an "artificial exclusion" against using a flipped shape. So, they added little notches or curves to the edges, ensuring that only the non-flipped version could be used, creating "the spectre". In Figure 2, the original hat shape (left) and the new spectre shape (right) can be easily seen.

In the 1970s, Nobel-winning mathematician Roger Penrose discovered that two shapes could form a non-repeating tiling pattern together, prompting hopes that a single shape may be found to do this one day. Researchers have been able to make other non-repeating patterns in the past, but the challenge has been finding a shape that can only make a non-repeating pattern. The shape of "the hat" can also be morphed to form additional tile shapes that make non-repeating patterns.

This new finding could lead to materials science investigations - for example, shapes that form non-repeating tilings could help design stronger materials. The elusive shape might also spark creative inspiration for new decorative designs or art.

Sources:

1. *https://www.theguardian.com/science/2023/apr/03/new-einstein-shape-aperiodic- monotile*
2. *ttps://en.prothomalo.com/science-technology/science/qq3lo25tnu*
3. *https://www.smithsonianmag.com/smart-news/at-long-last-mathematicians- have- found- a -shape-with-a-pattern-that-never-repeats-180981899/*

4.5 Existence Of (N,K,T) Subspace Design

The study of designs can be traced back to 1850, when Thomas Kirkman, a priest in a community in the north of England who experimented in mathematics, posed a seemingly straightforward problem in a magazine called the Lady's and Gentleman's Diary. Say 15 girls walk to school in rows of three every day for a week. Can you arrange them so that over the course of those seven days, no two girls ever find themselves in the same row more than once? Soon, mathematicians were asking a more general version of Kirkman's question: If you have n elements in a set (our 15 schoolgirls), can you always sort them into groups of size k (rows of three) so that every smaller set of size t (every pair of girls) appears in exactly one of those groups? Such configurations, known as (n,k,t) designs, have since been used to help develop error-correcting codes, design experiments, test software, and win sports brackets and lotteries.

In 2014, Peter Keevash showed that even if you do not know how to build such designs, they always exist, so long as n is large enough and satisfies certain simple conditions.

For decades, mathematicians have translated problems about sets and subsets - like the design question - into problems about the so-called vector spaces and subspaces.

The subspace design problem deals with n-dimensional vector spaces and their subspaces. In such vector spaces - again, so long as n is sufficiently large and satisfies simple conditions - can you find a collection of k-dimensional subspaces such that any t-dimensional subspace is contained in exactly one of them? Such an object is called an (n,k,t) subspace design.

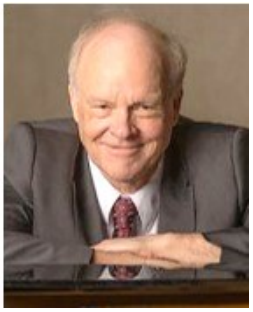
In the 50 years, since mathematicians started thinking about this problem, they have found only one nontrivial example: In a 13-dimensional vector space, it is possible to cover two-dimensional subspaces with three-dimensional ones exactly once. The result required a massive computer-based proof, because even for such small values of n, k and t, you end up working with millions of subspaces.



Now (left to right) *Mehtaab Sawhney, Ashwin Sah*, graduate students at MIT, and *Peter Keevash* (Professor of Mathematics at University of Oxford), have proved the existence of "subspace designs" whose existence is not at all obvious. For that they tested the limits of several well-known methods in combinatorics.
To do so, they had to revamp Keevash's original approach - which involved an almost magical blend of randomness and careful construction - to get it to work in a much more restrictive setting.
In the end, the work illustrated yet another counterintuitive way that mathematicians can attach the forces of randomness to search for hidden structure.
Source: *https://www.quantamagazine.org/mathematicians-find-hidden-structure-in-a- common -type -of-space-20230412/*

4.6 The Invariant Subspace Problem Solved?

Two weeks ago, a modest-looking paper was uploaded to the arXiv preprint server with the unassuming title "On the invariant subspace problem in Hilbert spaces." The paper is just 13 pages long and its list of references contains only a single entry. The paper implicates to contain the final piece of a jigsaw puzzle that mathematicians have been picking away at for more than half a century: the invariant subspace problem.



The author of this short note, Swedish mathematician *Per Enflo*, almost 80, has made a name for himself solving open problems, and has quite a history with the problem at hand. The invariant subspace problem is about spaces with an infinite number of dimensions, and it asks whether every linear operator in those spaces must have an invariant subspace. More precisely: the invariant subspace problem asks whether every bounded linear operator T on a complex Banach space X admits a non-trivial invariant subspace M of X, in the sense that there is a subspace $M \neq \{0\}$, of X such that $T(M)$ is contained back in M. Stated in this way, the invariant subspace problem was posed during the middle of last century, and eluded all attempts at a solution. Many variants of the problem have been solved, by restricting the class of bounded operators considered or by specifying a particular class of Banach spaces. The problem is still open for separable Hilbert spaces (in other words, each example, found so far, of an operator with no non-trivial invariant subspaces is an operator that acts on a Banach space that is not isomorphic to a separable Hilbert space).
The first breakthrough was made by Enflo in the 1970s (although his result was not published until 1987). He answered the problem in the negative, by constructing an operator on a Banach space without a non-trivial invariant subspace.
Well, Enflo settled the problem for Banach spaces in general. However, resolving the invariant subspace problem for operators on Hilbert spaces has been stubbornly difficult, and it is this which Enflo claims to have achieved. This time Enflo answers in the affirmative: His paper claims to prove that every bounded linear operator on a Hilbert space does have an invariant subspace. Since earlier some attempts of mathematicians to prove the statement have turned out to be flawed, one would have to wait for a due pear review process to confirm the present proof to be correct.

If correct, it will be a remarkable achievement, especially for someone who has already produced so many remarkable achievements over such a large span of time.
Enflo has been one of the great problem-solvers in functional analysis. His many contributions to mathematics, and his answers to many open problems, have made a big impact on the field, generating new techniques and ideas. Aside from his work on the invariant subspace problem, Enflo solved two other major problems - the basis problem and the approximation problem - both of which had remained open for more than 40 years.
Source: *https://phys.org/news/2023-06-mathematician-invariant-subspace-problem.html*

4.7 US Teens Say they Have New Proof For Pythagorean Theorem

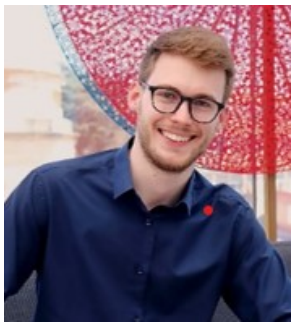
Two high school students *Calcea Johnson* and *Ne'Kiya Jackson* from New Orleans are making major waves in the world of academia after solving a mathematical equation involving the Pythagorean Theorem that has stumped mathematicians for the last 2,000 years. They found a new way to prove the Pythagorean Theorem using trigonometry without circular logic and presented their findings at the American Mathematical Society's Annual Southeastern Conference on March 18, 2023 in Atlanta.

According to their presentation, they noted that even the most notable mathematicians had thought that solving the Pythagorean Theorem without using circular logic was impossible. There are no trigonometric proofs, because all the fundamental formulae of trigonometry are themselves based upon the truth of the Pythagorean Theorem. Yet Johnson and Jackson knew that was not true, and they proved it.
"In our lecture we present a new proof of Pythagoras' Theorem which is based on a fundamental result in trigonometry - the Law of Sines - and we show that the proof is independent of the Pythagorean trigonometric identity $\sin^2 x + \cos^2 x = 1$", they said. In short, they claim to have proven the theorem using trigonometry and without resorting to circular reasoning.

