The Mathematics Consortium

BULLETIN

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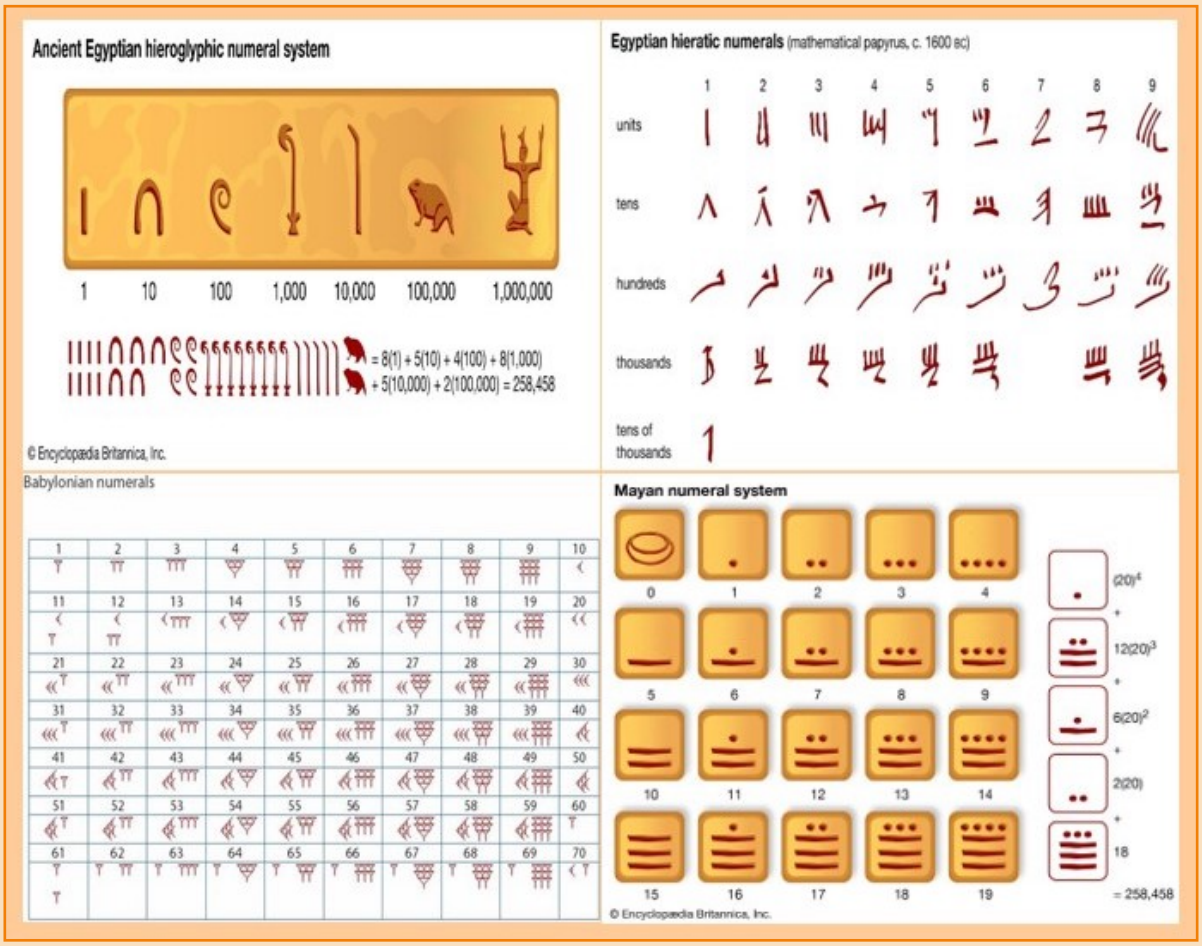
VARIOUS NUMERAL SYSTEMS



Chief Editor: Ravindra S. Kulkarni

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The Mathematics Consortium

Bulletin

April 2023 Vol. 4, Issue 4

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About the Cover Page: The figure on the front cover includes four different number representation systems which were adopted in the different parts of the world at different periods before the spread of the Decimal Place-Value system. The images involved are taken from Encyclopaedia Britannica Inc.

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From the Editors' Desk

India's last Education Policy was passed and implemented in 1986. After thirty-four years, Ministry of Education has announced the new National Education Policy 2020 (NEP 2020) on 29th July, 2020. This New Policy was recently approved by the Indian government in 2023. And now educational institutions across India are preparing to implement NEP-2020.

The 'Gurukul' was the first education system in India in the ancient times. In the modern context a school system was introduced in India in 1830 by Lord Thomas Macaulay, which introduced classroom learning, examinations. The system continued in independent India. The first significant educational reforms were done by the Kothari Commission in 1966, by introducing the structure of the $10+2$ schooling and $3+2$ higher education leading to graduate and postgraduate degrees. Since then, India has seen a series of reforms like NEP 1968, 1986, POA 1992, RTE 2009, and NCF 2005 which restructured our education system to a great extent but still followed the same structure.

The motto of NEP 2020 is to educate, encourage and enlighten, and is based on four pillars Access, Equity, Quality, and accountability. The major reforms of the policy include (1) subdivision of schooling of students into four stages namely, 5 years Foundational stage (3 years of pre-schooling + class 1-2) + 3 years Preparatory stage (class 3-5) + 3 years Middle stage (class 6-8) + 4 years Secondary stage (class 9-12). (2) The exams for students will be held only in classes 2nd, 5th, and 8th instead of every academic year. (3) Proposal of a four-year multi-disciplinary, holistic bachelor's degree program, with multiple exits. (4) PG degrees will be limited to 1 to 2 years and not mandatory for joining Ph.D. program and in particular, the M.Phil. program is abolished. (5) A digital storage of credits earned by the students will be maintained which can be utilized for further education across the institutions.

The NEP 2020 has many useful features. However, there are many challenges in implementing the Policy in its true spirit. Major challenges in higher education sector will be: Designing Imaginative and flexible curricular structures; Developing educational approaches in UG education that integrate the humanities and arts with Science, Technology, Engineering and Mathematics; Orient the Graduate-level, master's and doctoral education in the universities, so as to provide rigorous research-based specialization, and provide opportunities for multidisciplinary work, including in academia, government, industry and the service sector. All the stake holders in the education sector will have to strive hard to achieve the goals set forth by the NEP 2020. We at TMC will also be eager to contribute a bit in this process.

Article 1 is the second part of multi-article series initiated by Professor S. G. Dani in the previous issue of TMCB. In part 1 the Vedic roots of the decimal place-value systems were discussed and in the second part he demystifies the major steps involved in moving from those "roots" to the mature decimal representation system.

In the second Article Prof. Mohan Joshi illustrates how different ramifications of best approximates which appear in several areas of science and engineering and thereby highlights the linkage between theory and practice.

In Article 3, Dr. D. V. Shah gives an account of significant developments in the Mathematical world during recent past, including brief write-ups on the winners of the 2023 Wolf Prize, and Infosys prize 2022 and Tributes to Mathematicians who passed away during the last quarter. Prof. Vijayakumar Ambat writes about his reminiscences of Prof. Vera T Sós in Article 4.

In the Problem Corner, Dr. Udayan Prajapati presents two solutions to the problem posed in the January 2023 issue. These solutions are given by a student Shrestha Suraiya, and by Prof. J. N. Salunke. A problem on Number theory is also posed for our readers. Dr. Ramesh Kasilingam gives a calendar of Academic events, planned during May, 2023 to July, 2023 in Article 6.

We are happy to bring out the fourth and final issue of Volume 4 in April 2023. We thank all the authors, all the editors, our designers Mrs. Prajkta Holkar and Dr. R. D. Holkar, and all those who have directly or indirectly helped us in bringing out this issue on time.

Chief Editor, TMC Bulletin

1. The Decimal Place-value System Part II : Advancing to Maturity

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We saw in the first part of this series that the Vedic literature manifests deep preoccupation with large numbers and the various choices made with regard to setting the pattern for number recognition were in consonance with our present system of number representation. Nevertheless, as was briefly mentioned at the end of the last part, moving on from those "roots" to the mature decimal representation system involved some major steps, and several centuries were to pass before they were taken. In hindsight some of the factors responsible for this seem to be the following. As we recalled earlier, the Vedic culture was a largely oral, and though large numbers were conceived, communication relating to them was in terms of their names, in spoken or recited words, and not through symbolic or written representation. Among the immediate consequences of this the following may be especially noted. The composite numbers, involving multiple decuple terms, were expressed by describing the sizes of the decuple terms in flexible order, rather than in the decreasing order as we would do today (in any systematic treatment), a feature that is intricately linked with the place value systems. Sometimes this could be for metrical reasons in presenting the words involved, or perhaps just convenience in pronouncing the string of words to be presented[1], but at other times it could simply be because of not associating much significance to the heirarchical role of the individual decuple terms. Many of the large numbers found from that period turn out to be creative outpourings, unrelated to practical contexts that typically instill a fuller grasp of the number. Another consequence is that the role played by 0 in the written form of decimal representation was absent in their context; thus if there were no bundles corresponding to a decuple size, one would just drop the mention of it, like in saying "three thousand and eight", involving no reference to hundreds and tens. Indeed there is no evidence from the vedic corpus for a mathematical notion of zero - it was not needed (cf. [25])[2]. Thus in effect the complete development of the decimal place value system had to await the stage when writing them became common, and for it to incorporate the wisdom accumulated through the earlier phase, which, it would seem, lived in different quarters of the rather stratified society.

1.1 The Role of Written Numbers

There is a discussion in [6] on some small digits having had a symbolic representation during the Vedic period. A hymn in Ṛgveda mentions "*sahasraṃ me dadato aṣṭakarṇyaḥ*"[3] which may be translated as "gave me thousand *aṣṭakarṇyaḥ* [cows]", ([6], I, p. 18) and the term *aṣṭakarṇyaḥ* has been interpreted to mean cows with their ears marked with the digit eight (viz. with symbol used for it) ([6], I, p.18) - according to *Sāyaṇabhāṣya*, a much-cited commentary on Vedic texts by the acclaimed 14th century commentator Sāyaṇācāraya, however, in the context of the hymn *aṣṭa* does not mean eight, but rather it stands for the adjective *vistṛta* (wide), and the correct meaning of the part is "gave me a thousand robust cows"; (see [24], p. 493, or [16])[4]. The authors of [6] also recall

[1] It may be recalled here that in Sanskrit the order of words in a sentence is more flexible than in most contemporary languages, which may also have bearing on this phenomenon.

[2] It is noted in [25] that there was no term for zero in the Vedic oral tradition, and that the word *kha* which later came to be used for denoting zero occurs in Vedic literature only in the senses of "hole", "opening", "vacancy" or "space".

[3] For convenience, for technical terms we adopt the style of academic literature; this involves in particular some diacritical marks which a casual reader may ignore, and also a convention by which "ch" in common usage is represented only by "c" - thus in particular *candra* stands for *chandra* in normal usage.

[4] As explained by Sāyaṇācāraya, in the hymn the word *aṣṭa* has its root, etymologically, in the verb *aś*, which means "to pervade" or "to fill completely", and *aṣṭakarṇyaḥ* corresponds to "with wide ears", and the latter is a suggestive

an instance of a gambler lamenting “having staked on one”, with “one”referring to the number stamped on the dice. On the whole, however, the evidence in respect of numbers being written is rather thin, and whatever the status in the matter, it may not have made a serious difference towards adoption of symbols, when the practice of writing emerged.

The substantive and reliable history of written numbers, as would be relevant to our context here, starts only with the inscriptions of emperor Aśoka (ca. 300 BCE). By that time writing seems to have become a well-set mode of communication, especially in practical contexts such as indicating distances with markings on (equivalent of) milestones that have been mentioned, as also in accounting systems as discussed in Kauṭilya’s *Arthaśāstra* etc.(see [6], p.21-footnote). Little specific detail about the mode of writing numbers can be garnered from these other sources however, as the information about them is indirect. The inscriptions of Aśoka, and especially those from the succeding period of the Sakas, Pārthians, and Kuṣānas, from around the 1st century BCE thus form our main source on the issue.

Most of these inscriptions are in a script called *Brāhmī*, while some others follow another script known as *Kharoṣṭhī*. The latter have been found mostly in the Gāndhāra region (which was situated in eastern part of the present Afghanistan and northern Punjab) where it was apparently a popular script used by clerks and businessmen (cf. [6], I, p. 21). On the other hand the Brāhmī script was very widespread across the country, and inscriptions in Brāhmī are found distributed all over India. While the language and the script are argued to be much older [6], I, p.25), the epigraphic evidence goes only as far back as the Aśokan period. The period of the inscriptions is seen to extend upto about the 4th century CE.

Both the systems of number representation are seen to have witnessed substantial evolutionary changes over the period from the 3rd century BCE until about the 2nd century CE, while retaining the underlying structure.

In the Kharoṣṭhī inscriptions of Aśoka only four numerals are found, corresponding to 1, 2, 4 and 5. Figure 1 shows some of the number representations from the later evolved phase (see[6]). It should be borne in mind here that the script was written from right to left.

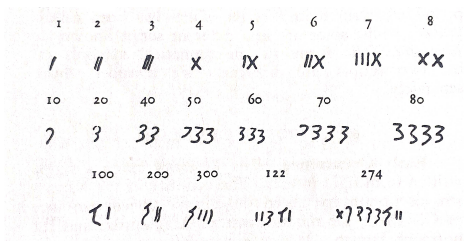


Figure 1: Kharośṭhī numerals

It may be observed that the numeration was made up of the following: The numbers upto 99 are represented via basic symbols for 1, 4, 10, 20, used *additive* convention (just as in the Roman numerals that are still used in certain contexts); for numbers upto 3 the stroke for one is repeated, the numbers 5 to 7 are represented with the symbol for 4, with additional strokes *on the left* for the remaining part to be added, and so on - the multiples of 20 are obtained by repeating the symbol for 20 the appropriate number of times; interestingly the symbol for 20 is a (cursive) double of that for 10, with the two joined together, but the same principle is not extended further on to 3, though it may not be so impractical. For the numbers 100 and above, yet another principle is brought in, using an independent notation in the form of a special symbol (seen in the figure at the beginning of the last line) with the convention that the number to the *right* of it is to be thought of as being hundred-fold. The number 274 appearing at the end of the list is conveyed as $4 + (10) + (20) + (20) + (20) + (\mathrm{H2})$, with the last H2 (which corresponds to 2 H in left to

description (*upalakṣaṇaṃ*) meant to convey “robust” cows; the author would like to thank Prof. Shailaja Katre, Pune, for help in following the meaning of Sāyaṇācārya’s comment on the hymn.

right writing as we practice) standing for 2 hundred, following their convention. It may be noted that though 10 has a special role in the notation, there is no "place value" accorded to it, and the multiples of it are to be represented by putting down the requisite number of symbols for 20 and an additional one for 10 if needed.

The Brāhmī numeral system had individual signs, as seen in Figure 2, for the digits, multiples of 10, multiples of 100, and multiples of 1000, and values corresponding to various signs written in juxtaposition were meant to be added; thus the number 274 would be represented by setting down the symbol for 200, followed by the one for 70 and followed by that for 4, on the *right* in each case - unlike Kharoṣṭhī the Brāhmī script was written from left to right; what brought such a shift is completely unclear.

Figure 2: Brāhmī numerals of the second century

Thus we see that while there are differences in detail between the two systems they both rely on adding values of symbols and do not use the place value technique. The efficiency in representing numbers is sought to be increased, as one can see from the pattern in evolution, by having more symbols.

