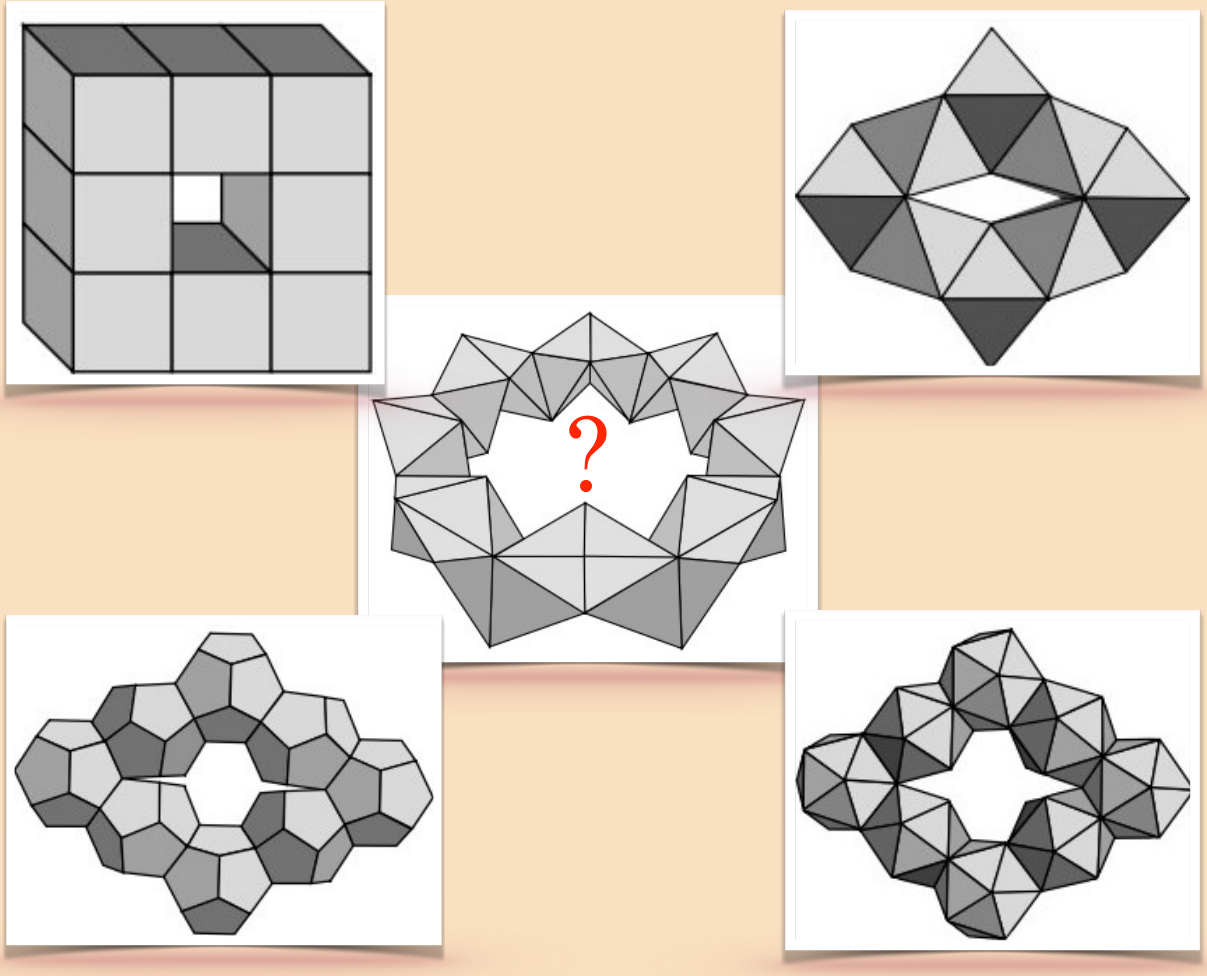
The Mathematics Consortium

BULLETIN

January 2023 | A TMC Publication | Vol. 4, Issue 3

RINGS OF REGULAR POLYHEDRA



Chief Editor: Ravindra S. Kulkarni

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The Mathematics Consortium

Bulletin

January 2023 Vol. 4, Issue 3

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Contents

1 Steinhaus's Problem on Chains of Regular Tetrahedra
Ian Stewart **1**

2 Prisoners and Hats Problem
Introducing the Binary Hamming Codes
Sharad S. Sane **9**

3 The Decimal Place-Value System
Part I : The Vedic Roots
S. G. Dani **14**

4 What is happening in the Mathematical world?
Devbhadra V. Shah **19**

5 Problem Corner
Udayan Prajapati **30**

6 International Calendar of Mathematics Events
Ramesh Kasilingam **33**

7 The 10th Heidelberg Laureate Forum (HLF)
T. R. Ramadas **34**

8 TMC Activities
S. A. Katre **35**

About the Cover Page: The figure on the front cover shows the closed rings of regular polyhedra, viz. cube, octahedron, dodecahedron, icosahedron and tetrahedron. Given a set of identical regular polyhedra, a challenge is to join them so that faces meet exactly and they form a closed ring. Question mark in the middle asks whether the feeling that one can form a tetrahedral ring is correct or not. This question is discussed in Article 1 of this issue.

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From the Editors' Desk

Padma Awards - the much coveted civilian Awards of the country, for the year 2023, were announced on the occasion of Republic Day the 26th January, 2023. These awards are conferred in three categories, namely, Padma Vibhushan, Padma Bhushan and Padma Shri, and are given for exceptional and distinguished service in various disciplines / fields of activities including science and engineering, and literature and education. We are happy to note that among the 106 awardees this year, there are three mathematicians, and one physicist who works in Statistical physics and Stochastic processes.

Awardees for the year 2023 in the discipline of Science and Engineering include Sathamangalam Ranga Iyengar Srinivasa Varadhan, USA - Padma Vibhushan; theoretical physicist Deepak Dhar - Padma Bhushan; and Sujatha Ramdorai, Canada, - Padma Shri. An Indian historian of mathematics Radha Charan Gupta is awarded Padma Shri in the discipline of literature and education. On behalf of The Mathematics Consortium, we pay our regards and congratulate these eminent personalities from the mathematics world.