Calcea Johnson and Ne'Kiya Jackson are being encouraged to submit their work to a peer-reviewed journal.
Sources:

1. *https://theblackwallsttimes.com/2023/03/28/new-orleans-teens-solve-impossible- mathematical -equation/*
2. *https://www.theguardian.com/us-news/2023/mar/24/new-orleans-pythagoras- theorem -trigonometry-prove*

4.8 First Year Graduate Student Finds Paradoxical Set



Mathematicians rejoice when they prove that seemingly impossible things exist. Such is the case with a new proof posted online in March by *Cédric Pilatte*, a first-year graduate student at the University of Oxford. Pilatte proved that it is possible to create a set that satisfies two apparently incompatible properties. The first is that no two pairs of numbers in the set add up to the same total. For example, add together any two numbers in 1, 3, 5, 11 and you will always get a unique number. It is easy to construct small "Sidon" sets like this one, but as the number of elements increases, so too does the likelihood that the sums will coincide, destroying the Sidon-ness of the set.
The second requirement is that the set must be very large. It must be infinite, and you should be able to generate any sufficiently large number by adding together at most three numbers in the set. This property, which makes the set an "asymptotic basis of order 3", requires a large, dense

set of numbers. Sidon sets are constrained to be small, and an asymptotic basis is constrained to be large.
The question of whether such a set exists has remained open for decades, ever since it was posed by the prolific Hungarian mathematician Paul Erdös and two collaborators, in 1993. Erdös' fascination with Sidon sets can be traced to a conversation he had in 1932 with their inventor Simon Sidon, who at the time was interested in understanding the growth rate of these sets.
Sidon sets arise in a variety of mathematical contexts including number theory, combinatorics, harmonic analysis and cryptography, but the simple question of how big they can get has been a lasting mystery that Erdös considered for much of his career. In 1969, Bernt Lindström have showed that largest possible Sidon set whose members are all less than some integer N, is smaller than $\sqrt{N} + \sqrt[4]{N} + 1$ and in 2021 another group of mathematicians tightened the bound to $\sqrt{N} + 0.998 \times \sqrt[4]{N}$. Sidon sets, in other words, have to be sparse.
It has long been known that a Sidon set cannot be an asymptotic basis of order 2, where any integer can be expressed as the sum of at most two numbers. It was generally believed that an asymptotic basis of order 3 could be constructed from a Sidon set, but proving this was another matter. In 2010, the Hungarian mathematician Sándor Kiss showed that a Sidon set can be an asymptotic basis of order 5, in 2013 Kiss and two of his colleagues proved the conjecture for an asymptotic basis of order 4. Two years later, the Spanish mathematician Javier Cilleruelo took these results a step further by proving that it is possible to construct a Sidon set that is an asymptotic basis of order $3 + \varepsilon$, meaning that any sufficiently large integer N can be written as the sum of four members of the Sidon set, with one of them smaller than $N\varepsilon$ for arbitrarily small positive ε. These findings were obtained using variations of a probabilistic method pioneered by Erdös. Pilatte realized that the probabilistic method had been pushed as far as it could go. So Pilatte took a different track, turning instead to a procedure that uses the logarithms of prime numbers as the building blocks of Sidon sets.
The search for a solution took Pilatte in an unexpected direction, away from additive number theory and into the world of algebraic geometry, a branch of mathematics that studies the relationship between geometric shapes, like curves and surfaces, and the equations that define them. Pilatte began by replacing numbers with polynomials, which immediately made the problem more tractable. Using a recent result by the Columbia University mathematician Will Sawin on the distribution of irreducible polynomials in arithmetic progressions, Pilatte was able to construct a set that possessed just the right amount of randomness and just the right density of numbers to satisfy Erdös' constraints.
Erdös problems have a strange ability for unearthing connections between supposedly unrelated branches of mathematics, and the discoveries mathematicians make while trying to answer them are often more meaningful than the answers themselves.
Sources: *https://www.quantamagazine.org/first-year-graduate-finds-paradoxical-number-set- 20230605/#: :text=Such%20is%20the%20case%20with,satisfies%20two%20apparently%20incompatible%20properties.*

4.9 Awards

4.9.1 Luis Caffarelli Awarded The 2023 Abel Prize For His Work On PDEs.

Argentinian-American *Luis A. Caffarelli*, 74 years old, a professor of mathematics at the University of Texas at Austin, is this year's winner of the Abel Prize - popularly known as Nobel prize of mathematics. The prize is accompanied by 7.5 million Norwegian kroner, or about $7,00,000. Caffarelli uncovered the interactions between solids and liquids, opening new doors to medicine, the automotive industry and even knowledge of the universe.
There is no Nobel Prize in mathematics, and for decades the most prestigious awards in mathematics were the Fields Medals, awarded in small batches

every four years to the most accomplished mathematicians who are 40 or younger. The Abel prize, named after Niels Henrik Abel, a Norwegian mathematician, is set up more like the Nobels. Since 2003 it has been given annually to highlight important advances in mathematics.
As a mathematician, Luis A. Caffarelli of the University of Texas at Austin tries to answer questions that sound simple, but are not easy to answer:

- How does the shape of a piece of ice change as it melts?
- Can a smooth flow of water ever spin out of control?
- What is the shape of an elastic sheet stretched around an object?

The behavior in these and many other phenomena in the world around us - including the gyrations of financial markets, the turbulence of river rapids and the spread of infectious diseases - can be described mathematically, using partial differential equations. The equations can often be written down simply, but finding exact solutions is devilishly difficult and indeed usually impossible. Yet, Caffarelli was able to make major progress in the understanding of partial differential equations even when complete solutions remain elusive. For those achievements, he is this year's winner of the Abel Prize.
The Norwegian Academy of Science and Letters, which awards the prize, has highlighted his results, especially in the so-called free boundary problems, such as those mathematical models of what happens at the contact surface between water and ice, or in an alloy of different molten metals that solidify at different rates. Caffarelli has also excelled by investigating into the Navier-Stokes equations, which describe the flow of a viscous fluid, such as oil. The applications of his work are incalculable: the analysis of a person's blood circulation, the prediction of the movement of oil, the manufacture of an automobile engine, financial mathematics, the refinement of the fundamental models that explain the universe.
His Majesty King Harald V presented the 2023 Abel Prize to Caffarelli at the award ceremony in the University of Oslo on May 23, 2023.
Source: *https://www.nytimes.com/2023/03/22/science/abel-prize-math-luis-caffarelli.html*

4.9.2 US Professors Shing-Tung Yau And Vladimir Drinfeld Awarded 2023 Shaw Prize

This year's Shaw Prize in Mathematical Sciences is awarded in equal shares to *Prof. Shint-Tung Yau* (left), director of Yau Mathematical Sciences Center at Tsinghua University and professor emeritus at Harvard University, and *Prof. Vladimir Drinfeld*, Harry Pratt Judson Distinguished Service Professor of Mathematics at the University of Chicago. The two laureates were chosen for their contributions related to mathematical physics, to arithmetic geometry, to differential geometry, and to Kähler geometry.
Prof. Yau is a Chinese-American mathematician and the William Caspar Graustein Professor of Mathematics at Harvard University. In April 2022, Yau announced retirement from Harvard to become Chair Professor of mathematics at Tsinghua University. He was awarded the Fields Medal in 1982.
Prof. Drinfeld is a renowned mathematician from the former USSR, who emigrated to the United States and is currently working at the University of Chicago. He was awarded the Fields Medal in 1990 and he received the Wolf Prize in Mathematics in 2018.
The Shaw Prize is awarded annually in three disciplines: astronomy, life sciences and medicine, and mathematical sciences. Each prize carries a gold medal, a certificate, and a monetary award of US $1.2 million. This will be the 20th year that the prize has been awarded, and the presentation ceremony is scheduled for November 12, 2023 in Hong Kong.
Source: *https://www.math.harvard.edu/harvard-professor-emeritus-shing-tung-yau-awarded- 2023-shaw-prize/*

4.9.3 Prof. Sourav Chatterjee has been Elected as a Fellow of The Royal Society

Indian origin *Prof. Sourav Chatterjee*, Professor of Mathematics and Statistics, Stanford University has been elected as a Fellow of the Royal Society (FRS). Fellowship of the Royal Society is a recognition granted by the Royal Society of London to individuals who have made a "substantial contribution to the improvement of natural knowledge, including mathematics, engineering science, and medical science". Fellowship of the Society, the oldest known scientific academy in continuous existence, is a significant honour. It has been awarded to many eminent scientists throughout history.