We may now tear our hair over why they did not think of twenty as two tens, thirty as three tens etc., and just write them in the next place as we do. A reason is likely to be the following. For the people who wrote them, the numbers had their "identity" as twenty, thirty etc., aside from being two or threes of tens - it may be likened to there being currency notes for 20 and 50 rupees. And there were not too many to remember after all - *their* number world, presumably circumscribed by practical contexts, did not extend, even with the increasing needs that must have arisen over time, farther than they could accommodate by introducing additional symbols, or combinations of the symbols, for them.

It may be noted however that when it came to hundreds, and thousands, in the developed phase of Brāhmī, the individual symbols for the multiples are no longer entirely distinct independent symbols for each. Rather, they involve basic symbols for 100 and 1000 and the multiples are indicated by added features corresponding to the digits involved; thus 200 has one additional horizontal stroke attached to the symbol for the hundred, 300 has two of them and the later multiples also involve hooking to the symbol for 100 an extra feature corresponding to the digit - interestingly the added features in the case of multiples of 1000 are the same strokes as in the corresponding multiples of 100 for lower multiples and are similar for multiples by 5. At this stage it is close to say writing the number 3576, as somewhat like T3H5[76], with T a symbol for thousand, H a symbol for hundred, each with the corresponding digit hooked to it, but with the remainder consisting of a number less than 100 being left in the earlier familiar mode (in terms of the 20s, 10s, 4s and 1s, following additive convention). It may be borne in mind that in our present usages also, in verbal communication the numbers upto hundred have their own individual identity as a pair.[5]

One may surmise that if they were to move on to higher powers of 10, from 100 and 1000, they would have extended the same principle, and also eventually realized the futility of having

[5]This aspect is even more prominent in Sanskrit than in English, especially on account of the order in which the pair of digits appear in the verbal rendering.

the individual symbols appear each time, as they were always being written in the same order![6] The transition to the place-value system, however, seems to have taken place sooner than may be expected through such a process. The presence of people around who were already thinking and talking of large numbers would have made a difference in this respect, expediting the emergence of the decimal place-value system, an all-encompassing standard system of representation of numbers.

1.2 Minding the Zero

As noted earlier, in representing numbers in the place-value systems in written form, "zero" is an important ingredient; what is involved in this respect from a mathematical point of view, as may be seen from the description at the beginning of Part I of this article, is how to keep track of the situations when there may be no remainder at one of the stages (including possibly the first one, but not the last) of grouping iterations, and how to deal with them in the course of the arithmetic to be carried out in various contexts. The old Babylonians of the second millenium BCE, with their sexagesimal place-value system, initially managed with blank spaces for them. Usage of a sign for 0 is found from around 500 BCE. The Babylonians of the Seleucid period (312 BCE - 64 BCE) were using multiple signs for zero, and though there was no standardization of notation, from a mathematical point of view there was a fully functional concept of zero, in their sexagesimal place-value system (see [7], p. 49).

In India "*śūnya*", the term that came to stand for the zero when the decimal place-value representation of numbers took shape, is known to have emerged earlier in the studies on philosophy (in terms of contemplation of "void" or "nothingness" and more specifically in the work of the renowned grammarian Paṇinī from around the 5th century BCE (as *lopa*)[7], and it seems plausible that this fact may have facilitated conceiving the mathematical zero, when the need for it arose in the course of representation of numbers in written form (see, in particular, [25]). Nothing further need be read into a connection with these topics, with regard to the development of zero as a mathematical concept.[8] Also, not much can reliably be said concerning the specific details of the genesis of adoption of the term for the mathematical zero, or for the choice of the notation for it. It may be noted however that the original symbol for zero was a dot, which later acquired the shape of a ring, originally of a size smaller than the other characters in the text, turning later into the regular- sized zero 0 (of the same size as the other characters).

1.3 The Early Hints

The issue as to when the place value system for representation of numbers would have come to be practised is difficult to ascertain, even within a margin of a century or so. Earlier it was believed (cf. [18], p. 61 and [21], p. 5), based on a work of David Pingree, that representation of numbers through *bhūtasaṅkhyā* is found in *Yavana-jātaka* ("Greek horoscopy") of Sphujidhvaja, and that an expression in its final verse is a *bhūtasaṅkhyā* rendering of the śaka year 191, which corresponds to 269 or 270 CE, as the year of composition of the work. However, the interpretation of the phrase in question as the number 191, and hence also the inference that it is the śaka year of its

[6]There is still the issue of the numbers upto 100 being written in a different way, but once they were to choose to drop the notation for a hundred, thousand, etc. (H,T as above), gradually the "tens" would have fallen in line!

[7]The term *śunyā* also appears in Piṅgala's *Chandaḥ sūtra* (from the second century BCE), but the reference is an isolated one and it is not clear whether it corresponds to zero in decimal representation of numbers, and whether it had a symbol - it may, for instance, refer to a blank space (see [7], p. 56), or a token marking an operation.

[8]There has been a huge discourse on "zero" and its history, dominated mostly to philosophical and grammatical issues that are peripheral to the mathematical context of the concept and execution of operations with it. While the general theory no doubt has its relevance, unwarranted references to it (in deep voices!) often seem to obfuscate discussions from the point of view of history of mathematics.

composition, have recently, in [15], been comprehensively argued to be untenable; the conclusion has been cited also in [19] (p. 143)(conveying endorsement).

There are nevertheless various indications to the effect that the usage may have begun sometime in the first half of the first millennium. Firstly it may be recalled that ancient Jaina scriptures involve calculations featuring very large numbers and their reckoning, like in the Vedic tradition was in terms of powers of 10, some of which go back to the millennium BCE (see [8] and [12], and other references there). However, prior to the fifth century this activity was in oral tradition, for which as we noted before the place-value system does not hold much relevance. To preserve the originality of the scriptures the Jaina *acāryas* organised Councils from time to time. Five Councils were held, starting with the first one in about 300 BCE. It was at the 5th (the last) of these Councils, held in 454 at Valabhi (in Gujarat), the Āgamas were edited and compiled, and were given some sort of book form (see [8], p. 3). Given the overall context, these written versions would in all probability have involved the decimal palce-value system, but this is difficult to ascertain since our access to their texts is only through commentaries from a later period, in which a change of format can not be ruled out. On the whole, there is not enough clarity on these historical aspects in respect of the Jaina works, and while it seems likely that they played an important role in the development of the decimal place-value system (and use of zero) any definitive statement about it would depend on further exploration.[9]

The possibility of prevalence of the decimal place-value system in the fifth century is also supported by indirect references such as the following: a commentary on Patañjali's *yoga-sūtra*, which is believed to be from the 5th century, contains a simile along the following lines (cf. [18], p. 46); "Just as a line ("*yathā ekā rekhā...*") in the hundreds place stands for hundred, in the tens place ten, and one in the ones place, so one and the same woman is called mother, daughter, and sister.", which indicates that the concept of place-value was a familiar one by that time.

1.4 Arrival - in Different Avatars

When we move a little forward in time however, we witness a torrent of direct and verifiable, source-specific evidence concerning decimal place-value system being used on a regular basis. This is however not actually in terms of numbers written through *digits* themselves, but rather in terms of words or letters of Sanskrit alphabet standing for the digits; these nevertheless carry an unmistakable stamp of representations of number in the place-value system. The point about using words or letters in place of digits themselves is the following. Much of the context for representation numbers in this respect came from situations warranting simulataneously a verbal rendering, such as compositions (including scientific works) in verse form, or formulae or charts, especially in astronomy, which used to be memorized for ready recall, etc. Symbolic figures of numbers or digits would not fit into this framework. Even if one is to choose to put digit-names in place of the digits themselves it is still not convenient, since the digits would have unique names, or at best provide a small number of options for them, bringing in rigidity in expressing them in the contexts as above. Two separate solutions are seen to have come up towards resolving the issue: the *bhūtasaṅkhya* and the *katapayādi* system. Of the two, the *bhūtasaṅkhya* system is older, and independent in its development from the emergence of the decimal place-value system. It had been in use since more ancient times[10] though evidence of its application together with the decimal place-value system is from a later period.

The basic idea involved in the *bhūtasaṅkhya* system is that in place of a digit-name we could

[9] Similarly, the *Sūryasiddhānta*, an important source work in the Siddhānta tradition of astronomy, whose original version is believed to go back to the 4th century (see [2], p. 9), may be expected to have contained numbers in the decimal place, but the available versions are only from later centuries.

[10] It is noted in [22] that symbolic words representing numbers are used sporadically in the *Śrauta-sūtras* and *Vedāṅga-jyotiṣa* and that it is employed rather extensively, with about 100 occurrences, in Pingala's *Chandaḥ sūtra*, from the second century BCE.

instead put the name of an object or entity which, in terms of common parlance of the time or general knowledge of the reader/listener, with which the corresponding digit may be associated; thus in place of "one"the same could be understood if the verse mentions say "moon"(*śaśi*), or various other things which are unique, like earth (*bhūmi*), in any of the synonyms (of which there are plenty in Sanskrit for most words!); in place of "two", "eye"(*netra, cakṣu, ...*) or "hand"(*hasta, kara, ...*) could be mentioned; for "four", *veda* or *yuga* (corresponding to the four epochs *satya, tretā, dvāpara* and *kali* of hindu mythology) would be natural candidates, and so on.

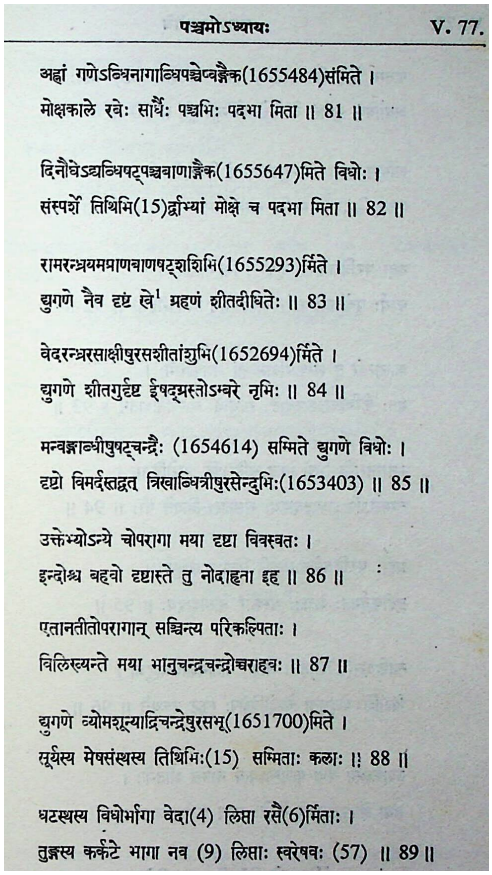
We shall not go into any extensive details on the variety of entities adopted in *bhūta-saṅkhyas*; some more examples will appear in the narrative below (see [6], I, p. 53 - 57 for a detailed exposition). Here are however a few points about the system which it may be worth keeping in mind. While here we motivated it in terms of its use as alternative names for *digits*, the system, having had an independent life of its own, extended also to various composite numbers (especially, 2 or 3-digit numbers) by familiar object-names; thus *raśi* (constellation) or *māsa* (month) were used for 12, *tithi* was used for 15, *danta* or *rada* (tooth) for 32, etc.. The digits, or groups of digits, involved were arranged in the ascending order of their significance, opposite to the modern practice - viz. in writing the number the digit corresponding the highest decuple term is on the right; the reader is referred to [21] for a discussion on this aspect from an historical perspective. The groups of digits stand at the due places, for the appropriate number of digits in the number being represented in the decimal place-value system; thus "*Veda-tithi-śaśi*" stands for 1154 (and not 4151), with *śaśi* for one, *tithi* for 15 and *Veda* for 4. Finally, it may be borne in mind that understanding some of the terms involved calls for good familiarity of the scriptures, and even with that sometimes it may be hard to understand how certain associations may have arisen; the reader is referred to the discussion in this respect in [21] (pages 2 - 3), speculating on how the term *rūpa* may have come to be assigned the digit 1, as is found to be the case in various texts, starting with *Vedaṅga jyotiṣa*. Also, sometimes ambiguities could arise, with the same word having been used for different numbers, at different places or at different times as the conventions involved could differ.

The classic trio of works of Siddhānta astronomy from the immediately succeding period, *Pañcasiddhāntikā* of Varāha Mihira, *Mahābhāskarīya* of Bhaskara I, and *Brāhmasphuṭasiddhānta* of Brahmagupta are seen to routinely express numbers in the decimal place-value system in the *bhūtasaṅkhyā* format[11] and, as may be expected from the subject of astronomy, they have numerous quite large numbers expressed in this way along the main narrative, woven into the verses. *Mahābhāskarīya*, written as a commentary on *Āryabhaṭīya*, was composed in 629, and *Brāhmasphuṭasiddhānta*, which, incidentally, is critical of *Āryabhaṭīya* in many respects, was composed in 628. *Pañcasiddhāntikā* is older than these and in particular is quoted in *Brāhmasphuṭasid-*

[11]Though in some popular literature the decimal place-value system, and the zero, are linked with the name of the renowned astronomer mathematician Āryabhaṭa (born 476), there is actually no concrete evidence towards either of it. While we do not have any information on how he made his calculations - which indeed involve quite large numbers - his work *Āryabhaṭīya* does not quite involve the decimal place-value system. Though he does introduce decuple terms, for powers upto 10^9 (which by itself does not involve "place-value"), for expressing composite numbers he introduces his own system of representation, and while the system is impressive in being concise, it is based on *additive* conventions; thus *makhi* stands for 225 with 200 coming from *khi* and 25 from *ma*; similarly, the number 4320000, which is the number of eastward revolutions of the Sun in a *yuga*, is represented by *khyughṛ*, with *ghṛ* representing 4×10^6 and *khyu* contributing 320000. His verse concerning an algorithm for computing square-roots (*Gaṇitapāda*, 4) is often interpreted in terms of the decimal representation of numbers. However, the verse manifests no clear connection with the decimal place-value system, and seems open to an alternative interpretation, akin to the approximation procedure involving the formula $\sqrt{a^2+b} \approx a + \frac{b}{2a}$; the latter is known after Theon of Alexandria (see [10], I, p. 61), but was arguably known in more ancient times in various cultures, including to the geometers of Śulba works. The text in [20] following the verse, on page 36, is an *interpretation* of the verse offered by the editors, and not a faithful translation of the original, and is apparently based on the knowledge of how the procedure involved was practised in later times; the later works concerned are listed in the footnote under it. The translation given in [4], page 22, is more authentic, and though the interpretation in the commentary there coincides with that in [20], it is clear from the discussion that there *is* issue of interpretation involved concerning the decimal place-value system being deployed.

dhānta, but it has not been possible to determine its year of composition as precisely as these. However, the work involves as "epoch date", the reference year for the starting for calculations, which corresponds to 505 CE, and since the epoch dates were nomally chosen, in order to simplify calcula- tions, as a past date close to the year of composition, it may be inferred that the work was composed in the 6th century.

Here is an example from *Pañcasiddhāntikā* [26] (see p. 3 of the Devanagari part and p. 5 of the translation part): "*varṣayute dhṛtighne nava-vasu-guṇa-rasa-rasaḥ syuradhimāsaḥ*"stands for "in 180000 years there are 66389 intercalary months"; in the first part *dhṛtighne* means multiplied by 18, and *ayuta* is 10000; in the second part 66389 is fully represented in the decimal place-value format, where *rasa* stands for 6 (*rasa* means "taste" or "flavour" and its association with 6 arises from the classification of tastes as sweet, sour, salty, pungent, bitter, and astringent), *guṇa* for 3 (corresponding to qualitative classification into 'good', 'bad' and 'indifferent'), *vasu* for 8, and *nava* is in pure digit form. Figure 3 below gives a sample of verses from *Mahābhāskarīya*, illustrating the point, drawn from [14], p. 331; the edition shows in parenthesis the numbers appearing in the original in *bhūtasaṅkhyā*. To this let me add here one example from *Brahmasphuṭa-siddhānta*, with some explanation. In Chapter 18, *kuṭṭakādhāya*, verse 94, the number 3249624 is expressed as *jinarasago'bdhirada*, with *rada* (tooth) standing for 32, *abdhi* (oceans) for 4, *go* for 9, *rasa* for 6, and *jina* for 24; (while the common meaning of *go* is cow, the word has a host of other meanings in Sanskrit including "earth", and viewing the earth as one of the planets the number 9 is associated with it; the last term involved, *jina* is after the 24 *tīrthaṅkaras* of Jainism; as this comes from Brahmagupta we see the usage transcending religious affiliations!).


पञ्चमोऽध्यायः V. 77.

अह्नां गणेऽब्धिनागाब्धिपञ्चेष्वङ्कैक(1655484)संमिते ।
मोक्षकाले रवेः सार्धैः पञ्चभिः पदभा मिता ॥ 81 ॥

दिनौघेऽद्र्यब्धिषट्पञ्चबाणाङ्कैक(1655647)मिते विधोः ।
संस्पर्शे तिथिभि(15)र्द्वाभ्यां मोक्षे च पदभा मिता ॥ 82 ॥

रामरन्ध्रयमप्राणबाणषट्शशिभि(1655293)र्मिते ।
द्युगणे नैव दृष्टं खे[1] ग्रहणं शीतदीधितेः ॥ 83 ॥

वेदरन्ध्ररसाक्षीषुरसशीतांशुभि(1652694)र्मिते ।
द्युगणे शीतगुर्दृष्ट ईषद्ग्रस्तोऽम्बरे नृभिः ॥ 84 ॥

मन्वङ्गाब्धीषुषट्चन्द्रैः (1654614) सम्मिते द्युगणे विधोः ।
दृष्टो विमर्दस्तद्वत् त्रिखाब्धित्रीषुरसेन्दुभिः(1653403) ॥ 85 ॥

उक्तेभ्योऽन्ये चोपरागा मया दृष्टा विवस्वतः ।
इन्दोश्च बहवो दृष्टास्ते तु नोदाहृता इह ॥ 86 ॥

एतानतीतोपरागान् सञ्चिन्त्य परिकल्पिताः ।
विलिख्यन्ते मया भानुचन्द्रचन्द्रोच्चराहवः ॥ 87 ॥

द्युगणे व्योमशून्याद्रिचन्द्रेषुरसभू(1651700)मिते ।
सूर्यस्य मेषसंस्थस्य तिथिभिः(15) सम्मिताः कलाः ॥ 88 ॥

घटस्थस्य विधोर्भागा वेदा(4) लिप्ता रसै(6)र्मिताः ।
तुङ्गस्य कर्कटे भागा नव (9) लिप्ताः स्वरेषवः (57) ॥ 89 ॥

Figure 3: A page from Mahābhāskarīya; ([14], page 331)

The *kaṭapayādi* system, on the other hand, depends on using the letters of the Sanskrit alphabet in place of the digits, with multiple options of letters for each digit. The consonants *ka, ṭa, pa*, and *ya* are all assigned the digit 1, which explains the name kaṭapayādi, "beginning with *ka, ṭa, pa, ya*". The letters following them in the standard listing, until the next one in the above group, are assigned successive values 2 to 9 and then 0; the group starting with *pa* ends with *ma*, with the latter standing for 5, and the group starting with *ya* ends with *ha* which stands for 8. The vowels (note that these include *ṛ*) are accorded no numerical significance in this system; thus, for instance, any of *ta, tā, ti, ...* may be used to denote the digit 6.

Also, the consonants appearing in conjunctions are treated as equivalent to the last consonant that is paired with a vowel; thus *priya* and *riya* would carry the same value 12. On account of the considerable leeway this offers, the system enabled verbal representation of numbers through crisp, meaningful, and even amusing or thought provoking words, phrases or clauses, which could be easily memorized. As in *bhūtasaṅkhyā* the numbers are usually read starting with the least significant digit. As it follows a fixed algorithm, unlike with *bhūtasaṅkhyā*, the representation is unambiguous. The reader is referred to [6] and [22] for a variety of other details on the *kaṭapayādi* system.

The first reliably datable occurrence of numbers in the decimal place-value system in the *kaṭapayādi* format is found in the *Graha-cāra-nibandhana*, composed by Haridatta, from Kerala, in 683 (cf. [22], p. 41). Haridatta may not have been the inventor of the system, though a 15th century commentator Sundararāja expressly associates the system with him (see [22], p. 42). Nevertheless, the system certainly had its origin in Kerala, and its usage

was confined to that region for several centuries, and only gradually spread over to other parts of South India. Whether it made serious inroads into north India anytime is unclear (see [22]).

There is a belief that the earliest text employing the *kaṭapayādi* system is *candra-vākyāṇi* ("moon sentences"). The *candra-vākyas* are crisp phrases describing the daily angular position of the moon, for 248 successive days (after which the cycle repeats) in *raśis*, degrees and minutes; thus each of them consists of three numbers, expressed in the *kaṭapayādi* format; the last two, corresponding to degrees and minutes are uniformly represented with two digits (by inserting a zero for the ten's place, in case of single digit figures), while the *rāsi* term is omitted if it is zero and presented either in single digit or with a 0 in the ten's place, if less than 10; let me note also that the three numbers are viewed in succession, and a word in the "sentence" may have letters corresponding the successive numbers from the triple involved. Here are some examples (which in particular illustrate these points): the triple $2^r\ 27^\circ\ 13'$, the seventh entry, is rendered as *gṛhyā surā rajñā* ("the king may consume wine"!), $2^r\ 10^\circ\ 04'$ (the 33rd) as *vanāni yatr* ("where there are woods"), $6^r\ 23^\circ\ 36'$ (the 70th) as *te bālā bhrāntāḥ* ("those confused children") etc.. While it would of course involve considerable literary creativity to compose such representations, this should also illustrate to the reader potential in-built into the system to make it possible.

Unfortunately there is no clarity about when the *candra-vākyās* may have been composed. They are attributed to Vararuci, a legendary figure from Kerala who is supposed to have flourished in the fourth century, but there is no substantive evidence to that effect, and in terms of the astronomical details involved there seem be doubts whether the work may be that old (see [22]).

The above examples of use of the decimal place-value system are of course tied to the academic community, and one would be curious about its presence outside that milieu. Such usage is witnessed in various stone inscriptions and grant-plates in copper - these, additionally, provide *physical* exemplars from the times under consideration. A list of 33 such exemplars is found in [6], p.40 - 43, together with brief details about them. These are in numerals, viz. symbols for the digits concerned, the script involved being one of the derivatives of Brāhmī. The oldest extant stone inscription with a number in decimal representation and with a zero in it, is on a stele (a broken doorjamb in this instance), named K-127. It was discovered in 1891 at Sambor on Mekong, Cambodia, and is now displayed in the National Museum in Phnom Penh. The inscription, aimed primarily at commemorating donations made to god Amareśvara, has on it the year of the occasion engraved, as śaka 604, which corresponds to 682 - 83 CE (see [1] for details). The oldest specimen in India with a decimal number with zero is in an inscription in the Chaturbhuja temple at the Gwalior Fort in Madhya Pradesh. The inscription, which is from 876 CE, states in particular that a garden of 187 *hastas* by 270 *hastas* was planted and it yielded 50 garlands for the temple every day, and the numbers are seen in decimal place-value notation, together with the zero.

Along with the inscriptions in stone, copper plates that were used as a typical medium for records of transactions such as grants and deeds (where the document is sought to be portable but permanent) would be a source material to explore usage of numbers from the early period. The list from [6] contains several instances of these, and the earliest entry from the list, from 595, is of a copper grant-plate from Sankheda, Gujarat, recording in the place-value system the year-number 346 of the so-called Kalacuri era at the time. Unfortunately, while copper plates incorporating property deeds and grants have the advantage of durability as physical specimen, they are vulnerable to being forged by unscrupulous individuals for pecuniary gains, and it has been persuasively argued that the Gujarat plate mentioned above is a spurious one, actually inscribed at a later date ([18], p. 45).

1.5 The Bakhshālī Manuscript - a Celebration

Before concluding this part of the article I ought to discuss an interesting, and rather enigmatic, manuscript in which numbers in decimal place-value system occur aplenty, in down to earth numerals, involving down to earth calculations. Indeed, with its multitude of numbers, including some breathtakingly large ones as we shall see below, it may indeed be looked upon as a celebration of

the decimal place-value system in its most natural and basic form! I am alluding here to what is known as the Bakhshālī manuscript.

Though it would be a bit of a digression from our main theme let me go over its history in some detail, which is not commonly discussed. The manuscript was found at Bakhshālī (after which the manuscript is named), a village, about eighty kilometers away from the present city of Peshawar in Pakistan; it may also be mentioned here that the village is about 110 kilometers from Takṣaśilā (or Taxila as the Greeks referred to it), the seat of the renowned ancient university which flourished from around 5th century BCE to 5th century CE, in the Gāndhara region. According to an old letter on record (see [13], p.1) the manuscript was apparently found "while digging in a ruined stone enclosure on one of the mounds near Bakhshālī"[12] and the finder who happened to be a tenant of a local Inspector of Police passed it onto him. The manuscript subsequently reached the Lt. Governer of the Punjab, who on the advice of General Cunningham, forwarded it to A. F. R. Hoernle, a renowned Indologist based in Calcutta (now Kolkata), for examination and publication (see [13], p. 2). Hoernle published accounts on it in 1880s, including [11], and in 1902 presented the manuscript to the Bodleian Library at Oxford, which has served as its home since then. At the instance of Hoernle, G. R. Kaye undertook an extensive study of the manuscript, producing [13], which he mentions, in the preface to the book, would have been published as joint work with Hoernle, had the latter been alive when the book was completed. It so happens Kaye did not appreciate the value of the manuscript as a historical find, and while the valuable service rendered by him in carefully editing the book is appreciated, his comments are often considered prejudicial. Over the years there have been numerous studies of the manuscript and publications related to it. I may especially mention the following works in this respect, for reference: Bibhutibhushan Datta [5], Svāmi Satya Prakash Saraswati [23] and Takao Hayashi [9].

The manuscript consists of 70 of birch-bark folios, of size 15 cm by 9 cm; an image of Folio 46-Recto is seen in Figure 5 below, where we discuss a bit also about its content; the manuscript was already in a rather deteriorated condition when discovered; damage may be noticed in the image. There has been much confusion about its period from very early on, with a whole range of periods from early centuries of CE to the 12th century being proposed by different authors. In [9] Hayashi, after a detailed analysis of the earlier proposals and evidence from paleographic inputs and comparisons with other works in the Sanskrit pāṭīgaṇita genre, tentatively assigned the date of composition of the work to the 7th century and the date of the present manuscript (which is apparently a copy of a previous manuscript) to be between the 8th and 12th centuries ([9], p.148-149). A radiocarbon dating of 3 folios in the manuscript was carried out by the Bodleian Library in 2017. Though one had hoped that it would play a crucial role in resolving the issue of date of the manuscript, the result has unfortunately led to further chaos in the matter with the three folios being assigned three widely different historical periods, the date-ranges of the three folios being put out as 224-383, 680-779, and 885-993 ! Though perhaps one could come up with an explanation that the folios dated to be from later may be substitutes in a manuscripts which indeed belongs to the 3rd or 4th century, it has also been persuasively argued in [19] that the manuscript is homogeneous in its features, precluding the folios being from vastly different periods, and that the results are so much in conflict with what we know about the manuscript, that one ought to question the reliability of the findings.

Let us now return to our original theme of the decimal place-value system, in the context of the Bakhshālī manuscript; for obvious reasons we shall not go into any comprehensive account of the mathematical contents of the manuscript here and will limit to occurrence of large numbers in the manuscript using the decimal place-value system; [3] would be a good reference as well as supplement for the discussion.

As mentioned earlier, the manuscript contains some very large numbers repesented in the decimal place-value system. The format is the same as we adopt today, including the orientation from left to right, the only difference being the notation for the digits. Figure 4 below shows the

[12]Kaye however has expressed doubts about this description, and especially the word "ruined".

correspondence between the symbols used in the manuscript with the respective digits.

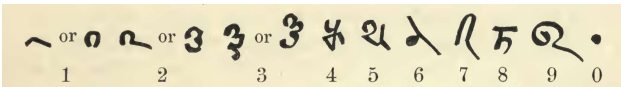


Figure 4: The Bakhshālī numerals

To explain the context for the large numbers I need to recall something of the pattern adopted in the manuscript. It is in the nature of a collection of problems, and consists of units involving a 3-step format where firstly a rule for solving a mathematical problem is stated or alluded to, it is then followed by some examples of how the solution is carried out, and then after each example, a verification is carried out. There is a class of problems which on the face if it concern the question of finding n for which the sum, $a + (a + d) + \cdots + (a + [n - 1]d)$, of an arithmetical progression with given values of a and d attains a given value S. The rule for working it out is given and it corresponds in our notation to the appropriate formula

$$n = \frac{\sqrt{8dS + (2a - d)^2} - (2a - d)}{2d}$$

Curiously however in none of the problems on the theme is S a value that actually holds for some integer n. Also they do not seem to be content to find, by using the formula, when the sum crosses the given value S the first time. For some inexplicable reason the task is taken to be to "determine" the value of the right hand side, using for the purpose a formula they had for square-roots, even though it is not an integer. The formula for square roots involved, which is unique to the Bakhshālī manuscript (and often known after it) gives, for $x > 1$ an expression for $\sqrt{x}$ as

$$m + \frac{a}{2m} - \frac{(r/2m)^2}{2(m + \frac{r}{2m})},$$

where m is taken to be the largest integer for which $m^2 \leq x$ and $r = x - m^2$.[13]

The calculations with these formulae led them to fractions with huge numerators and denominators; the fractions were written with the denominator under the numerator, as we do, but without the bar in between; also no cancellation of common factors seems to be practised unless needed for verification. Problems of this type occur on folios 45, 46, 56, 64, and 65 (involving one or both sides of the folio), and the computations of the above, and the verification part, involve fractions with large numerator and denominator. For instance one of the problems (starting on folio 45), has the given values as $a = \frac{3}{2}$, $d = \frac{3}{2}$ and $S = 7000$. Now, S is between 569^2 and 570^2, so in this case $m = 569$ and $r = 7000 - (569)^2 = 768$. Applying the above formula for square roots they compute the requisite square-root to be $\dfrac{12516007433}{43183750}$, and substituting that value in the preceding expression they get the desired solution n to be $\dfrac{6225615904}{64775625}$; see [3] for a more detailed account. In the verification part, starting with that value of n, through some magical steps they confirm S to be $\dfrac{50753383762725000000000}{7250483394675000000}$, which is indeed 7000 (!).

The page in Figure 5 is seen to be full of calculations with numbers. The numbers seen in the box in the middle near the top (marked in green) are the numerator and denominator of the fraction $\dfrac{110895125022}{1554615000}$, appearing as a part of the calculation, expressed in the decimal place-value system! (As mentioned earlier, it was not the practice yet to insert a bar in between). The nine dots in the bottom line, underlined in yellow, are the surviving part of the numerator of the fraction 50753383762725000000000, discussed in the example here.

[13]The expression may be seen to be the 2-step iterate for the approximation formula known after Theon of Alexandira, which was recalled in an earlier footnote in another context. The manuscript does not say anything about the derivation, nor is there any indication that the formula was meant to be an approximate one.