Sourav Chatterjee received the Bachelor of Statistics and Master of Statistics degrees from the Indian Statistical Institute, Calcutta, in 2000 and 2002. He received the Ph.D. degree in Statistics from Stanford University in 2005. He joined UC Berkeley as a Neyman Assistant Professor of Statistics in 2005, and started as a tenure-track assistant professor in the UC Berkeley Statistics department in 2006. He moved to New York University as an associate professor of Mathematics in the Courant Institute of Mathematical Sciences in 2009. Since 2013, he has been a professor of Mathematics and Statistics at Stanford University, California. His areas of interest are probability theory, statistics, and mathematical physics.

Dr. Chatterjee was awarded a Sloan Research Fellowship in mathematics in 2007, the 2008 Tweedie New Researcher Award from the Institute of Mathematical Statistics, the 2010 Rollo Davidson Prize for work in probability theory, the 2012 Doeblin Prize from the Bernoulli Society, the 2012 Young Researcher Award from the International Indian Statistical Association, and the 2013 Line and Michel Loeve Prize in probability from UC Berkeley. He gave a Medallion Lecture of the Institute of Mathematical Statistics in 2012 and was an invited speaker at the International Congress of Mathematicians in 2014. He was elected a Fellow of the Institute of Mathematical Statistics in 2018, and received the 2020 Infosys Prize in Mathematical Sciences.

Source: *https://royalsociety.org/people/sourav-chatterjee-36210/*

□□□

Indian FRS in Mathematical Sciences

1	S. Ramanujan (1918)	9	M.S. Narasimhan (1996)
2	C. Chandrashekhar (1944)	10	S.R.S. Varadhan (1998)
3	P.C. Mahalanobis (1945)	11	M.S. Raghunathan (2000)
4	S.N. Bose (i958)	12	C.S. Khare (2012)
5	C.R. Rao (1963)	13	Subhash Khot (2017)
6	Harishchandra (1973)	14	Manjul Bhargava (2019)
7	C.S. Seshadri (1988)	15	Akshay Venkatesh (2019)
8	Roddam Narasimha (1992)	16	Sourav Chatterjee (2023)

5. Kalyanapuram Rangachari Parthasarathy (25 June 1936 - 14 June 2023)

Kalyan B. Sinha
Emeritus Professor, ISI, INSA Senior Scientist, JNCASR Jakkur, Bangalore.
Email: kbs@jncasr.ac.in

Professor K.R. Parthasarathy ("Partha" to his western friends and collaborators and KRP to the Indian Mathematical Community) passed away on June 14, 2023. One of the handful of the legendary creators of schools of excellence of modern Mathematics in India, KRP was an expositor-par-excellence (both in lectures and in written articles) - a "great teacher". Like many very distinguished mathematicians he was often 'terse', he would sometimes say that "Mathematics is about the economy of thought and the economy of expression". His absence will be felt for a long time in the Indian Mathematical scene.

KRP was born in Chennai, Tamilnadu, had his early education (including B.Sc. honours in Mathematics) in Chennai before joining the 3-year 'advanced professional statisticians' training course' in the Indian Statistical Institute (I.S.I. for short), Kolkata in 1956. The academic climate in I.S.I. in 1950's and 1960's was somewhat unstructured and liberal and three bright young budding mathematicians (V.S. Varadarajan, R. Ranga Rao and KRP), to be expanded a few years later to include S.R.S. Varadhan, and named for posterity "the Famous Four", taught each other much of modern mathematics. Nearly two decades later when I met KRP for the first time in I.S.I., Delhi, he would often regale us with many stories of his time in Kolkata and talk about the 'underlying unity in Mathematics'; for example, among the Kolmogorov's decomposition theorem, the Gelfand-Naimark-Segal construction and Bochner's theorem on positive definite functions. This point of view had a major influence on many of us, including myself. However, this expansive (wide-canvas) view of Mathematics needed a demanding and relatively mature mathematical background, which was often missing in average graduate students at that time, and many of them had some difficulty in keeping up with KRP's zeal and intensity.

Under the supervision of C. R. Rao, KRP completed his Ph.D. dissertation entitled "Some problems of ergodic theory and information theory" in 1962 and started a career in I.S.I. itself. After a tradition instituted by the visionary founder of I.S.I., P.C. Mahalanobis, many distinguished Statisticians and Mathematicians from the world over visited I.S.I.. Professor A. N. Kolmogorov, the founder of modern theory of probability and a great mathematician, visited I.S.I. in 1962 and KRP was deputed to accompany and guide him during his stay in India. This led to KRP visiting Kolmogorov for a year during 1962-63 in the Steklov Mathematics Institute in Moscow. Though life in Moscow at that time was quite difficult for a vegetarian KRP from a warm tropical country, he was very happy about having the opportunity of attending the seminars of great minds like those of E.B. Dynkin, I.M. Gelfand and others. When KRP returned from Moscow in 1963, Varadarajan had also returned from the United States. Though Varadhan left for the United States in 1963, this period (1962-65) saw a burst of academic activities among the "famous four", resulting in 5-6 top-class publications, culminating with the "Representations of complex semi-simple Lie groups and Lie algebras" by KRP, Ranga Rao and Varadarajan (Ann. Math. (2) 25, 1967). It is in this period, 1964-65, that Varadarajan started giving a course of lectures on "Mathematics of Quantum Mechanics" (now available as book in Springer-Verlag publications, 1984), which KRP attended, and this may have sown the germ of interest in this area in KRP's mind. This interest stayed with KRP till his last days and may have been instrumental in my getting together with him.

Another landmark event happened to KRP: He got married to Shyamala (Shyama to friends) in 1965 and left for the United Kingdom. After several years of teaching in the Universities of Sheffield (1965-68) and of Manchester (1968-1970), he decided to return to India for good and following a

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This obituary is being co-published with Current Science. It will appear there in the 10^{th} August issue.

few years in the University of Bombay (1970-73), in the Indian Institute of Technology, Delhi (1973-76), he settled in the new campus of the Delhi Centre of I.S.I., from which he superannuated in 1996. There he continued as Professor Emeritus till his last days. His celebrated book "Probability Measures on Metric Spaces" (AMS 1967, reprinted 2005) was written during this period as was also the beautifully-presented more basic book "Introduction to Probability and Measures" (Macmillan 1977, Hindustan Book Agency 2005). I learnt my basics in Probability theory from that book and sometimes used theorems from this book to prove results in Operator theory.

The study of Central Limit theorems and of infinitely divisible probability distributions on topological spaces or groups were of great interest to the probabilists during the 60-70's and KRP was no exception; he wrote several important articles on these topics. In structures admitting some kind of continuous convolution as in the case of groups (much later quantum groups as well), it was known from KRP's (as well as many other probabilists') work that infinitely divisible distributions can be canonically associated with one-parameter semigroups of maps on a suitable space of functions on the structures. On the other hand, KRP's studies and investigations in the underlying structures arising in Quantum Mechanics (including his mastery of Mackey's theory of induced representations and the systems of imprimitivity) led to the natural query of the possibility of dropping the underlying space/groups in the space of functions and replace them by some kind of $*$-algebras in a Hilbert space, which is often the vessel carrying the description of quantum systems. This, "large canvas", more formally the construction of Markov Processes over $*$-algebras in a Hilbert space, occupied KRP and many of his collaborators for nearly four decades. This process of replacing classical objects like "the commutative family of functions on a suitable set" by non-commutative family of "operators in a suitable Hilbert space" would become (at the hands of KRP and Robin L. Hudson), the central tool in creating new non-commutative family of stochastic processes, driving the action of a one-parameter semigroup of (completely positive and unital) maps on a $*$-algebra in a Hilbert space. As is often the case, KRP first looked at finite dimensional case, in which classical observables (random variables) were mapped into the set of real diagonal matrices there, while the quantum ones went into the set of all Hermitian matrices. In this setup, the classical processes driven by classical Markov semigroups were constructed naturally by Fokker-Planck type equation associated with stochastic matrices. This point of view allows a quick natural generalization to the more interesting infinite-dimensional case, reinforcing the thought, held strongly by KRP and many others, that the Quantum theory is fundamentally a theory of probability, albeit of non-Kolmogorovian variety.

This gradual, yet striking change in the point of view in the research canvas of KRP took place over the years (1972-onwards) via the paper on the representation of current groups and the Araki-Woods embedding theorem (coauthored with K. Schmidt, Acta Math. 128 (1972)) and the lecture note on positive definite kernels, continuous tensor products and central limit theorems of probability theory (with K. Schmidt. LNM 272, Springer, 1972). The continuous tensor product of an indexed family of Hilbert spaces $\{H_{s,t} |\ 0 \leq s \leq t \leq \infty\}$, turned out to be the right vehicle to incorporate, the semigroups driven by continuous–time quantum stochastic processes. After some interesting interludes including attempts by KRP (with R.L. Hudson and P.D.F. Ion, in Comm. Math. Phys. 83, 1982) on Feynman-Kac-like formulae, as time-orthogonal product integral, KRP along with Hudson realized that the Fock–space representation of the unitary Weyl-Segal system on $L^2(R_+)$ provides an ideal setup to implement the continuous tensor product structure explicitly and develop a nice kind of stochastic calculus, and the rest is history.

At about this time in January 1982, as a part of the Golden Jubilee celebrations of the I.S.I., a conference on the "Theory and Applications of Random Fields" was organized in the Bangalore Centre of the I.S.I., and a large number of luminaries in Probability theory, including E.B. Dynkin, T. Hida, P.A. Meyer, S. Watanabe, K. Bichteler, J. Jacod lectured there. Both KRP and Hudson, of course, were there and they talked about their nascent theory and the quantum Ito-table (with the annihilation and creation processes only) made its appearance, implicitly, there for the first time.

This non-commutative calculus led to the beautiful paper "Quantum Ito's formula and stochas-

tic evolutions" (Comm. Math. Phys. 93, 1984) and to a more elaborate and abstract presentation of the theory in the lectures of KRP in 1988 in the Delhi Centre of ISI. These lectures of KRP appeared in 1992 as a monograph "An Introduction to Quantum Stochastic Calculus". Birkhauser, 1992 and are presented in a delectable style with the right kind of mixture of abstract ideas and their concrete manifestations. Suitable linear combinations of the mutually conjugate position and momentum processes lead to the annihilation and creation processes and the now familiar complete Quantum Ito table appears for the first time, which was incomplete in earlier attempts in the absence of the "Conservation or Number process". In fact, this process makes its appearance in the simple example of the "second quantization" of an operator of multiplication by an "adapted family" of function $\left\{f_{\mathcal{X}_{[0,t]}} |\ 0 \leq t \leq \infty, f \in nL^2(R_+)\right\}$. These names led KRP, in his book to compare them with the trilogy of Gods in Hindu scriptures: Shiva (the annihilator), Brahma (the creator), and Vishnu (the conservator). This "completion" led KRP and Hudson to formulate and solve quantum stochastic differential equation (qsde) with bounded operator-valued coefficient in the Hilbert tensor-product of the initial Hilbert space (in which the operator-coefficients act) and the Fock space mentioned earlier. The unitarity of the solution, giving the quantum stochastic evolution of the Hilbert space vectors are of great importance (just as in the so-called 'Schrodinger picture' of ordinary quantum mechanics) and the authors solved this problem completely for the case of constant bounded operator coefficients. The book "Quantum Probability for Probabilists" by P. A. Meyer appeared soon after in 1992 and was written in a style perhaps closer to that of classical probabilities. Curiously, Robin Hudson earlier and Accardi, Frigerio and Lewis in their article "Quantum Stochastic Processes" (Publ. RIMS, Kyoto Univ. 18, 1982) had emphasized the corresponding "Heisenberg picture", obtained by conjugating an observable (or any bounded operator) in the initial Hilbert space by these unitaries mentioned above. This looked like an aesthetically satisfactory situation in the development of the theory.

However, for the theory to be more useful to serve as a possible model to describe "dissipative phenomena in Quantum Physical systems", two further generalizations were needed: (i) a suitable class of unbounded operator coefficients, mentioned above, needs to be admitted in the analysis of the qsde and (ii) "the stochastic or the noise" part, as modeled in the Fock space over $L^2(R_+)$, needed to be expanded in variety.

In the same paper (CMP, 1984) KRP and Hudson also showed that the two abelian $*$-algebras generated by functions of classical Brownian motion and by classical Poisson processes are subalgebras of the non-abelian algebra of all bounded operators in the Fock space and classical notions of "independence" in each of these cases are subsumed by the tensor-independence in the new structure. Thus, in a sense, the tensor-independence used in the so called Hudson-Parthasarathy (or HP for short) stochastic calculus is the simplest generalization of the concept of independence in classical probability; the one farthest from the classical independence is that of "free independence" introduced by Voiculescu. Thus, the germ of idea in KRP's mind that quantum theory is a new kind of theory of Probability grew in four decades into a mature tree having more than one 'colour' of probability theories, giving rise to multiple possible choices of 'noises', relating to the question(ii) raised above. The partial resolution of the mainly analytical question in (i) took another decade or so to be sorted out.

In this way, the Delhi Centre of the I.S.I. during 1980-2000 (2 decades) became one of the major centres (with KRP as the driving force) in the world for research and dissemination of quantum probability theory. Just as it happened during the younger days of KRP in I.S.I., Kolkata, many international researchers visited I.S.I., Delhi regularly. The list includes Professors T. Hida, Robin Hudson, Wilhelm von Waldenfels, P.A. Meyer, L. Accardi, and Professors V.P. Belavkin, Martin Lindsay, F. Fagnola, S. Attal, N. Obata, M. Schurmann, U. Franz, and many others of the younger generation.

All these visits, back and forth, fostered all-round collaborations not only with KRP, but also amongst the whole group of researchers including students; the world of quantum probability, at the end of the millennium, grew to be of significant strength. It was at the beginning of this

period of excitement and growth that Prof. Accardi, along with a few others formed an association, which came to be named "Association for QPIDA (Quantum Probability and Infinite Dimensional Analysis)" of which KRP was the first president. Furthermore, this association became quite active in holding annual conferences all over the world; several in the universities of Heidelberg and of Greifswald (Germany), Oberwolfach Institute (Germany), Levico (Italy), Nottingham (U.K.), Marseilles (France), Universities of Columbia, N. Y., and Ohio State University (USA), Oxaca (Mexico), Chungbuk and Yeosu (South Korea), Kuantan (Malaysia) and of course, Delhi and Bangalore Centres of the I.S.I.. Of these, the one in 1990 in I.S.I. Delhi had KRP in his prime, both in academic and organizational fronts, and in all these conferences the research-talks on the HP-calculus formed a major component.