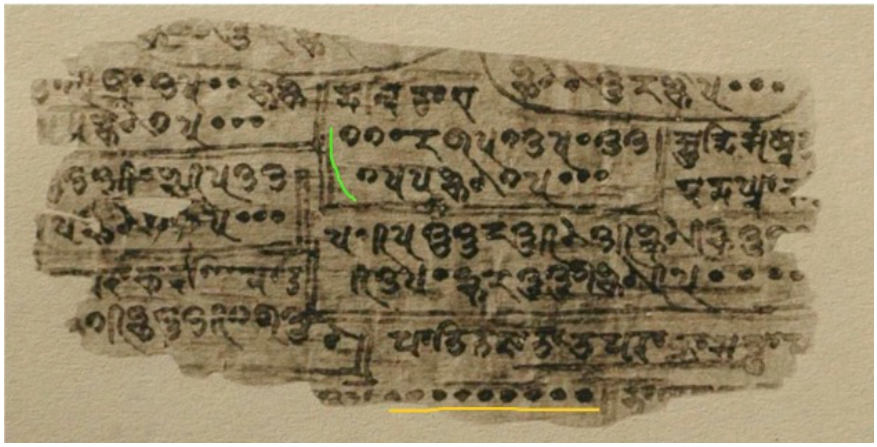


Figure 5: Folio 46 (recto) of the Bakhshālī manuscript

While the numbers as above are impressively large, the largest number found in the manuscript is not one of these. It is a number, found on folio 58, whose context is not clear[14]; the number is

$$2653296226447064994283211 87.$$

This would indeed be a good point to conclude this part of the exposition on the decimal place-value system of representing numbers!

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[14]The same number is written on the folio in words at the top and in numerals at the bottom of the folio, so one may wonder whether it concerned practising conversion from numers written in words to those represented in numerals.

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□□□

2. Many Avatars of "Best Approximates"

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2.1 Introduction

The main aim of this article is to illustrate how different ramifications of best approximates -termed as Avatars in this article- which appear in several areas of science and engineering actually emanate from one of the most elementary theorems taught at the graduate level program in mathematics. This is concerning existence and uniqueness of a point of minimum distance in a subspace of a Hilbert space from any given point in the space.
We show how the main theorem helps in unifying different facets of Avatars. By doing so, one can also visualize the linkage between theory and practice.
From the modern day perspective of high tech it is imperative that a student of mathematics learns how to set up linkages of these types.
In our discussion we have diluted the mathematical rigour of definitions and proofs to make the article more accessible to a general audience.

2.2 Best Approximates - Avatar in Finite Dimensional Spaces

We begin with a linear system of the form

$$Ax = z, z \in \Re^m, \tag{2.1}$$

where $A : \Re^n \to \Re^m$ is an $m \times n$ matrix with $m > n$. If $z \notin R(A)$, the range space of A, the matrix equation (2.1) has no solution.
Problem 2.1
Let $z \notin R(A)$. Let $[A^\top A]$ be invertible. Obtain an element $u^* = Ax^*$, such that the following holds.

$$||z - Ax^*|| = \inf_{x \in \Re^n} ||z - Ax||. \tag{2.2}$$

If this element $u^* = Ax^*$ exists satisfying (2.2), it is called the best approximate to z.

<u>Dual Problem</u>

Assume that $n > m$ and $(AA^\top)$ is invertible. Then (2.1):

$$Ax = z$$

has infinitely many solutions for a given $z \in \Re^m$.
We wish to find out the solution x^* of minimum norm. That is, we need to investigate the following problem.

<u>Problem 2.2</u>

Find $x^* \in \Re^n$ such that

$$||x^*|| = \inf_{\{x \in \Re^n : Ax = z\}} ||x||. \tag{2.3}$$

As all solutions x of the solutions set $S = \{x \in \Re^n : Ax = z\}$ are of the form $x = x^{(0)} + u$, where $x^{(0)} \in S$ is a particular solution and $u \in N(A)$, the Null space of A, we can restate the above dual problem as under.
Find $u^* \in N(A)$ such that

$$||x^{(0)} + u^*|| = \inf_{u \in N(A)} ||x^{(0)} + u||, \tag{2.4}$$

with the assumption that $AA^\top$ is invertible. Here $Ax^{(0)} = z$.

Key Theorem

As indicated in the introduction, we shall obtain ready answers to Problem 2.1 and Problem 2.2 by resorting to the following simple but elegant theorem in a Hilbert space H (real or complex).

Definition 2.1. Let H be a Hilbert space and let $M \subset H$ be a subspace. Let $z \in H$. An element $u \in M$ is said to be the point of minimum distance from z if the following holds.

$$||z - u|| = \inf_{w \in M} [||z - w||]. \tag{2.5}$$

We also call this point $u \in M$ as the best approximate of z.

Theorem 2.2. Let H be a Hilbert space and let $M \subset H$ be a closed subspace and $z \in H$. Then

(i) there exists a unique best approximate $u \in M$ of z.

(ii) Further, this best approximate u is characterized by the following orthogonal property (**P**):

$$(\mathbf{P})(z - u) \in M^{\perp}. \tag{2.6}$$

Remark 2.1. The first part of the above theorem holds true even if M is a closed convex subset of the space H. We have a detailed discussion on this aspect in the last section of this article.

Revisit Problem 2.1 and Problem 2.2

Let A be as defined in Problem 2.1. We have $R(A)$, a closed subspace of $\Re^m$ and hence by Theorem 2.2, we get existence and uniqueness of the best approximate $u^* = Ax^*$ for Problem 2.1. Further, by property (P) it follows that the best approximate satisfies the condition that $z - Ax^* \in [R(A)]^{\perp}$. One can easily show that: $R(A)^{\perp} = N(A^{\top})$. Hence, we get $A^{\top}(z - Ax^*) = 0$. As $(A^T A)$ is invertible, we get the following representation for x^*

$$x^* = [A^{\top} A]^{-1} A^T z, \tag{2.7}$$

and hence the best approximate u^* is given by

$$u^* = Ax^* = A[A^{\top} A]^{-1} A^T z. \tag{2.8}$$

Problem 2.2 has equivalent formulation given by (2.4) in terms of the best approximate in $M = N(A)$. Again, $N(A)$ is a closed subspace of $\Re^n$ and hence we get existence and uniqueness of the best approximate $u^* \in N(A)$ to $x^{(0)}$.

Further the property (P) implies that $x^{(0)} - u^* \in [N(A)]^{\perp} = R(A^{\top})$. This implies that there exists $w \in \Re^n$ such that $x^{(0)} - u^* = A^T w$. This gives $u^* = x^{(0)} - A^{\top} w$ and hence $0 = Au^* = Ax^{(0)} - AA^{\top} w = z - AA^{\top} w$. There by giving us $w = [AA^{\top}]^{-1} z$. Hence, we get solution x^* to Problem 2.2:

$$x^* = x^{(0)} - u^* = A^{\top} w = A^{\top} [AA^{\top}]^{-1} z.$$

2.3 Avatar in Action - Application to Parameter Estimation & Neural Network

Parameter Estimation- Product Analysis

In many processes (chemical, biological etc.), a set of products, say A and B combine together linearly to give a new product C; $A + B \to C$. More generally a set of products, say $A_1, A_2, \ldots, A_n$ combine together linearly to give a new product C.

$$A_1 + A_2 + \cdots + A_n \to C.$$

Representing these products in terms of their molecular values $\{x_i\}_{i=1}^n$ and y, we get the linear model

$$y \equiv [\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n] = \sum_{i-1}^{n} [\alpha_i x_i],$$

with the parameter set $\{\alpha_i\}_{i=1}^n$ as the known set of weights. We wish to estimate the parameter set $\{\alpha_i\}_{i=1}^n$, given that the observed values $\left\{y^{(k)}\right\}_{k=1}^m$ of the new molecule y are as close as possible to the model values

$$\left\{\sum_{i=1}^{n} \alpha_i x_i^{(k)}\right\}_{k=1}^m$$

in the excercise of conducting m number of experiments. This amounts to estimating the collection $\{\alpha_i\}_{i=1}^n$ so that the 2-norm of the deviation vector $d = (d_k)$:

$$d_k = \left[y^{(k)} - \sum_{i=1}^{n} \alpha_i x_i^{(k)}\right], 1 \leq k \leq m,$$

is as small as possible. Thus, in finite dimensional space setting it reduces to the following minimization problem. The minimizing set will be the collection of all elements $w = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \Re^n$,

$$Min_{w \in \Re^n} \left[\sum_{k=1}^{m} \left[y^{(k)} - \sum_{i=1}^{n} \alpha_i x_i^{(k)}\right]^2\right].$$

Equivalently, we can describe this minimization problem as

$$Min_{w \in \Re^n} \begin{pmatrix} (y^{(1)} & -(\alpha_1 x_1^{(1)} + \alpha_2 x_2^{(1)} + \cdots + \alpha_n x_n^{(1)}))^2 + \\ (y^{(2)} & -(\alpha_1 x_1^{(2)} + \alpha_2 x_2^{(2)} + \cdots + \alpha_n x_n^{(2)}))^2 + \\ \vdots & \\ (y^{(m)} & -(\alpha_1 x_1^{(m)} + \alpha_2 x_2^{(m)} + \cdots + \alpha_n x_n^{(m)}))^2 \end{pmatrix}.$$

We denote the observed vector $(y^{(1)}, y^{(2)}, \cdots, y^{(m)}) \in \Re^m$ as z. Thus the parameter estimation problem reduces to the following best approximation problem in $\Re^m$:.
Find $w^* \in \Re^n$ such that

$$||z - Aw^*|| = \inf_{w \in \Re^n} ||z - Aw||,$$

where the matrix A is defined as:

$$A = \begin{pmatrix} x_1^{(1)} & x_2^{(1)} & \cdots & x_n^{(1)} \\ x_1^{(2)} & x_2^{(2)} & \cdots & x_n^{(2)} \\ . & . & . & . \\ x_1^{(m)} & x_2^{(m)} & \cdots & x_n^{(m)} \end{pmatrix}.$$

Invoking the result concerning Problem 2.1 (refer (2.7)), we compute the weights vector $w^* = (\alpha_i)$ through the following formula

$$w^* = (A^\top A)^{(-1)} A^\top z.$$

<u>Neural Network</u>

Generalizing the above linear model, we get a series of outputs $\{y_1, y_2, \ldots y_r\}$ by linearly combining inputs $\{x_1, x_2, \ldots x_n\}$ through the synaptics weights $\{a_{ij}\}$, as we see below:

$$\begin{array}{ll} y_1 & \leftarrow \alpha_{11} x_1 + \alpha_{12} x_2 + \cdots + \alpha_{1n} x_n \\ y_2 & \leftarrow \alpha_{21} x_1 + \alpha_{22} x_2 + \cdots + \alpha_{2n} x_n \\ \vdots & \\ y_r & \leftarrow \alpha_{r1} x_1 + \alpha_{r2} x_2 + \cdots + \alpha_{rn} x_n \end{array}.$$

Our task is to train the synaptic weights by subjecting the network to a large set of input-output data as we see below.

Individual 1

$$\begin{array}{llllll} y_1^{(1)} & \leftarrow \alpha_{11}x_1^{(1)}+ & \alpha_{12}x_2^{(1)}+ & \cdots & + & \alpha_{1n}x_n^{(1)} \\ y_2^{(1)} & \leftarrow \alpha_{21}x_1^{(1)}+ & \alpha_{22}x_2^{(1)}+ & \cdots & + & \alpha_{2n}x_n^{(1)} \\ & & & & \vdots & \\ y_r^{(1)} & \leftarrow \alpha_{r1}x_1^{(1)}+ & \alpha_{r2}x_2^{(1)}+ & \cdots & + & \alpha_{rn}x_n^{(1)} \end{array}$$

Individual 2

$$\begin{array}{llllll} y_1^{(2)} & \leftarrow \alpha_{11}x_1^{(2)}+ & \alpha_{12}x_2^{(2)}+ & \cdots & + & \alpha_{1n}x_n^{(2)} \\ y_2^{(2)} & \leftarrow \alpha_{21}x_1^{(2)}+ & \alpha_{22}x_2^{(2)}+ & \cdots & + & \alpha_{2n}x_n^{(2)} \\ & & & & \vdots & \\ y_r^{(2)} & \leftarrow \alpha_{r1}x_1^{(2)}+ & \alpha_{r2}x_2^{(2)}+ & \cdots & + & \alpha_{rn}x_n^{(2)} \end{array}$$

Individual m

$$\begin{array}{llllll} y_1^{(m)} & \leftarrow \alpha_{11}x_1^{(m)}+ & \alpha_{12}x_2^{(m)}+ & \cdots & + & \alpha_{1n}x_n^{(m)} \\ y_2^{(m)} & \leftarrow \alpha_{21}x_1^{(m)}+ & \alpha_{22}x_2^{(m)}+ & \cdots & + & \alpha_{2n}x_n^{(m)} \\ & & & & \vdots & \\ y_r^{(m)} & \leftarrow \alpha_{r1}x_1^{(m)}+ & \alpha_{r2}x_2^{(m)}+ & \cdots & + & \alpha_{rn}x_n^{(m)} \end{array}$$

We denote

$$\begin{array}{rcl} \bar{w}_i & = & (\alpha_{i1}, \alpha_{i2}, \cdots, \alpha_{in})^T \in \Re^n \\ \bar{z}_i & = & (y_i^{(1)}, y_i^{(2)}, \cdots, y_i^{(m)})^T \in \Re^m. \end{array}$$

The matrix A is as defined before:

$$A = \begin{pmatrix} x_1^{(1)} & x_2^{(1)} & \dots & x_n^{(n)} \\ x_1^{(2)} & x_2^{(2)} & \dots & x_n^{(2)} \\ . & . & . & . \\ x_1^{(m)} & x_2^{(m)} & \dots & x_n^{(m)} \end{pmatrix}.$$

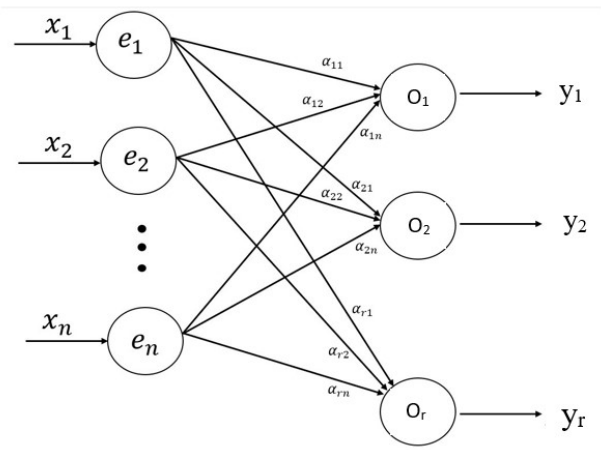

Figure 1: A graphical representation of neural network

By carrying out an analysis as before, we get the best weights vectors $\bar{w}_i$ by the following formula:

$$\bar{w}_i = (A^\top A)^{-1} A^\top (z_i),$$

$$\bar{z}_i = (y_1^{(i)}, y_2^{(i)}, \cdots, y_n^{(i)})^T.$$

So, we get the collection $\{a_{ij}\}$ of trained synaptic weights.