As one would expect for such a distinguished personality, KRP won many laurels and awards in his career - the Shanti Swarup Bhatnagar award (1977), the Ramanujan medal of the Indian National Science Academy (INSA, 2013) for lifetime achievement in Mathematical Sciences, TWAS award in Mathematics (1996), and the fellowships of INSA, TWAS and the Indian Academy of Sciences, Bangalore. Furthermore, he received Doctorate Honoris-causa from the Nottingham Trent University, UK and from the I.S.I..

I came to the Delhi Centre of I.S.I. in late 1978 and till mid–1985, was mostly occupied with pursuing my earlier interests in "Spectral Theory of Schrödinger Operators", though I had already collaborated with KRP in three articles, in one of which entitled "A Random Trotter-Kato Product Formula" (in Statistics and Probability, Essays in honor of C.R. Rao, North-Holland, 1982), the question of unitary stochastic evolutions was discussed with only classical Brownian 'noise'. Our offices were essentially next door to each other and KRP would often drop in to pose one problem after another - his enthusiasm was infectious and I was soon drawn into a dance of ideas - computations - discussions, leading to many jointly authored articles of which I shall mention only two. The earlier attempts by Hudson and Lindsay for a martingale representation theorem in non-Fock case (in which the conservation process does not appear) led KRP and me (Jour. Funct. Anal. 67, 1986) to investigate a general class of bounded regular martingales in Fock space which admits quantum-stochastic integral representation in terms of all three processes, viz. annihilation, creation and conservation processes. Another idea of Hudson to characterize a "quantum stop time" as a non-negative self-adjoint operator in Fock space whose spectral family $\{S(t)|\ t \geq 0\}$ is adapted with respect to the Fock space tensor-filtration, led KRP and me (Prob. Theory and Related Field, 75, 1987) to create a theory of stopped Weyl processes and prove the strong Markov property of the Fock space. By the end of the millennium, my interests moved more towards non-commutative geometry and my move to Kolkata and then to Bangalore reduced my collaboration with KRP though we would often discuss over phone and exchange newly written articles.

After more than two decades of development in the stochastic calculus, there was a bit of lull in this area. At the turn of the new millennium the restless creativity of KRP made him turn to his existing knowledge of classical information theory and use it in the newly emerging fields of the quantum information theory. As was often the case with KRP, after he (by himself alone or along with few others) has established a substantial body of mathematical knowledge in an area, he would write a book on the subject and this new area was no exception. His book "Coding Theorems of Classical and Quantum Information Theory" (Hindustan Book Agency, 2013) was written in his inimitable style.

For more than 2 decades, KRP and I lived in the campus of I.S.I. Delhi as (vertical) neighbours and I had seen at first hand his ideal of "simple living and high thinking", his sense of discipline, and his dedication to Mathematics. He was also a scholar - would often recite Sanskrit verses, learnt in childhood. KRP had an impish sense of humour as well - at the 'drop of a hat', he could reproduce the speeches of P.C. Mahalanobis and of some of the political leaders of erstwhile Soviet union, complete with appropriate inflexions!

In the last five years or so, KRP started losing effectively his eyesight and yet, true to his nature, took up detailed study and analysis of the well-investigated area of quantum Gaussian

states. But his 'large-canvas vision' brought a great sense of clarity and completeness to this whole area, leading to several beautiful articles (in collaboration with much younger collaborators).

I guess it is fair to say that KRP could achieve so much not only because of his intellectual heights, but also because of the immense, patient support he received generously from his better half, Shyama. They had two sons Ramesh and Harish; and we have fond memories of many get-togethers and of the wonderful musical renderings by Ramesh in the Indian Flute.

Unlike many other mathematicians-some ex-colleagues of mine from the I.S.I., some my professional friends-KRP's breadth of mathematical interest was staggeringly large and that suited me very well. In my view, he was temperamentally a bit of a mathematical physicist like me - impatient in solving a mathematical problem before jumping onto another one, both possibly having origin in some distant physics question and then bringing to bear on it whatever "imagination and mathematical artillery" that may be needed. It is a unique privilege - for me to have known and worked closely with such a mathematical giant, and for the generations of Indian mathematical community to have him walk amongst them.

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6. A Tribute to Dr Mangala Narlikar

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We are sorry to mark the passing away of Dr. Mangala Narlikar on 17^{th} July 2023 in Pune, at the age of 80 years, following a battle with Cancer.

Born on 17^{th} May 1943, Mangala Narlikar did her M.A. in Mathematics from the University of Mumbai in 1964, securing 1st Rank and earning Chancellor's Gold medal. Following that she joined the Tata Institute for Fundamental Research (TIFR), Mumbai and pursued research in mathematics. In 1966 she married the well-known Cosmologist Prof. Jayant Naralikar, who later founded IUCAA. This led in particular to her moving to Cambridge, England, where she was during 1966 -72. While in Cambridge she actively engaged herself in teaching of mathematics.

Returning to Mumbai with her husband in 1972, she resumed research in mathematics and did her Ph.D. in 1981, under the guidance of Prof. K. Ramachandra, TIFR, in Analytic Number Theory. After Ph.D. she continued at TIFR as Pool Officer, from 1982 to 1985, during which she also taught advanced level courses at the University of Mumbai, alongside pursuing research in mathematics. Unfortunately, around then she had her first tryst with cancer. Notwithstanding the debilitating effects of the disease, she continued valiantly to be an active academic and in particular continued her research, resulting in publications in international journals. With the founding of IUCAA, the Narlikars moved to Pune, where they were to settle. At Pune, during 1989-2010 she taught from time-to-time MSc courses such as Advanced Calculus, Complex Analysis, in University of Pune, Bhaskaracharya Pratishthana and Abeda Inamdar College. Alongside, she also got interested in mathematics education in schools, an area in which she made stellar contributions in the last decades, especially through her work for Bal Bharati in the text book committee as member and Chairperson. She was very active in community work for the underprivileged, including mathematics coaching for school girls in slum areas in Pune and has written several books facilitating better understanding of mathematics at the school level. She also contributed several articles on mathematics in Science Age, to create interest in mathematics among lay people.

With her combination of human and intellectual qualities, she was an inspiring figure for fellow colleagues and students. As a trustee of Bhaskaracharya Pratishthana, she supported many activities of Bhaskaracharya Pratishthana, such as Bhaskaracharya Talent Search Competition (BMTSC) at the High School Level for the last several years. The teachers and students of mathematics will no doubt sorely miss her in the years to come.

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7. Blaise Pascal - Legacy of 400 Years

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This year '2023' marks the 400^{th} Birth Anniversary of one of the most influential mathematicians in the history - Blaise Pascal. Blaise Pascal was born on June 19, 1623 at Clermont in Auvergne. His father Etienne Pascal was president of the court of aids at Clermont. His mother, Antoinette Begone died when Pascal was four years old. He had two sisters Madame Perier and Jaqueline.

At the age of seven Pascal moved from Clermont to Paris with his father and sisters. Étienne, who was respected as a mathematician, devoted himself henceforth to the education of his children. At a tender age of 14, Pascal began participating in the meetings of a mathematical academy in Paris. He learned different languages from his father, Latin and Greek in particular. Blaise was very brilliant but having a weak physique. Because of this reason, his father prohibited Pascal from studying mathematics which needs more effort and strain. But this aroused in him curiosity towards mathematics.

He began to study geometry by himself and went on to experiment with geometrical figures. He proved that the sum of angles of a triangle is a straight angle. Realizing the talent of his son, Pascal senior gave him a copy of *Euclid's Elements*. Before the age of sixteen, Pascal proved theorems in geometry. A special form of the theorem can be described as follows: Let P, Q, R be any three distinct points on a line L and P', Q', R' be three distinct points on another line L'. Join P to Q' and P' to Q, Q to R' and Q to R' and finally R to P' and R' to P. Then the two lines in each of the pairs intersect and the points of intersection lie on a straight line.

Another important contribution of Pascal to geometry is related to conic sections. Consider any conic section, say for definiteness, ellipse. Mark on it six points A, B, C, D, E, F and join them by straight lines. Then the pairs of sides AB and DE, BC and EF, and CD and FA are pairs of opposite sides. Extending the opposite sides, they will intersect at a point and there are three such points G, H, K. These three points lie on a straight line (See Figure 1). This is called Pascal's theorem and the hexagon ADBFCE is called a mystic hexagon. In 1640, at the age of 16 he wrote an important short work on projective geometry, "Essay on Conics".

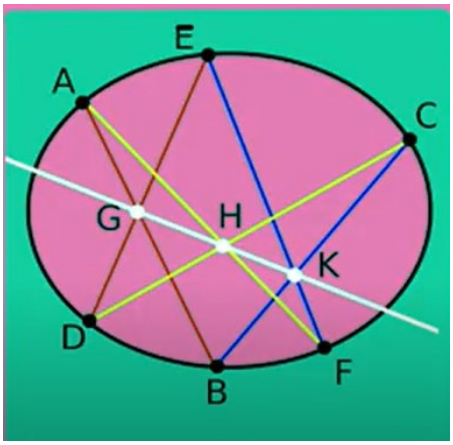

Figure 1: Pascal's theorem

Figure 2: Pascaline

The young man's work, which was highly successful in the world of mathematics, aroused the envy of no less a personage than the great French Rationalist and mathematician René Descartes, who was twenty seven years older than Blaise.

Between 1642 and 1644, Pascal conceived and constructed a calculating device, the Pascaline, to help his father, who was then a tax collector for the French city of Rouen, in his tax computations. This mechanical calculator was able to add and subtract and used a numerical wheel with movable dials. Pascal continued to refine and improve his calculator for ten years, although his calculator was not a commercial success. However, The computer language Pascal is named after him in recognition of his early computing machine.

During the 1640's Pascal contributed to hydrostatics. He tested the theories of Galileo and Evangelista Torricelli (an Italian physicist who discovered the principle of the barometer). To do so, he reproduced and amplified experiments on atmospheric pressure by constructing mercury barometers and measuring air pressure, both in Paris and on the top of a mountain overlooking Clermont-Ferrand. In the course of his experiments, he invented the syringe and the hydraulic press (see Figures 3 and 4) an instrument based upon the principle that became known as Pascal's principle (Pascal's law) which states that a pressure change at any point in a confined incompressible fluid is transmitted throughout the fluid such that the same change occurs everywhere. His publications on the problem of the vaccum (1647-48) added his reputation. Thus Pascal had significant contribution to physics in the field of fluid mechanics and pressure. In honour of his contribution to physics, a name *Pascal* has been given to SI unit of pressure.

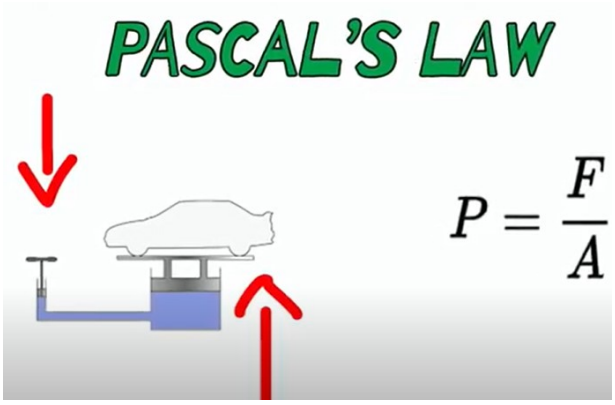

Figure 3: Hydraulic press

Figure 4

Pascal wrote Traité du triangle arithmétique in 1654, which was was published posthumously in 1665, in which he constructed 'arithmetic triangle', now famously known as Pascal's triangle (see Figure 5). This triangle he used in problems of combinatorial analysis and the theory of probability. In the triangle, the numbers in any row after the first two are obtained by adding two numbers directly above it. The number in the n^{th} row represents the coefficients of the binomial expansion of $(a+b)^n$. The number of n things taken r at a time which Pascal correctly stated to be $n!/r!(n-r)!$ in the present notation. The notation $n!$ was introduced in 1808 by Christian Kramp of Strasbourg.

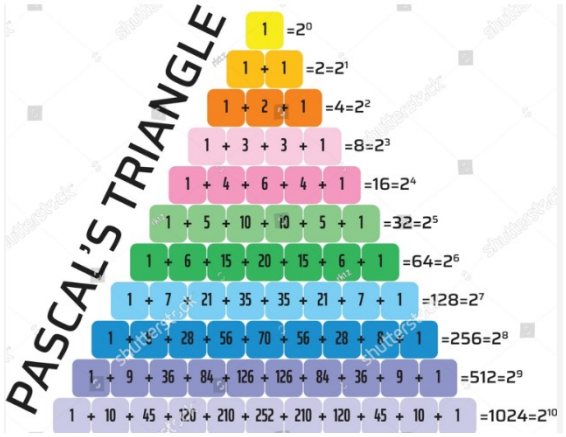

Figure 5

The mathematical theory of probability was founded by Pascal and Fermat. Originally it was applied to gambling problems. Suppose two players want to finish a game early, given the current circumstances of the game, they want to divide the stake fairly based on the chance of winning the game from that point. The notion of expected value was introduced from this discussion.

Pascal's last mathematical work was on cycloid. This curve is formed by a point on the circumference of a circle while it is rolling along a straight line. In his effort to create a perpetual motion

machine, Pascal inadvertently devised the now well-known gambling device, the roulette wheel. Later in life, he wrote his two most famous works, the "Lettres provinciales" and the "Pensées", showing his religious views. They remain Pascal's best-known legacy, and he is usually remembered today as one of the most important authors of the French Classical Period and one of the greatest masters of French prose, much more than for his contributions to mathematics.
Throughout his life, Pascal was in weak health, especially after the age of 18. On 19 August 1662, Pascal went into convulsions and died. His last words being 'May god never abandon me'. He was buried in the cemetery of Saint-Etienne-du-Mont in Paris.

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International Mathematical Olympiad 2023
India Secures 9^{th} Spot with Six Medals

Six-member Indian team won six medals - two gold, two silver, and two bronze at the 64^{th} International Mathematical Olympiad (IMO) 2023, which was held in Japan's Chiba city from July 2 to July 13. The team finished ninth out of the 112 countries that competed. The gold medal was won by Atul Shatavart Nadig and Arjun Gupta, who both scored 37 points. Ananda Bhaduri and Siddharth Choppara grabbed silver with 29 points apiece, while Adhitya Mangudy and Archit Manas won bronze with 22 and 20 points, respectively.

This is the fourth time the Indian team has finished in the top 10. According to reports, India has finished seventh in 1998, seventh again in 2001, and ninth in 2002.

The Indian team was joined by Prithwijit De of the Homi Bhabha Centre for Science Education in Mumbai, Sahil Mhaskar of the Chennai Mathematical Institute, Anant Mudgal of the University of California in the United States, and Pranjal Srivastava of the Massachusetts Institute of Technology (MIT) in the United States.