2.4 Avatar in Space Science - Controlling the Trajectory of A Satallite

A satellite can be thought of as a unit mass orbiting around the earth under inverse square law. The satellite has thrusting capacity with radial thrust u_1 and tangential thrust u_2. Thrust is applied through booster rockets. The problem of practical interest is stated as follows:

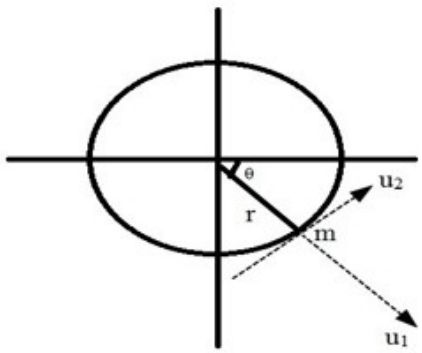

Figure 2: Radial and tangential thrusts

Problem 4.1. Find radial thrust u_1 and tangential thrust u_2 so as to put the satellite in a prescribed position with minimum fuel.
Equating forces in the tangential and normal directions, we get

$$\frac{d^2r}{dt^2} - r(t)\left[\frac{d\theta}{dt}\right]^2 = -\frac{k}{r^2} + u_1(t),$$

$$r\frac{d^2\theta}{dt^2} + 2\frac{d\theta}{dt}\frac{dr}{dt} = u_2(t).$$

This gives a pair of second order differential equations

$$\frac{d^2r}{dt^2} = r(t)\left[\frac{d\theta}{dt}\right]^2 - \frac{k}{r^2} + u_1(t), \qquad (4.1a)$$

$$\frac{d^2\theta}{dt^2} = -\frac{2}{r}\frac{d\theta}{dt}\frac{dr}{dt} + \frac{u_2(t)}{r}. \qquad (4.1b)$$

If $u_1 = 0 = u_2$, one canshow that the solution of the above pair of differential equation is given by $r(t) = \sigma$ and $\theta(t) = wt$ (σ and w are constants with $\sigma^2 w^2 = k$). In view this observation, making the following changes of variables

$$x_1 = r - \sigma,\ x_2 = \frac{dr}{dt},\ x_3 = \sigma(\theta - wt),\ x_4 = \sigma(\frac{d\theta}{dt} - w)$$

we get

$$\begin{aligned}
\frac{dx_1}{dt} &= x_2(t),\\
\frac{dx_2}{dt} &= (x_1+\sigma)(\frac{x_4}{\sigma} + w)^2 - \frac{k}{(x_1+\sigma)^2} + u_1,\\
\frac{dx_3}{dt} &= x_4(t),\\
\frac{dx_4}{dt} &= -\frac{2x_2\sigma}{(x_1+\sigma)}(\frac{x_4}{\sigma} + w) + \frac{u_2\sigma}{(x_1+\sigma)}.
\end{aligned}$$

The above system is a nonlinear system of differential equations which involve the forcing functions u_1 and u_2 (controls). In a compact form, we can write the system as

$$\frac{dx}{dt} = f(x,u);\ x(t) \in \Re^4,\ u(t) \in \Re^2.$$

Linearizing the above system around $x = 0$ and $u = 0$, we get a system of the form

$$\frac{dx}{dt} = Ax + Bu, \tag{4.2}$$

where the matrices A and B are given by

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3w^2 & 0 & 0 & 2w \\ 0 & 0 & 0 & 1 \\ 0 & -w & 0 & 0 \end{bmatrix},\ B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

$u(t)$ is called a control with $x(t)$ as the trajectory corresponding to this control.
We denote the initial and final times of the trajectory by t_0 and t_f, respectively. Then, Problem 4.1 is restated for the linear system (4.2) as under.

Problem 4.2. Given any pair $(x_0, x_f) \in \Re^n \times \Re^n$ of initial and final state, find a control u^* of minimum norm, which will steer the trajectory from the initial state to final state.
We denote the transition matrix of the linear dynamical system (4.2) by $\phi(t,s) = e^{A(t-s)}$. The solution of this system is represented as

$$x(t) = \phi(t,t_0)x_0 + \int_{t_0}^{t} \phi(t,s)Bu(s)ds. \tag{4.3}$$

Evaluating (4.3) at $t = t_f$, we get

$$z_f = [x_f - \phi(t_f,t_0)x_0] = \int_{t_0}^{t_f} \phi(t_f,s)Bu(s)ds. \tag{4.4}$$

Define the operator $L : L_2[t_0,t_f] \to \Re^n$:

$$Lu = \int_{t_0}^{t_f} \phi(t_f,s)Bu(s)ds. \tag{4.5}$$

If L^* denote the dual operator of L. Then L^* maps $\Re^n$ to $X = L_2[t_0,t_f]$. If we further assume that (LL^*) is invertible, then (4.4) has infinitely many solutions for a given $z_f = [x_f - \phi(t_f,t_0)x_0] \in \Re^n$.
Our final reformulated control problem is described as under:
Problem 4.3. Find $u^* \in L_2$ such that:

$$||u^*|| = \inf_{u \in L_2 : Lu = z_f} ||u||.$$

We invoke the theory developed earlier for the solution of the dual problem defined by (2.3). The defining operator L under consideration is given by (4.5). It is not difficult to see that dual problem theory goes through. Hence the solution u^* of Problem 4.3 is given by

$$u^* = L^*[LL^*]^{-1}z_f. \tag{4.6}$$

Note that LL^* maps $\Re^n$ to $\Re^n$ and hence we denote it by a matrix $[w(t_0),t_f]$. This matrix is represented as

$$w(t_0,t_f) = \int_{t_0}^{t_f} \phi(t_f,s)BB^T\phi^T(t_f,s)ds,$$

and is called controllability Grammian of the above dynamical linear system. The functional representation of control is then given by

$$u^*(t) = B^\top \phi(t_f,t)[w(t_0,t_f)]^{-1}[x_f - \phi(t_f,t_0)x_0].$$

We note that the above representation is actually used in computing the booster controls (u_1, u_2).

2.5 Avatar as Optimal Estimator-Probabilistic

Let $X_1, X_2, \ldots, X_n$ and Y be random variables with zero mean and with finite second moment. It is possible to observe the random variables $\{X_i\}$ but not Y.

Problem 5.1. Predict Y by a linear combination of the random variables $X_1, X_2, \ldots, X_n$.
For our purpose, we set up the Hilbert space H as the space $L_2(\Omega), (\Omega \subset \Re)$ of all square integrable random variables (finite second order moment). We assume that $M = L[X_1, X_2, \ldots, X_n]$. Note that in the space $L_2(\Omega)$, we have

$$||X||^2 = E(X^2);\ \langle X, Y\rangle = E[XY].$$

Definition 5.1. We define the best approximate $\hat{Y}$ of Y in the closed subspace M of $H = L_2(\Omega)$ as the optimal estimator (predictor) of Y, given the random variables $X_1, X_2, \ldots, X_n$.
We shall make use of the following theorem, which is a restatement of Theorem 2.2.

Theorem 5.2. Let H be a Hilbert space and M a closed subspace of H with $z \in H$. Let $u \in M$ be the best approximate of z. Denote the correspondence $z \mapsto u$ by an operator P. Then the following properties hold:

(i) P is linear with $P^2 = P$.

(ii) $v = z - Pz \in M^{\perp}, \forall\, z \in H$.

(iii) $\langle Pz, w\rangle = \langle z, Pw\rangle = \langle Pz, Pw\rangle, \forall\, z, w \in H$.

(iv) Each $z \in H$ has a unique representation of the form $z = u + v, u \in M, v \in M^{\top}$.

(v) The predicted Error $||z - Pz||$ is of the form: $||z - Pz||^2 = ||z||^2 - ||Pz||^2$.

We proceed as under.

$$\hat{Y} = a_1X_1 + \cdots + a_nX_n;\ a_i \in \Re. \tag{5.1}$$

The computation of $\hat{Y}$ will be done by using property (ii) of Theorem 5.2: $\hat{Y}$ is the best estimator of Y if $(Y - \hat{Y})\perp M$. This gives the following linear system in unknowns $\{a_i\}_{i=1}^n$.

$$\begin{aligned}
\langle Y, X_1\rangle &= a_1\langle X_1, X_1\rangle + a_2\langle X_2, X_1\rangle + \cdots + a_n\langle X_n, X_1\rangle \\
\langle Y, X_2\rangle &= a_1\langle X_1, X_2\rangle + a_2\langle X_2, X_2\rangle + \cdots + a_n\langle X_n, X_2\rangle \\
&\vdots \\
\langle Y, X_n\rangle &= a_1\langle X_1, X_n\rangle + a_2\langle X_2, X_n\rangle + \cdots + a_n\langle X_n, X_n\rangle.
\end{aligned}$$

We use the following notations:

$$\sigma_{xx} = Cov(X_i, X_j), \sigma_{YX} = Cov(Y, X_j), a = (a_j).$$

If we assume that the covariance matrix Σ_{XX} is invertible, the solution of the above linear system is given by $a = \Sigma_{XX}^{-1}\Sigma_{YX}$. Using this representation of a in (5.1), we get the optimal estimator of $\hat{Y}$ of Y as under:

$$\hat{Y} = \Sigma_{YX}^{\top}\Sigma_{XX}^{-1}X, X = (X_j). \tag{5.2}$$

By using property (v) of Theorem 5.2, the predicted error is given by

$$||Y - \hat{Y}||^2 = ||Y||^2 - ||\hat{Y}||^2.$$

The estimator norm $||\hat{Y}||^2 = ||PY||^2$ is computed as

$$\begin{aligned} ||\hat{Y}||^2 = ||PY||^2 &= \langle PY, PY\rangle = \langle Y, P^2Y\rangle = \langle Y, PY\rangle \\ &= \langle Y, \hat{Y}\rangle = \langle Y, \Sigma_{YX}^\top \Sigma_{XX}^{-1} X\rangle \\ &= \langle \Sigma_{XX}^{-1}\Sigma_{YX}, \Sigma_{YX}\rangle_{\Re^n}. \end{aligned}$$

So, for the general case of random variables - not necessarily with zero mean - we summarize the result concerning the optimal predictor in terms of the following theorem.

Theorem 5.5. Let $Y, X_1, X_2, \ldots, X_n$ be random variables with finite second moment and let M be the linear span of $X_1, \ldots, X_n$. Then the optimal predictor $\hat{Y}$ of Y in the linear span M is given by

$$\hat{Y} = E[Y] + \Sigma_{YX}^\top \Sigma_{XX}^{-1}[X - E[X]].$$

Further, this optimal predictor $\hat{Y}$ is unique and it satisfies the following properties:

(i) Unbiased Estimator: $E[Y - \hat{Y}] = 0$.

(ii) Orthogonality of the prediction error: $E[(Y - \hat{Y})X] = 0, X = (X_i)$.

(iii) Predicted Error:

$$||Y - \hat{Y}|| = Var(Y) = \langle \Sigma_{XX}^{-1}\Sigma_{YX}, \Sigma_{YX}\rangle_{\Re^n}.$$

2.6 Avatar as Conditional Expectation

Let $H = L_2(\Omega)$ be as defined in the previous section and let $X \in H$ be a random variable. We consider the subspace of all functions $h(X)$ of X (with $h(X)$ measurable) having the finite second moment as our space M:

$$M = \left\{h(X) : [E[h(X)]]^2 < \infty\right\}.$$

Our main problem is stated as under:

Problem 6.1. Compute the optimal predictor $\hat{Y}$ of Y as a function of X (that is $\hat{Y} \in M$). Here the optimal predictor is the best approximate of Y in M.

As M is a closed subspace of $H = L_2(\Omega)$ the existence of optimal estimator is guarateed by Theorem 2.2. Further, by invoking the property (P), $\hat{Y}$ is the optimal estimator of Y if the following property holds:

$$(P_1) : (Y - \hat{Y}) \perp M.$$

We have the following key lemma, which restates the orthogonality property.

Lemma 6.1. $\hat{Y}$ is the optimal predictor of Y in M if

$$E[(Y - \hat{Y})Z] = 0 \quad \forall Z \in M.$$

The proof follows immediately from the above property (P_1).

This leads to the following theorem, connecting the optimal predictor with conditional expectation.

Theorem 6.1. Let X and Y be random variables on Ω such that $E[X^2], E[Y^2] < \infty$. The optimal predictor $\hat{Y}$ of Y in M is the conditional expectation $E[Y|X]$.

Proof: $\hat{Y} = E[Y|X]$. By Jenson's inequality, we have

$$\hat{Y}^2 = [E[Y|X]]^2 \leq E[Y^2|X]. \tag{6.1}$$

(6.1) gives

$$E[\hat{Y}^2] \leq E[E[Y^2|X]] = E[Y^2] < \infty \implies \hat{Y} \in M.$$

We are through if we show that $(Y - \hat{Y}) \in M^{\perp}$ (in view of the above stated lemma). Consider $E[(Y - \hat{Y})Z], Z \in M$. We have

$$\begin{aligned} E[(Y - \hat{Y})Z] &= E[E(Y - \hat{Y})Z|Z] \\ &= E[E(Y|Z)Z - \hat{Y}Z] \\ &= 0. \end{aligned}$$

This proves that $E[Y|X]$ is the best estimator of Y, in the space M.

2.7 Avatar in a Closed and Convex Subset of Hilbert Space - No Longer Nice

Problem 7.1. What happens to the best approximate if we have only a closed convex subset K in a Hilbert space H?
As pointed out in the remark in Theorem 2.2, we get the best approximate but not the orthogonality property.

Theorem 7.1. Let K be the closed convex subset of the Hilbert space H. Let $x \in H$. Then there exists a unique $u \in K$, called the best approximate of x, such that

$$||x - u|| = \inf_{w \in K}[||x - w||]. \tag{7.1}$$

Variational Analysis

Let P denote the projection operator assigning the unique approximate $u \in K$ corresponding to $x \in H$. The operator P need not be linear and hence its properties need to be investigated separately.
Let us do some calculus involving the norm functional $J : H \to \Re$,

$$J(w) = \frac{1}{2}||x - w||^2. \tag{7.2}$$

One can show that functional J is strictly convex, weakly lower semicontinuous and coercive ($||Jw|| \to \infty$ as $||w|| \to \infty$) and hence it has a unique minimizer $u \in K$. This proves Theorem 7.1. Further, one can show that J is differentiable. We denote the gradient of J at u by $\nabla J(u) \in H$. We obtain the following theorem for the minimizer of J in K, giving us the key variational inequality.

Theorem 7.2. $u \in K$ is the minimizer of J if the following inequality holds:

$$\langle \nabla J(u), u \rangle \leq \langle \nabla J(u), v \rangle, \forall v \in K. \tag{7.3}$$

Proof: Step 1

Assume that (7.3) holds. By convexity of J, we have,

$$J(u + t(v - u)) = J((1 - t)u + tv) \leq (1 - t)Ju + tJv, t \in (0, 1)$$

and hence,

$$\begin{aligned} \frac{J(u + t(v - u)) - Ju}{t} &\leq Jv - Ju \\ \Rightarrow \langle \nabla Ju, v - u \rangle &\leq Jv - Ju. \end{aligned}$$

In view of (7.3), the LHS of the above inequality is always non-negative and hence

$$Jv - Ju \geq 0, \forall v \in H.$$

$\Rightarrow u$ is a minimizer of J in K.

Step2:

If u is a minimizer of K, then it is easy to show that (7.3) holds. (Observe $J(u) \leq J((1t)u + tv) \forall v \in Ht \in (0,1)$). It is straight forward to compute the gradient $\bigtriangledown J$ of J at $u \in H$. It is given by

$$\langle J(u), v \rangle = \langle u - x, v \rangle \quad \forall v \in K. \tag{7.4}$$

Plugging-in the above representation in (7.3), we get the following characterization of the minimizer of J in K.

Theorem 7.3. The minimizer $u \in K$ of J is characterised by the following variational inequality

$$\langle x - u, u - v \rangle \geq 0 \quad \forall v \in K \tag{7.5}$$

Projection Operator - Properties

In view of (7.5), we have

$$\langle x - Px, Px - v \rangle \geq 0 \tag{7.6}$$

$$\langle y - Py, Py - v \rangle \geq 0 \tag{7.7}$$

$\forall \in K$.

Set $v = Py$ in (7.6) and $v = Px$ in (7.7) to get the following

$$\langle x - Px, Px - Py \rangle \geq 0$$

$$\langle y - Py, Py - Px \rangle \geq 0.$$

Combining these two inequalities, we get

$$\langle x - y - (Px - Py), Px - Py \rangle \geq 0.$$

So, the following final theorem emerges giving the full variational analysis of the Problem 7.1.

Theorem 7.4. Let H be a Hilbert space and let $K \subseteq H$ be a closed convex set. To each $x \in H$, there exists a unique element $Px = u \in K$, called the best approximate of x:

$$||x - u|| = \inf_{w \in K} ||x - w||.$$

1. The best approximate Px is characterised by the following variational inequality

$$\langle x - Px, Px - v \rangle \geq 0 \quad \forall v \in K. \tag{7.8}$$

2. The projection operator $P : H \to H$ is no longer linear but it is non-expansive and monotone:

$$||Px - Py|| \leq ||x - y|| \quad \forall x, y \in H \tag{7.9}$$

$$\langle x - y, Px - Py \rangle \geq ||Px - Py||^2 \quad \forall x, y \in H. \tag{7.10}$$

Note: For more on analysis in finite and infinite dimensional Hilbert spaces refer Krantz [5], Lax [6] and Limaye [7]. Fernandez-Cara and Zuazua [2], Klamka [4] and Sontag [9] discuss control theory in a nice way. Controls modelled by ordinary and partial differential equations are dealt with in Joshi [3] and Zuazua [10]. Athreya and Lahiri [1] and Nagel and Steyer [8] describe the probabilistic theory of optimal estimator in a detailed set up.

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3. Mohan Joshi, Ordinanry Differential Equations - Modern Perspective, Narosa Publishing House, 2007.
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9. Eduardo D. Sontag, Mathematical Control Theory: Deterministic Finite Dimensional Systems, Vol 6, Text in Applied Mathematics, Springer Verlag, 1998.
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□□□

Does the pursuit of truth give you as much pleasure as before? Surely it is not the knowing but the learning, not the possessing but the acquiring, not the being-there but the getting there that afford the greatest satisfaction. If I have exhausted something, I leave it in order to go again into the dark. Thus, is that insatiable man so strange: when he has completed a structure it is not in order to dwell in it comfortably, but to start another.
— ***Carl Friedrich Gauss***

We must admit with humility that, while number is purely a product of our minds, space has a reality outside our minds, so that we cannot completely prescribe its properties a priori.
— ***Carl Friedrich Gauss***

3. What is happening in the Mathematical world?

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3.1 Quest to build 'Spherical Cubes' completed

Is it possible to fill space "cubically" with shapes that are more spherical? A proof at the intersection of geometry and theoretical computer science says YES!!

In the fourth century, the Greek mathematician Pappus of Alexandria praised bees for their "geometrical forethought". The hexagonal structure of their honeycomb seemed like the optimal way to partition two-dimensional space into cells of equal area and minimal perimeter. For millennia, nobody could prove that hexagons were optimal - until finally, in 1999, Thomas Hales showed that no other shape could do better. Today, mathematicians still do not know which shapes can tile three or more dimensions with the smallest possible surface area. This "foam" problem has turned out to have wide-ranging applications - for physicists studying the behavior of soap bubbles (or foams) and chemists analyzing the structure of crystals, for mathematicians exploring sphere-packing arrangements and statisticians developing effective data-processing techniques.

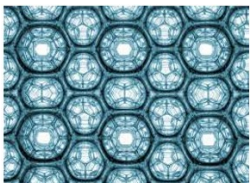


Figure 1

Among all shapes of a given volume, the one that minimizes surface area is the sphere. As the dimension n grows the sphere's surface area increases in proportion to the square root of n. But spheres can not tile a space without leaving gaps. On the other hand, an n-dimensional cube of unit volume can! The catch is that its surface area is $2n$, growing in direct proportion to its dimension. A 10,000-dimensional cube of volume 1 has a surface area of 20,000 - much bigger than 400, the approximate surface area of a 10,000-dimensional unit sphere. And so, researchers wondered if they could find a "spherical cube" - a shape that tiles n-dimensional space, like a cube, but whose surface area grows slowly, like that of the spheres.

Now, *Assaf Naor* (right) and *Oded Regev* (left), in their recent paper posted on the site arxive.org, proved the existence of spherical cubes in a geometrically natural way, and showed that one can partition the n-dimensional space along integer coordinates with a convex shape whose surface area is pretty close to that of the sphere. More specifically, they proved that for any $n \in N$ there is a convex body $K \subseteq R^n$ whose surface area is at most $n^{1/2+o(1)}$, yet the translates of K by the integer lattice Z^n tile R^n. And they did it entirely geometrically - returning the problem to its mathematical roots.

Sources:

1. *https://www.quantamagazine.org/mathematicians-complete-quest-to-build-spherical- cubes - 20230210/?mc_cid=f0ed562e28&mc_eid=df6d259cb7*
2. *Assaf Naor and Oded Regev: An Integer Parallelotope with Small Surface Area, arXiv:2301. 02862v1 [math.MG] 7 Jan 2023.*

3.2 Hidden patterns within 'chaotic' crowds discovered

A large crowd of people may seem chaotic. But as one surfs through the crowd, even in deepest moments of confusion and frustration, order may be found all around. Because, just like in nearly every corner of nature, mathematics is universal.

A new theory developed by a team of mathematicians at the University of Bath in the UK led by Professor Tim Rogers explains this phenomenon, and is able to predict when lanes will be curved or straight. The theory can even describe the tilt of a wonky lane when people are in the habit of passing on one side rather than the other (for instance, in a situation where they are often reminded to 'pass on the right').

This mathematical analysis unifies conflicting viewpoints on the origin of lane formation, and it reveals a new class of structures that in daily life may go unnoticed. The discovery, reported

recently (on March 3) in the journal *Science*, constitutes a major advance in the interdisciplinary science of 'active matter' – the study of group behaviours in interacting populations ranging in scale from bacteria to herds of animals. The Bath team used a new analytical approach, inspired by Albert Einstein's theory of Brownian motion, which makes predictions that can be tested.
To test their theory, the researchers asked a group of volunteers to walk across an experimental arena that mimicked different layouts, with changes to entrance and exit gates.

One arena was set up in the style of King's Cross Station in London. When the researchers looked at the video footage from the experiment, they observed mathematical patterns taking shape in real life.
Professor Rogers said: "At a glance, a crowd of pedestrians attempting to pass through two gates might seem disorderly but when you look more closely, you see the hidden structure. Depending on the layout of the space, you may observe either the classic straight lanes or more complex curved patterns such as ellipses, parabolas, and hyperbolas."
This study is likely to have implications for a range of scientific disciplines, particularly in the fields of physics and biology. Similar structures can also be formed by inanimate molecules, such as charged particles or organelles in a cell.
Encouraged by the way their theory agreed with the numerical simulations of colliding particles, they then teamed up with Professor Bogdan Bacik – an experimentalist from the Academy of Physical Education in Katowice, Poland – and ran a series of experiments (such as the one modelled on King's Cross) using human crowds.
Lead author Dr Karol Bacik said: "Lane formation doesn't require conscious thought – the participants of the experiment were not aware that they had arranged themselves into well-defined mathematical curves."
"The order emerges spontaneously when two groups with different objectives cross paths in a crowded space and try to avoid crashing into each other. The cumulative effect of lots of individual decisions inadvertently results in lanes forming."
Sources:

1. *https://www.fastcompany.com/90860429/mathematics-of-crowds-patterns-theory*
2. *https://www.sciencedaily.com/releases/2023/03/230303105331.htm*

3.3 Application of the Bernoulli process to gluing problem for 4D spheres

The École polytechnique fédérale de Lausanne (EPFL) mathematician *Prof. Nicolas Monod*, who leads the Ergodic and Geometric Group Theory research unit at EPFL, in collaboration with Purdue University, has settled a 30-year-old question about spheres and 4-dimensional spaces. The answer has been published in the mathematics journal *Inventiones mathematicae*. The results bring new light to the "Euler class", one of the most powerful tools to understand complicated spaces by cutting them into simpler pieces. It is named after Swiss mathematician Leonard Euler who was the first to consider the idea.
In 1958, Fields medalist John Milnor noticed a problem when trying to build spaces using only circles and two-dimensional surfaces: There was a limit on how complicated the Euler class can be in two dimensions. The observation mounted into an entire field of research in higher dimensions, and mathematicians quickly realized that Milnor's "complexity bound" did not apply for spaces in all dimensions.
A question that had remained open for decades was, what about gluing spheres on 4-dimensional spaces? Is there also here a limit on how they fit together? Gluing spheres on 4-dimensional spaces is a particularly important construction because this is precisely how the very first 'exotic

spheres' were constructed. Classical approaches of understanding spaces have been inadequate in solving this 4-dimensional question. So, the EPFL mathematicians turned to the Bernoulli process, named after Swiss mathematician Jacob Bernoulli, for inspiration. The Bernoulli process, which is a model of tossing coins, was combined with the study of spheres and the Euler class to finally solve the question.

As a game with a 50-50 chance to guess the right side of a coin, this might seem very simple, but its simplicity is misleading. "The Bernoulli process already includes many of the advanced features of probability theory when we set out to repeat the toss more and more often," says Monod. "In fact, the Central Limit Theorem–which is a kind of Law of Large Numbers–even tells us that this simple model can emulate many of the most complicated random phenomena of nature if we are willing to toss enough coins for long enough."

Earlier this year, Monod and his team published the discovery that Bernoulli's random coin games can help solve some difficult algebraic questions, very much non-random questions. This has now been combined with the study of spheres and the Euler class to finally solve the old question about 4-dimensional spaces. They confirmed that, there is no limit at all to the size of the Euler class for spheres in four dimensions.

Sources: *https://phys.org/news/2023-02-mathematicians-spheres-4d-spaces-scope.html*

3.4 Breakthrough on a question about union-closed families of sets

The union-closed conjecture is about sets. A collection, or family, of sets is said to be "union-closed" if the union of any two sets in the family belongs to the family.

Mathematicians talked about versions of the union-closed conjecture as far back as the 1960s, but it received its first formal statement in 1979, in a paper by Hungarian mathematician Péter Frankl. Frankl conjectured that 'if a family of sets is union-closed, it must have at least one element (or number) that appears in at least half the sets.' It was a natural threshold for two reasons. First, there are readily available examples of union-closed families in which all elements appear in exactly 50% of the sets, like the family of all subsets of a given finite set. And second, at the time Frankl made the conjecture no one had ever produced an example of a union-closed family in which the conjecture did not hold. So, 50% seemed like the right prediction. That did not mean it was easy to prove. In the years since Frankl's paper, there have been only a few results.

Now, *Justin Gilmer* of Google has made a breakthrough in this problem using the tools of information theory. Information theory got developed in the first half of the 20th century, most famously with Claude Shannon's 1948 paper, "A Mathematical Theory of Communication". The paper provided a precise way of calculating the amount of information needed to send a message, based on the amount of uncertainty around what exactly the message would say. Information theory appears often in combinatorics, an area of mathematics concerned with counting objects, which is what Gilmer had studied as a graduate student.

Gilmer's strategy was to imagine a union-closed family in which no element appeared in even 1% of all the sets – a counterexample that, if it really existed, would falsify Frankl's conjecture.

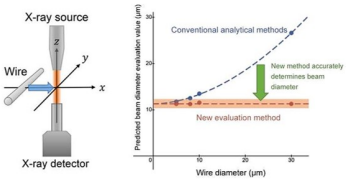
Gilmer deduced that if there is a union-closed family in which no element appears in at least 1% of all the sets, then there is more information in the union of two sets than in the sets themselves. But in general, the union must contain less information. Therefore there must be at least one element that appears in at least 1% of the sets.

When Gilmer posted his proof, he included a note that he thought it was possible to use his method to get even closer to the full conjecture, potentially raising the threshold to 38%.

Five days later, three different groups of mathematicians posted papers within hours of each other that built on Gilmer's work. Additional papers followed, but the initial burst seems to have taken

Gilmer's methods. For some of the authors of the follow-up papers, getting to 38% was relatively straightforward. However, getting to 50% will likely take new ideas.
Sources: *https://www.quantamagazine.org/long-out-of-math-an-ai-programmer-cracks-a-pure-math-problem-20230103/*

3.5 Evaluating the size of X-ray beams using new mathematical techniques



X-ray fluorescence (XRF) enables nondestructive analysis of elements present in samples, such as archeological samples, artworks, and electronics samples. Micro X-ray fluorescence spectrometry (micro-XRF) is conducted for elemental imaging and analysis of small regions, by focusing on primary X-rays. An appropriate beam diameter evaluation is essential as is directly related to the spatial resolution of micro-XRF imaging. The X-ray beam size was experimentally determined from the XRF intensity with respect to the scanning distance (thin-wire scanning method) or its differentiated curve (knife-edge scanning method). However, a beam diameter is not currently evaluated in a mathematically appropriate method and is known to differ significantly from the actual value under certain conditions. X-ray beam diameter can be calculated more accurately with the new evaluation method than with conventional methods used previously. The new method shows the same X-ray beam diameter, which remained consistent during this test, when using different wires with variable thickness.

Smaller the diameter of the X-ray microbeam, more accurate the elemental distribution can be. Because X-ray beams are not visible, an accurate method is needed to determine beam diameter. A research group led by *Prof. Kouichi Tsuji* of the Graduate School of Engineering at Osaka Metropolitan University has developed a new method to evaluate the diameter of X-ray microbeams based on mathematical analysis. While validating the new method, it was found that it can be used to calculate beam diameters more accurately than previously used conventional methods.
Currently, no uniform evaluation method for X-ray beam diameter has been established. Since this evaluation method was derived using mathematical analysis, it is expected to be widely adopted as an international standard. Potential applications of X-ray fluorescence analysis include materials development, environmental analysis, forensic science, biological sample analysis, and analysis of archaeological and culturally significant objects.
Sources:

1. *https://phys.org/news/2023-02-size-x-ray-successfully-mathematics.html*
2. *https://analyticalsciencejournals.onlinelibrary.wiley.com/doi/10.1002/xrs.332*

3.6 Researchers use AI to discover new planet outside the solar system

A University of Georgia research team has confirmed evidence of a previously unknown planet outside of our solar system, and they used machine learning tools to detect it. A recent study by the team showed that machine learning can correctly determine if an exoplanet is present by looking in protoplanetary disks, the gas around newly formed stars.
This is the initial step of AI in the world of astronomy and planet-hunting as it establishes itself as a potent tool across a range of fields.
"We confirmed the planet using traditional techniques, but our models directed us to run those simulations and showed us exactly where the planet might be," said Jason Terry, doctoral student in the UGA Franklin College of Arts and Sciences department of physics and astronomy, and lead

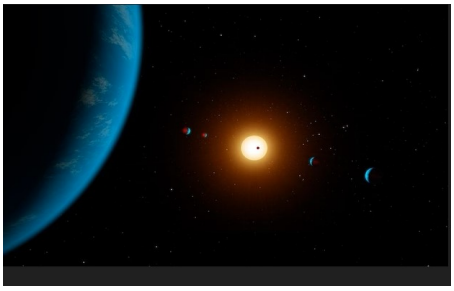


Figure 1: New processes underlying the formation and evolution of planetary systems.