Sahil represented India at IMO 2005, whereas Anant represented India at IMO 2015, 2016, 2017, and 2018. While Pranjal attended the IMO 2018, IMO 2019, IMO 2021, and IMO 2022. The Homi Bhabha Centre for Science Education (HBCSE)-TIFR trains and selects students for the Olympiads.

CONGRATULATIONS TO THE INDIAN TEAM

8. Problem Corner

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In the April 2023 issue of TMC Bulletin, we posed a problem from Number Theory for our readers. We have received two solutions to that problem, one from Shrestha Suraiya, a student from Dhirubhai Ambani International School, Grade 10, Mumbai is published here, the solution by the problem proposer Prof. Rajendra Pawale, Mumbai University is also presented below.
Also, in this issue we pose a problem from Geometry for our readers. Readers are invited to email their solutions to Udayan Prajapati (Udayan.prajapati@gmail.com), Coordinator, Problem Corner, before 15^{th} September, 2023. The most innovative solution will be published in the subsequent issue of the bulletin.
Problem posed in the previous issue (Prof. Rajendra Pawale, Mumbai University, Mumbai):
Show that $3(3+2^{2+n}+2^{2+2n})$ is a perfect square only for $n=1$.
Solution 1 by Shresth Suraiya: Suppose for contradiction that there exists an integer $n>1$ for which the expression $3(3+2^{2+n}+2^{2+2n})$ is a perfect square.

So, $3(3+2^{2+n}+2^{2+2n})=a^2$ for some positive integer a.

So, $3|a^2$. Hence $3|a$, so $a=3b$ for some positive integer b.

Therefore $3+2^{2+n}+2^{2+2n}=3b^2$. $\qquad(\star)$

Note that b has to be strictly greater than 1.

So, $2^{2+n}(1+2^n)=3(b^2-1)$, giving $2^{2+n}|3(b^2-1)$. Hence $2^{2+n}|(b-1)(b+1)$. Since $n>1$, both $b-1$ and $b+1$ are even. However, $gcd(b-1,b+1)=gcd(2,b+1)=2$, so at most one of the two can be divisible by 4. We now consider two cases.

Case 1. $4|(b-1)$.

It follows that $\frac{b+1}{2}$ is odd. So, $2^{1+n}|(b-1)\Rightarrow b=2^{1+n}q+1$ for some positive integer q.

Substituting this in $(\star)$, we get $3+2^{2+n}+2^{2+2n}=3(2^{2+2n}q^2+2^{2+n}q+1)$.

This implies $2^{2+n}(2^n+1)=3\cdot 2^{2+n}q(2^nq+1)\geq 3\cdot 2^{2+n}(2^n+1)$,

which is a contradiction.

Case 2. $4|(b+1)$.

It follows that $\frac{b-1}{2}$ is odd, so $2^{1+n}|(b+1)$. This implies $b=2^{1+n}q-1$ for some positive integer q.

Substituting this in $(\star)$, we get $3+2^{2+n}+2^{2+2n}=3(2^{2+2n}q^2-2^{2+n}q+1)$.

Hence $2^{2+n}(1+2^n)=3\cdot 2^{2+n}q(2^nq-1)\geq 3\cdot 2^{2+n}(2^n-1)$.

Therefore $1+2^n\geq 3\cdot 2^n-3$, which gives $2^n\leq 2\Rightarrow n\leq 1$,

a contradiction to $n>1$.

We conclude that there does not exist any positive integer other than $n=1$, for which $3(3+2^{2+n}+2^{2+2n})$ is a perfect square.■

Solution 2 by Prof. Rajendra Pawale:

Suppose $3(3+2^{2+n}+2^{2+2n})=a^2$ for some positive integer a.

So $a^2-(3+2^{n+1})^2=2^{2n+3}$. Thus

$\left(a-(3+2^{n+1})\right)\left(a+(3+2^{n+1})\right)=2^{2n+3}$. Here, $a>3+2^{n+1}$.

Hence $a-(3+2^{n+1})=2^{\alpha_1}$ and $a+(3+2^{n+1})=2^{\alpha_2}$, where α_1 and α_2 are positive integers such that $\alpha_2>\alpha_1>0$.

So $2^{\alpha_2}-2^{\alpha_1}=2(3+2^{n+1})$. Hence $\alpha_1\geq 1$. Hence $2^{\alpha_2-1}-2^{\alpha_1-1}=(3+2^{n+1})$.

As the right hand side is an odd integer $\alpha_1=1$.

This implies $2^{\alpha_2-1}=4+2^{n+1}$ or $2^{\alpha_2-3}=1+2^{n-1}$.

To maintain the parity on the both side $n = 1$.

Problem for this issue

Does there exist a scalene acute-angled triangle ABC with an altitude AD, an angle bisector BE and a median CF such that AD, BE and CF are concurrent?

□□□

9. International Calendar of Mathematics Events

Ramesh Kasilingam
Department of Mathematics, IITM, Chennai
Email: rameshk@iitm.ac.in

August 2023

- August 7-11, 2023, International Algebra Conference in Philippines, Mactan, Cebu, Philippines. *sites.google.com/g.msuiit.edu.ph/international-algebra-conferen/*
- August 23-27, 2023, The 6^{th} Mediterranean International Conference Of Pure & Applied Mathematics And Related Areas (MICOPAM2023), Université D'Evry Val D'Essonne, Paris, FRANCE. *micopam.com/*
- August 24-27, 2023, 5^{th} Canada-Mexico-US Conference In Representation Theory, Noncommutative Algebra, And Categorification, Centre De Recherches Mathématiques (CRM), Montreal, Canada. *www.crmath.ca/en/activities/#/type/activity/id/3879*

September 2023

- September 10-16, 2023, International Conference on Probability Theory and Number Theory 2023, Palanga, Lithuania. *https://www.numbertheory.lt*
- September 11-12, 2023, Bicocca, Milano, Italy: workshop "Geometria in Bicocca 2023", at the University of Milano-Bicocca. *https://geometryinbicocca.matapp.unimib.it*
- September 11-15, 2023, Dynamics and asymptotics in algebra and number theory, Bielefeld University, Bielefeld, Germany. *https://www.math.uni-bielefeld.de/daan23/*
- September 11-15, 2023, Automorphic Forms and L-functions of higher rank, Queen Mary University of London, UK. *https://sites.google.com/view/automorphic-conference*
- September 11-15, 2023, Algebra and Number Theory in Conversation (ANTIC), Manchester, UK. *https://sites.google.com/view/antic-manchester/*
- September 11-15, 2023, School on "K3 surfaces, hyperkähler manifolds, and cubic fourfolds", at the Hausdorff Institute of Mathematics, Bonn, Germany.
  https://www.him.uni-bonn.de/school-hyperk3s/
- September 11-15, 2023, Conference on Categorical Enumerative Geometry and Representation Theory, at EPFL, Lausanne, Switzerland.
  https://sites.google.com/view/workshopepfl/ home/
- September 18-22, 2023, Conference on "Characteristic Classes and Singular Spaces", at Kiel University, Kiel, Germany.
  https://www.math.uni-kiel.de/geometrie/de/essig/conferences /charclasses2023conf

- September 26-29, 2023, "Joy in Algebraic Geometry. A conference in honor of Rick Miranda for his 70^{th} birthday", at the Hotel, Bellavista, Levico Terme, Italy. *https://sites.google.com /view/joyinalgebraicgeometry23/home-page*

October 2023

- September 30 - October 1, 2023, 2023 Maine-Quebec Number Theory Conference in Orono, Maine, USA. *https://mainequebecnt.github.io*
- October 2-6, 2023, Conference on Arithmetic Algebraic Geometry on the occasion of Michael Rapoportâ 75 birthday, at the University of Munster, Munster, Germany. *https://www.uni-muenster.de/MathematicsMuenster/de/events/2023/arithmetic-algebraic-geometry.shtml*
- October 2-4, 2023, International Conference on Math Education and Technology (ICMET 2023), University of Aveiro, Aveiro, Portugal. *https://icmet.web.ua.pt*
- October 2-6, 2023, Workshop I: Quantum Algorithms for Scientific Computation, Institute for Pure and Applied Mathematics (IPAM), Los Angeles, CA. *www.ipam.ucla.edu/programs/workshops/workshop-i-quantum-algorithms-for- scientific-computation/*
- October 7-8, 2023, 2023 AMS Fall Central Sectional Meeting, Omaha, NE Creighton University, Omaha, NE. *https://www.ams.org/meetings/sectional/2307_program.html*
- October 11-12, 2023, The 2^{nd} International Conference on The Evolution of Contemporary Mathematics and their Impact in Science and Technology (ECMI-SciTech 2023), Brothers Mentouri University of Constantine, Algeria. *http://ecmi2023.ammlabo.net/call_for_papers. html*
- October 9-13, 2023, Workshop on "Vector bundles and combinatorial algebraic geometry", at Goethe University, Frankfurt, Germany. *https://sites.google.com/view/vectorbundlescomba lggeo/home*
- October 13-15, 2023, TORA (Texas-Oklahoma Representations and Automorphic forms) XII, University of Oklahoma, Norman, Oklahoma USA. *https://math.ou.edu/ kmartin/toraxii*
- Octrober 23-25, 2023, Bandoleros 2023, 7^{th} Algebraic Geometry meeting, Department of Mathematical Sciences of Politecnico di Torino, Italy. *https://sites.google.com/view/bandoler os2023/home-page*
- October 25-27, 2023, Spring School On Symmetries of Differential & Difference Equations and Their Applications, Stellenbosch, South Africa. *https://sites.google.com/view/spring-school-stellenbosch/home*
- October 30-November 3, 2023, Seoul, South Korea: Conference on "Fano varieties, their Geometry and Moduli", at Korea Institute for Advanced Study. *http://events.kias.re.kr/h/FVGM/ ?pageNo=5054*
- October 30-November 2, 2023, "Research School in Real Algebraic Geometry", at SISSA and ICTP, Trieste, Italy. *https://sites.google.com/view/rsrag/home*

□□□

Institutional Members
Of
The Mathematics Consortium as on 1st July, 2023

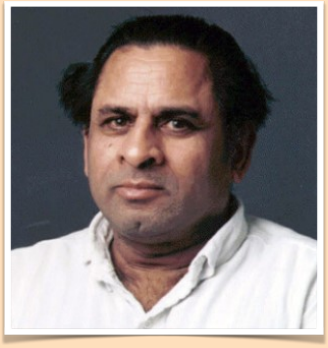
1. Bhaskaracharya Pratishthana, Pune
2. Jawaharlal Nehru University, New Delhi
3. Nowrosjee Wadia College, Pune
4. Dayalbagh Educational Institute, Agra
5. Atal Bihari Vajpayee Indian Inst. of Information Tech. and Management (ABV-IIITM, Gwalior)
6. Visvesvaraya National Institute of Technology, Nagpur
7. Ramakrishna Mission Vivekananda Centenary College, Rahara, Khardaha (W.B.)
8. Lal Bahadur Shastri Institute of Management, Delhi
9. National Institute of Technology, Kurukshetra
10. Modern College, Shivajinagar, Pune
11. School of Mathematics, Devi Ahilya University, Indore
12. Central University of Punjab, Bathinda
13. Indian Institute of Science Education and Research, Pune
14. Savitribai Phule Pune University, Pune
15. Mathematical Sciences Institute, Belgavi
16. Behala College, Kolkata
17. Mehsana Urban Institute of Sciences, Ganpat University, Kherva, Gujarat

Society Members
Of
The Mathematics Consortium as on 1st July, 2023

1. Ramanujan Mathematical Society, Tiruchirappalli
2. Academy of Discrete Mathematics and Applications (ADMA)
3. Vijnana Parishad of India, Allahabad, Orai, UP
4. Society for Special Functions & Their Applications (SSFA), Aligarh
5. Gwalior Academy of Mathematical Sciences, Gwalior
6. Shivaji University Mathematics Society, Kolhapur
7. Kashmir Mathematical Society, Srinagar
8. Ramanujan Society of Mathematics and Mathematical Sciences, Jaunpur
9. Indian Academy of Industrial & Applicable Mathematics (IAIAM), Pune
10. Jharkhand Society of Mathematical Sciences, Ranchi
11. Forum for Interdisciplinary Mathematics, Meerut
12. Calcutta Mathematical Society, Calcutta
13. Nepal Mathematical Society, Kathmandu, Nepal
14. Gujarat Ganit Mandal, Ahmedabad
15. Marathwada Mathematical Society, Chhatrapati Sambhaji Nagar (Aurangabad), Maharashtra

Financial Support for Mathematical Activities

Society members and Institutional members of TMC are encouraged to organize Mathematical events of short duration in collaboration with TMC such as workshops, conferences, lecture series, TMC Lectures etc., for which a moderate financial support will be given by TMC in terms of honorarium to experts or for some specific purpose as per availability of funds.

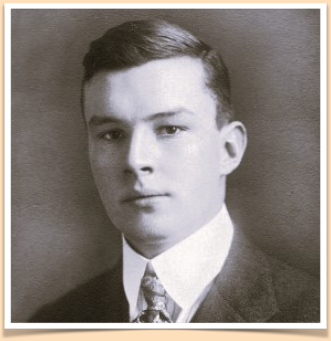


Shreeram Shankar Abhyankar (22 July 1930 - 02 Nov. 2012)

Indian American Mathematician. Known for his contributions to Algebraic Geometry and a conjecture in finite group theory, known after him. He also, worked on: Commutative algebra, local algebra, valuation theory, the Cayley-Hamilton theorem theory of functions of several complex variables, quantum Electrodynamics, circuit theory, invariant theory, combinatorics and robotics.

Sir William Rowan Hamilton (04 Aug. 1805 - 02 Sep. 1865)

An Irish mathematician and Royal Astronomer of Ireland. Worked in both pure mathematics and mathematics for physics. He made important contributions to optics, classical mechanics and algebra. Known for his reformulation of Newtonian mechanics, now called Hamiltonian mechanics. He is best known as the inventor of quaternions and the Cayley-Hamilton.



Haskell Brooks Curry (12 Sep. 1900 - 01 Sep. 1982)

An American mathematician and logician. Best known for his work in combinatory logic bringing much of the development. Also known for Curry's paradox and the Curry-Howard correspondence, three programming languages named after him, Haskell, Brook & Curry, the concept of currying (a technique for transforming functions in mathematics and computer science).

Publisher

The Mathematics Consortium (India),
(Reg. no. MAHA/562/2016 /Pune dated 01/04/2016),
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AUM Copy Point, G-26, Saffron Complex, Nr. Maharana Pratap Chowk,
Fatehgunj, Vadodara-390001; Phone: 0265 2786005;
Email: aumcopypoint.ap@gmail.com

Annual Subscription for 4 issues (Hard copies)

Individual	: TMC members: Rs. 700;	Others: Rs. 1000.
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The amount should be deposited to the account of
"The Mathematics Consortium", Kotak Mahindra Bank, East Street Branch,
Pune, Maharashtra 411001, INDIA.

Account Number: 9412331450, IFSC Code: KKBK0000721