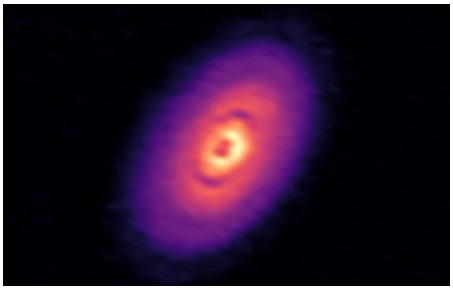


Figure 2: The models suggested a planet's presence in a specific region of the disk

author on the study. The new discovery adds to the growing catalogue of over 5000 exoplanets found outside the solar system.
The findings have been detailed in a study published in The Astrophysical Journal, which is found around a star dubbed HD 142666. Researchers said that AI and Machine Learning have revealed new processes underlying the formation and evolution of planetary systems.
The signal of the planet was discovered by the AI in the data that was previously analysed by astronomers, but the planet had gone unnoticed. The AI took just about an hour to comb through the data and find strong evidence for a new planet in a specific spot.
The model processed using AI suggested the presence of a planet, indicated by several images that strongly highlighted a particular region of the disk that turned out to have the characteristic sign of a planet – an unusual deviation in the velocity of the gas near the planet.
The discovery highlights how machine learning has the power to enhance scientists' work, utilizing artificial intelligence as an added tool to expand researchers' accuracy and more efficiently economize their time when engaged in such a vast endeavour as investigating deep outer space.
Sources:

1. *https://www.sciencedaily.com/releases/2023/04/230424133426.htm*
2. *https://www.indiatoday.in/science/story/ai-solar-system-exoplanet-discovery-machine- learning -artificial-intelligence-2364927-2023-04-26*

3.7 Awards

3.7.1 Ingrid Daubechies awarded the 2023 Wolf Prize in Mathematics

Duke Professor *Ingrid Daubechies* whose work is of tremendous importance in image compression, medical imaging, remote sensing, and digital photography has been awarded the 2023 Wolf Prize in Mathematics. Along with the Abel Prize and the Fields Medal, the Wolf Prize is considered one of the most prestigious awards in mathematics.

Ingrid Daubechies, Professor of Mathematics and Electrical and Computer Engineering, was named a recipient of the $100,000 prize for her work in wavelet theory and applied harmonic analysis. Daubechies has also made unparalleled contributions to developing real world applications of harmonic analysis, introducing sophisticated image processing techniques to fields ranging from art to evolutionary biology and beyond.
Born in a small coal mining town in Belgium, Daubechies got her start in research doing physics, not mathematics. It was after earning a bachelor's degree in theoretical physics and then a Ph.D. in quantum mechanics, both from the Free University of Brussels, that she got involved in applications of mathematics beyond physics. She was particularly interested in using Fourier transformation, which breaks down complicated signals that vary in time or space, such as sound waves or images, into simpler building blocks.

After earning her doctorate in 1980, she became one of the world's leading authorities on wavelet theory, a refinement of the Fourier technique frequently used to shrink digital photos and movies so that they take up fewer kilobytes without noticeably losing detail. In the mid-1990s, her work on wavelets was adopted by the FBI to squeeze down their hundreds of millions of fingerprints to make them easier to transmit and store.

Daubechies has also used wavelets to help geologists analyze the wiggly lines on seismograms from huge earthquakes and reconstruct what lies below the Earth's crust, and to help neuroscientists read MRI images of brain activity. In 2007, Daubechies and her team used wavelets and machine learning to identify subtle differences in brush strokes and distinguished copies or forgeries from true Vincent van Goghs. For that they analyzed more than 100 high-resolution scans of paintings, mostly by Vincent van Gogh. Thus began an ongoing collaboration between mathematicians, computer scientists, museum curators and art historians. Their algorithms are now being used to compare artists' styles, pinpoint when something was painted, even mathematically restore artwork that has cracked, faded, or been reduced to rubble by wartime bombing - all without laying a finger on the actual art.

Daubechies held positions at Bell Laboratories, the University of Michigan, Rutgers University and Princeton University before joining the Duke faculty in 2011.

Sources: *https://today.duke.edu/2023/02/duke-professor-wins-one-most-prestigious-awards- mathematics*

3.7.2 Prof. Mahesh Kakde has been awarded Infosys Prize 2022

The Infosys Science Foundation (ISF) awarded the Infosys Prize 2022 to researchers in six categories for their contributions to research. The winners, across the categories such as Engineering and Computer Science, Humanities, Life Sciences, Mathematical Sciences, Physical Sciences and Social Sciences, were felicitated with gold medal, a citation, and a prize purse of USD 1,00,000.

Prof. Mahesh Kakde, Professor of Mathematics, Indian Institute of Science, Bengaluru was awarded the Infosys Prize 2022 for his outstanding contributions to algebraic number theory, specifically the non-commutative Iwasawa main conjecture. Prof. Kakde's deep work on the noncommutative Iwasawa main conjecture, his work on the Gross-Stark conjecture (with Samit Dasgupta and Kevin Ventullo), and his work on the Brumer-Stark conjecture (with Samit Dasgupta), resolves outstanding conjectures at the heart of modern number theory.

Mahesh Kakde was born on 13 July 1983 in Akola. He obtained a Bachelor of Mathematics degree at the Indian Statistical Institute in Bangalore in 2004, and a Certificate of Advanced Study in Mathematics at the University of Cambridge in 2005. He completed his Ph.D. under the supervision of John Coates at the University of Cambridge in 2008. He subsequently worked at Princeton University, University College London, and King's College London, before joining as Professor at the Indian Institute of Science in 2019.

In 2019, Kakde was awarded a Swarnajayanti Fellowship. Together with Samit Dasgupta, Kakde was one of the invited speakers at the International Congress of Mathematicians 2022, in which they gave a joint talk on their work on the Brumer-Stark conjecture.

Sources:

1. *https://www.infosysprize.org/laureates/2022/mahesh-kakde.html*
2. *https://en.wikipedia.org/wiki/Mahesh_Kakde*

3.7.3 Dr. Sati Mazumdar and Dr. J. V. Deshpande awarded IISA Lifetime Achievement Award for 2021

International Indian Statistical Association (IISA) has honoured Dr. Sati Mazumdar from University of Pittsburgh and Dr. J. V. Deshpande from Savitribai Phule Pune University by the IISA Lifetime Achievement Award for 2021!

Dr. Mazumdar received her bachelor's and master's degrees in Statistics from the University of Calcutta, India and her doctoral degree in 1967 in Statistics from Cornell University, Ithaca, NY, USA. She joined as a faculty member in the Department of Biostatistics, and the Graduate School of Public Health, University of Pittsburgh. With over 40 years of professional expertise, Dr. Mazumdar retired as a Professor Emeritus from the University of Pittsburgh in 2013. She also held a secondary Professorship in University of Pittsburgh Medical School.

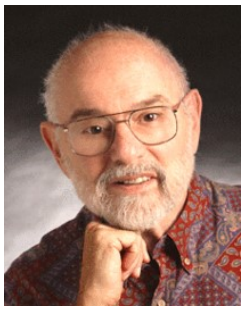
Dr. Mazumdar has served in various professional committees and panels of the University of Pittsburgh and US Government Agencies. She served as a member of an Advisory Committee of FDA (1983-1987). In 2007 Sati served on an expert panel organized by the Toxicology Excellence for Risk Assessment (TERA) of analysing correlation of toxicological effects with the polycyclic aromatic compound (PAC) content of petroleum substances.

Dr. J. V. Deshpande received his bachelor's degree from University of Patna and master's degrees in 1962 and Ph.D. in Statistics in 1965, both from Savitribai Phule Pune University. He started his meritorious teaching career in India in 1972, first at Panjab University, Chandigarh and then at University of Pune till his superannuation in 2002.

Prof. Deshpande was awarded Prof P. V. Sukhatme Lifetime Achievement Award by the Ministry of Statistics. He was elected a Fellow of the Institute of Mathematical Statistics in 1995 and a member of the Council of the International Statistical Institute in 2004. Professor Deshpande has co-authored three text books in the area of Survival Analysis and Nonparametric Statistics.

3.8 Obituary

3.8.1 Artificial Intelligence pioneer Dr. Stan Franklin passes away at the age of 91



American scientist *Prof. Stan Franklin*, a pioneer in Artificial Intelligence died on Jan. 23, 2023 at the age of 91. He was a Professor at the Univ. of Memphis in Memphis, Tennessee, and co-director of the Institute of Intelligent Systems. He was the author of Artificial Minds (MIT Press, 1995), and the developer of IDA and its successor LIDA, both computational implementations of Global Workspace Theory. He was founder of the Cognitive Computing Research Group at the University of Memphis.

Prof. Franklin was born in 1931. His graduate degrees are from UCLA, his undergraduate degree being from the University of Memphis. He has been on the faculties of the University of Florida, the Indian Institute of Technology Kanpur, Carnegie Mellon University, and the University of Memphis.

He was formerly a mathematician, who in 1965 introduced a new subfield of mathematics - Sequential Spaces. He then became first a computer scientist and then a cognitive scientist. Prof. Franklin has worked for some years on "conscious" software agents, which model the global workspace theory of consciousness. These autonomous agents computationally model human and animal cognition, and provide testable hypotheses for cognitive scientists and neuroscientists. This work was funded by the United States Navy and has been the subject of numerous papers in scientific journals and

conference proceedings. He created various software implementations related to AI that are used by the U. S. Navy, among others; a computer literacy program at FedEx that was one of the first of its kind in the nation. He wrote the landmark 1995 MIT Press book, "Artificial Minds", which at the time was the definitive guide to a growing field. Prof. Franklin and his Institute colleagues received about $100 million grant money. In 1987, he founded the Institute for Intelligent Systems as an interdisciplinary research center that united mathematicians, computer engineers, psychologists, linguists, neuroscientists and others in the pursuit of "cognitive science" (the study of the mind and its processes).

Sources:

1. *https://en.wikipedia.org/wiki/Stan_Franklin https://www.infosysprize.org/laureates/2022 / mahesh -kakde.html*
2. *https://www.commercialappeal.com/story/news/local/2023/01/26/stan-franklin-obit - dies - memphis -artificial-intelligence-institute-for-intelligent-systems/69843866007/*

3.8.2 Mathematician Norman Lebovitz, who unraveled the behaviors of stars and galaxies passes away at the age of 87

Norman Lebovitz, a professor emeritus of mathematics at the University of Chicago who worked to unravel the complex behaviors of stars and galaxies, died on Dec. 28, 2022 at the age of 87.

Born in 1935, Lebovitz grew up in Los Angeles and earned his BA from Univ. of Chicago in 1956 and received his Ph.D. in physics in 1961. His doctoral advisor was the famed astrophysicist Subrahmanyan Chandrasekhar, with whom he collaborated for decades. Lebovitz joined the faculty of the University of Chicago in 1963 as a founding member of the Applied Mathematics Program, which he chaired for many years.

Though a mathematician, Lebovitz was interested in mathematics as a way to understand the rules of the universe around us. He conducted mathematical analysis of the equilibrium and stability of rotating self-gravitating fluids that provide simple models of planets and stars. He worked closely with Chandrasekhar on "Ellipsoidal Figures of Equilibrium," a treatise on rotating liquid masses published in 1969, and became known as the world's expert on self-gravitating rotating fluid objects.

Lebovitz worked on applications of the Virial theorem for astrophysics, dynamical systems, differential equations, asymptotics, and Hamiltonian methods. Later in his career, he also became interested in the perturbations in flowing fluids that lead to turbulence. Lebovitz published more than 70 scientific papers and edited two books during his career.

Lebovitz was a fellow of the Alfred P. Sloan Foundation from 1967-69 and the John Simon Guggenheim Foundation from 1977-78. From 1983-1988 he was the managing editor of SIAM Journal on Applied Mathematics. In 2012, he was elected as a Fellow of the American Physical Society.

Sources: *https://news.uchicago.edu/story/norman-lebovitz-uchicago-mathematician-who- unraveled -behaviors-stars-and-galaxies-1935-2022*

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"Mathematics is the queen of sciences and number theory is the queen of mathematics. She often condescends to render service to astronomy and other natural sciences, but in all relations, she is entitled to the first rank." — ***Carl Friedrich Gauss***

4. Reminiscences of Vera T Sós (11 September 1930 - 22 March 2023)

Ambat Vijayakumar
Emeritus Professor, Department of Mathematics, Cochin University of Science and Technology
Cochin - 682 022, India. Email: vambat@gmail.com

Vera Turán Sós, a renowned number theorist and combinatorics expert breathed her last on 22nd March 2023 at her birth place Budapest, Hungary. Born as a daughter of a school teacher, she had her basic studies at the Abonyi Street Jewish high school in Budapest. She later graduated from the prestigious Eötvös Loránd University in 1952 with Physics and Mathematics majors. She married her collegemate Paul Turán, who later made significant contributions to extremal graph theory, in the same year and is blessed with two children, Gyorgy and Thomas.

Vera T. Sós taught at ELTE till 1987, laying the foundation for the regular teaching of combinatorics in Hungary. She later joined the Mathematical Research Institute of Hungarian Academy of Sciences (renamed as Alfred Renyi Institute of Mathematics in 1999). She had the good fortune of being introduced to the legendary Hungarian mathematicians Paul Erdős and Alfréd Rényi, at a very young age, which later resulted in several joint research papers, one of which is the proof of the Friendship Theorem (1966). Informally, it states that, if a group of people has the property that every pair of people has exactly one friend in common, then there must be one person who is a friend to all the others. Formally it states that the finite graphs with the property that every two vertices have exactly one neighbour in common are exactly the friendship graphs. A friendship graph (Windmill graph) can be constructed by joining n copies of the cycle graph C_3 with a common vertex, which becomes a universal vertex for the graph. Another contribution is the Kővári-Sós-Turán theorem in extremal graph theory concerning the maximum possible number of edges in a bipartite graph that does not contain certain complete subgraphs.

Sós has received the Academy Award, the Széchenyi Award, the Order of Merit of the Republic of Hungary, and the Academy Gold Medal. In 1999, the Austrian Academy of Sciences elected her as a corresponding member, and the Academia Europaea accepted her among its members in 2013. In 2018, she was awarded an honorary doctorate by the Hebrew University in Jerusalem.

I shall conclude this note with some personal reminiscences.

I first met Prof. Vera T. Sós in October 1994 during my first visit to the Renyi Institute, Budapest, under the cultural exchange program of the UGC, India. She welcomed me with great warmth and took me around the institute. In the library, there were multiple copies of the same volume of some journals in Combinatorics. Some faculty members were in its Editorial board and had the privilege of getting complimentary copies of the journal. I was quite excited to see my name printed in the prestigious journal 'Discrete Mathematics' Volume 131, 1994 along with my first Ph.D. student B. Radhakrishnan Nair in our first international publication entitled 'About triangles in a graph and its complement'. She immediately gave me a copy of that issue and some books in Combinatorics edited by Laslo Lovasz etc. and shared my excitement.

She attended my talk entitled 'A Conjecture of Kotzig on self-complementary graphs' based on our work . It was an unforgettable moment in my academic life.

The certificate of my visit and the talk, signed by her is still kept with me as a precious document.

I met her again at ICM Madrid in 2006 along with Prof. Laslo Lovasz who served as the President of IMU from 2007.

I pay my sincere tributes to this great lady who inspired me throughout my academic career.

□□□

5. Problem Corner

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In the January 2023 issue of TMC Bulletin, we posed a problem from Number Theory for our readers. We have received two solutions, one by Shresth Suraiya of Class 9 of Dhirubhai Ambani International School, Mumbai and another by Prof. J. N. Salunke from Latur. We present these solutions here.

Also, in this issue we pose a problem on Number Theory, which is proposed by Prof. Rajendra Pawale, Mumbai University, Mumbai, for our readers. Readers are invited to email their solutions to Udayan Prajapati (udayan64@yahoo.com), Coordinator, Problem Corner, before 15th June, 2023. The most innovative solution will be published in the subsequent issue of the bulletin.

Problem posed in the previous issue of the Bulletin:

Show that there is no positive integer n such that $(1000^n - 1)$ divides $(2023^n - 1)$.

Solution 1 by Shresth Suraiya: For any prime p and any nonzero integer n, we define $v_p(n)$ as the exponent of the highest power of p that divides n. (For example, $v_2(504) = 3$ and $v_5(-16) = 0$).

Some elementary properties of $v_p(n)$: Let a, b be integers such that $ab \neq 0$, then:

(i) If $a|b$, then $v_p(a) \leq v_p(b)$ for all primes p.

(ii) $v_p(ab) = v_p(a) + v_p(b)$ for all primes p.

We will prove that: For any positive integer n, $v_3(1000^n - 1) - v_3(2023^n - 1) = 2$, which shows that there exists a prime $p = 3$ such that, $v_3(1000^n - 1) > v_3(2023^n - 1)$, which sussequently proves the required result by (i).

Proposition: For any positive integer n, $v_3(1000^n - 1) - v_3(2023^n - 1) = 2$. ..... (1)

Proof : Note that:

$$(1000^n - 1) = 999(1000^{n-1} + 1000^{n-2} + \cdots + 1000 + 1), \qquad \ldots\ldots\ldots (2)$$

$$\text{and } (2023^n - 1) = 2022(2023^{n-1} + 2023^{n-2} + \cdots + 2023 + 1). \qquad \ldots\ldots\ldots (3)$$

Hence $999|(1000^n - 1)$ and $2022|(2023^n - 1)$.

Thus $v_3(1000^n - 1) \geq v_3(999) = 3$ and $v_3(2023^n - 1) \geq v_3(2022) = 1$.

Case 1 - $3 \nmid n$: We show that $v_3(1000^n - 1) = 3$ and $v_3(2023^n - 1) = 1$.

Consider the second factor of (2).

$$1000^{n-1} + 1000^{n-2} + \cdots + 1000 + 1 \equiv \underbrace{1 + 1 + \cdots + 1 + 1}_{n \text{ times}} \pmod 3 \equiv n \pmod 3$$

So $3 \nmid (1000^{n-1} + 1000^{n-2} + \cdots + 1000 + 1)$. Hence $v_p(1000^n - 1) = 3$.

Similarly, since for any positive integer r, $2023^r = (2022 + 1)^r \equiv 1 \pmod 3$,

$$(2023^{n-1} + 2023^{n-2} + \cdots + 2023 + 1) \equiv \underbrace{1 + 1 + \cdots + 1 + 1}_{n \text{ times}} \pmod 3 \equiv n \pmod 3$$

Hence $3 \nmid (2023^{n-1} + 2023^{n-2} + \cdots 2023 + 1)$. Thus $v_p(2023^n - 1) = 1$ by (3),

which proves (1) in this case.

Case 2 - $3|n$: First we will show:

If the equation $v_3(1000^n - 1) - v_3(2023^n - 1) = 2$ holds for $n = k$, then it holds for $n = 3k$. ...(4)

Computing $v_3(2023^{3k} - 1)$: Let $v_3(2023^k - 1) = t$. Hence $v_3(1000^k - 1) = t + 2$.

Now, $2023^{3k} - 1 = (2023^k - 1)(2023^{2k} + 2023^k + 1)$.

We show that

$(2023^{2k} + 2023^k + 1) \equiv 3 \pmod{\ }9$ (which will give $v_3(2023^{2k} + 2023^k + 1) = 1$).

Note that

$$2023^{2k} - 4^k = (2023^2 - 4)(2023^{2(k-1)} + \cdots) \equiv 0 \pmod 9,$$

$$2023^k - 7^k = (2023 - 7)(2023^{(k-1)} + \cdots) \equiv 0 \pmod 9.$$

Hence $2023^{2k} + 2023^k + 1 \equiv 4^k + 7^k + 1 \pmod 9$. .....(5)

Now, we show that $1^k + 4^k + 7^k \equiv 3 \pmod 9$, using induction.

The base case $(k = 1)$ is true.

Suppose the result holds for $n = m$. Then for $n = m + 1$, observe that

$$(1^{m+1} + 4^{m+1} + 7^{m+1}) - (1^m + 4^m + 7^m) \equiv 3(4^m + 2 \cdot 7^m) \equiv 3(4^m - 7^m) \equiv 0 \pmod 9$$

Therefore, $1^{m+1} + 4^{m+1} + 7^{m+1} \equiv 1^m + 4^m + 7^m \equiv 3 \pmod 9$ by induction hypothesis.

Hence the result holds for any positive integer k.

Thus by (5), $(2023^{2k} + 2023^k + 1) \equiv 3 \pmod 9$.

This gives $v_3(2023^{3k} - 1) = v_3(2023^k - 1) + v_3(2023^{2k} + 2023^k + 1) = t + 1$. .....(6)

Computing $v_3(1000^{3k} - 1)$: $(1000^{3k} - 1) = (1000^k - 1)(1000^{2k} + 1000k + 1)$.

However, $1000^{2k} + 1000^k + 1 \equiv 1 + 1 + 1 \pmod 9 \equiv 3 \pmod 9$.

Hence $v_3(1000^{2k} + 1000^k + 1) = 1$. And by assumption $v_3(1000^k - 1) = t + 2$.

So, $v_3(1000^{3k} - 1) = v_3(1000^k - 1) + v_3(1000^{2k} + 1000^k + 1) = t + 2 + 1 = t + 3$. .....(7)

Combining (6) and (7), we get

$v_3(1000^{3k} - 1) - v_3(2023^{3k} - 1) = (t + 3) - (t + 1) = 2$. This proves (4).

If $3|n$ then $n = 3^l m$, where $3 \nmid m$. Then the proposition holds for m by Case 1 and then it holds for n by (4).

Solution 2 by Prof. J. N. Salunke:

We have to prove that for any $n \in \mathbb{N} = \{1, 2, 3, \cdots, \}$, $(1000^n - 1)$ does not divide $(2023^n - 1)$....(1)

We have $2023 = 7 \times 17^2$, so 7 does not divide $(2023^n - 1)$, for any positive integer n. ...(2)

(Otherwise $7|(2023^n - 1), 7|2023 \Rightarrow 7|2023^n - (2023^n - 1) = 1$, a contradiction).

Now $1001 = 143 \times 7$. So $1000 \equiv -1 \pmod 7$. In general $1000^n \equiv (-1)^n \pmod 7$. ...(3)

Hence $7|(1000^n - 1)$ for every even positive integer n.

So (1) is true for every even positive integer n. ...(*)

Now $111|(1000^n - 1), \forall\, n \in \mathbb{N}$. ...(4)

As $\gcd(2023,\ 111) = 1$ and $\Phi(111) = 72$, where Φ is Euler's totient function which counts the positive integers up to a given integer n that are relatively prime to n.

Hence by Euler's theorem, $2023^{72} \equiv 1 \pmod{111}$.

Let k be the least positive integer such that $2023^k \equiv 1 \pmod{111}$. Then $k|72$ i. e. $k \in \{1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, 72\}$.

Now $2023 \equiv 25 \pmod{111}$, since $2023 = 18 \times 111 + 25$; $2023^2 \equiv 625 \equiv 70 \pmod{111}$.

It is easy to check that, $2023^3 \equiv -26 \pmod{111}$, $2023^4 \equiv 16 \pmod{111}$, $2023^6 \equiv 10 \pmod{111}$, $2023^8 \equiv 34 \pmod{111}$, $2023^9 \equiv -38 \pmod{111}$, $2023^{12} \equiv -11 \pmod{111}$, $2023^{18} \equiv 1 \pmod{111}$.

[18 is the order of 2023 modulo 111.]

Hence $2023^{18n} \equiv 1 \pmod{111} \forall\, n \in \mathbb{N}$, and

$2023^n \equiv 1 \pmod{111}$ if and only if $n \in \{18k | k \in \mathbb{N}\}$. ...(5)

By (4) and (5); $111|(1000^n - 1)$ but 111 does not divide $(2023^n - 1) \forall\, n \in \{2k - 1 | k \in \mathbb{N}\}$; which implies that (1) is true for all odd positive integers n. ...(**)

By (*) and (**), (1) is true for every positive integer n.

Problem for this issue

Show that $3(3 + 2^{2+n} + 2^{2+2n})$ is a perfect square only for $n = 1$.

□□□

6. International Calendar of Mathematics Events

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May 2023

- May 29, 2023 - June 1, 2023, Algebraic and geometric methods of analysis, International scientific online conference. *imath.kiev.ua/ topology/conf/agma2023*
- May 31, 2023 - June 2, 2023, Aspects of Nonlinear Evolution, Conference in Honour of the 60th Birthday of Joachim Escher, Leibniz Universität Hannover, Germany. *www.math-conf.uni-hannover.de/escher60*

June 2023

- June 1-3, 2023, International Conference of Young Mathematicians, Institute of Mathematics of NAS of Ukraine (online), Kyiv, Ukraine. *www.imath.kiev.ua/ young/youngconf2023/index.php?lang=ua*
- June 4-7, 2023, ADMA-Pre-conference Workshop on Spectral Graph Theory, Vidyavardhaka College of Engineering, Mysuru. *https://vvce.in.ngrok.in/adma/*
- June 5-7, 2023, 2nd International E-Conference on Mathematical and Statistical Sciences, A Selcuk Meeting, Selcuk University, Konya, Turkey. *icomss23.selcuk.edu.tr/*
- June 8-10, 2023, International Conference on Discrete Mathematics (ICDM 2023), Vidyavardhaka College of Engineering, Mysuru. *https://vvce.in.ngrok.io/adma/*
- June 18-24, 2023, 10th Slovenian Conference on Graph Theory, Kranjska Gora, Slovenia. *sicgt.si*
- June 19-23, 2023, Mini-courses in Mathematical Analysis 2023, University of Padova, Padova, Italy. *events.math.unipd.it/minicorsi/*
- June 22-24, 2023, The 2023 International Conference of the Honam-Chungcheong Mathematical Societies, Jeonbuk National University, Jeonju city, Korea. *honammath.or.kr/Conference/ConferenceView.asp?AC=1&CODE=CS20230201&CpPage=#CONF*
- June 25, 2023 - July 01, 2023, 11th PhD Summer School in Discrete Mathematics, University of Primorska, Koper, Slovenia. *https://conferences.famnit.upr.si/event/29/*
- June 26-29, 2023, The 9th International Conference "Mathematics & IT: Research and Education (MITRE)", Moldova state University, Chisinau/Moldova. *mitre.usm.md*

July 2023

- July 2-8, 2023, International Conference "Mathematics in Armenia", Yerevan State University, Yerevan, Armenia. *mathconf.sci.am/*
- July 3-7, 2023, Conference on Theoretical and Computational Algebra 2023, Pocinho, in the Douro valley, Portugal. *sites.fct.unl.pt/tca2023/home*
- July 3-7, 2023, Conference on TOPOLOGY, University of Patras and University of Peloponnese, Nafpaktos/Greece. *www.lepantotopology.gr*
- July 4-6, 2023, 12th IMA International Conference on Modelling in Industrial Maintenance and Reliability (MIMAR), Nottingham, United Kingdom. *ima.org.uk/20581/12th-mimar/*
- July 5-9, 2023, Seventh International Conference of Mathematical Sciences (ICMS 2023), Maltepe University, Istanbul, Turkey. *www.maltepe.edu.tr/icms23*
- July 6-8, 2023, 2nd International Workshop: Constructive Mathematical Analysis, Selcuk University, Konya, Turkey. *//iwcma.selcuk.edu.tr/*
- July 8, 2023 - July 11, 2023, 9th International IFS and Contemporary Mathematics and Engineering Conference (IFSCOM2023), Tarsus University, Mersin/ TURKEY. *ifscom.com/*

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Professor C. R. Rao honoured with the International Prize in Statistics in 2023

The 2023 International Prize in Statistics, regarded as the equivalent of the Nobel Prize in Statistics, has been awarded to Calyampudi Radhakrishna Rao, an Indian-American mathematician and statistician.

The prize, established in 2016, is presented once every two years to an individual or team that has made significant contributions to science, technology, and human welfare through the use of Statistics. Professor Rao will be honoured with the prize, which carries a $80,000 cash award, at the International Statistical Institute World Statistics Congress in Ottawa, Canada, in July.

Prof. C. R. Rao was born in Karnataka, India on 10 September 1920. He received an M.A. in Mathematics from Andhra University in 1940 and an M.A. in Statistics from the Calcutta University in 1943. He obtained a Ph.D. degree at King's College in Cambridge University, under R. A. Fisher in 1948, and a D.Sc. degree, also from Cambridge University, in 1965.

Prof. Rao was affiliated with the Indian Statistical Institute, Calcutta, from 1942 to 1978, and served as its Director from July 1972 to June 1976. He also held several other important positions including Jawaharlal Nehru Professor and National Professor in India, University Professor at the University of Pittsburgh and Eberly Professor and Chair of Statistics and Director of the Centre for Multivariate Analysis at Pennsylvania State University.

In addition to being awarded the Padma Bhushan in 1968 and the Padma Vibhushan in 2001 by Government of India, Prof. Rao has been honoured by numerous awards which include SS Bhatnagar Prize in 1963, the Wilks Medal of the American Statistical Association in 1979, US National Medal of Science in 2002, International Mahalanobis Prize in 2003 of theInternational Statistical Institute, Srinivasa Ramanujan Medal in 2003 of theIndian National Science Academy, India Science Award 2010 presented by Government of India, Guy Medal in Gold in 2011 of the Royal Statistical Society, Sardar Patel Lifetime Achievement International Award (Sardar Ratna) in 2014. He was elected a Fellow of the Royal Society in 1967.

The American Statistical Association has described him as "a living legend whose work has influenced not just statistics, but has had far reaching implications for fields as varied as economics, genetics, anthropology, geology, national planning, demography, biometry, and medicine.

Sources:

1. https://currentaffairs.adda247.com/c-r-rao-wins-international-prize-in-statistics-2023/
2. https://en.wikipedia.org/wiki/C._R._Rao#cite_note-Padma_Awards-19
3. https://www.isical.ac.in › C_R_Rao-biography-F

Carl Friedrich Gauss (30 April 1777 - 23 Feb. 1855)

A German mathematician, physicist, astronomer, & statistician. A child prodigy in mathematics. Proved the fundamental theorem of algebra. Made important contributions to Number theory. Credited with inventing the FFT algorithm. Introduced the Gaussian gravitational constant & the method of least squares. Invented the heliotrope in 1821, magnetometer in 1833.

Abraham de Moivre (26 May 1667 - 27 Nov. 1754)

A French mathematician & statistician, known for de Moivre's formula and for his work on the normal distribution and probability theory. Wrote a book on Probability theory, The Doctrine of Chances. First to discover Binet's formula, an expression for the nth Fibonacci number. First to postulate the central limit theorem.

Alonzo Church (14 June 1903 - 11 Aug. 1995)

An American mathematician and logician who made major contributions to mathematical logic and the foundations of theoretical computer science. Best known for the lambda calculus, Church-Turing thesis, providing the undecidability of the Entscheidungsproblem, Frege-Church ontology, and the Church-Rosser theorem. Also worked on philosophy of language.

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