A very short summary of the valuable contributions made by these mathematicians/physicists included on pages 8, 18 and 29 of this issue.

In Article 1, Professor Ian Stewart from the University of Warwick, UK, discusses the curious puzzle of tetrahedral rings posed by the Polish mathematician and educator Hugo Steinhaus. In simple words, given a set of identical regular solids, a challenge is to join them so that faces meet exactly and they form a closed ring. It is possible to form a ring of cubes, octahedra, dodecahedra, and icosahedra, using eight of them. However, in the article it is highlighted that this does not hold for identical tetrahedra.

In the second Article Prof. Sharad Sane presents the Prisoners and Hats Problem as a combinatorial game and connects it to the Binary Hamming Codes. He notes that despite the probabilistic formulation of the game, the framework in which a solution can be worked out is purely combinatorial (and deterministic).

Prof. S. G. Dani explores the Vedic roots of the Decimal Place-Value System in Article 3. He discusses how the fascination of the Vedic people for large numbers, and various choices they made in respect of counting, influenced and aided the development of the decimal place value system for representation of numbers that is universally adopted today.

In Article 4, Dr. D. V. Shah gives an account of significant developments in the Mathematical world during recent past. The article also includes brief write-ups on the winners of Breakthrough Prize in Mathematics, New Horizons prize in Mathematics, Maryam Mirzakhani New Frontiers Prizes and Steel prize, in 2023. Tributes to Mathematicians who passed away during the last quarter are included in the Obituary section of the article.

In the Problem Corner, Dr. Udayan Prajapati presents three solutions to the problem posed in the October 2022 issue. These solutions are given by Prof. M. R. Modak from Pune, Shrestha Suraiya, a student from Bhaskaracharya Pratishthana, Pune, and Prof. J. N. Salunke from Latur. A problem on Number theory is also posed for our readers.

Dr. Ramesh Kasilingam gives a calendar of Academic events, planned during February, 2023 to April, 2023 in Article 6. Prof. Ramdas gives information about 10th Heidelberg Laureate Forum in Article 7 and Prof. S. A. Katre gives an account of TMC activities in Article 8.

We are happy to bring out the third issue of Volume 4 in January 2023. We thank all the authors, all the editors, our designers Mrs. Prajkta Holkar and Dr. R. D. Holkar, and all those who have directly or indirectly helped us in bringing out this issue on time.

Chief Editor, TMC Bulletin

1. Steinhaus's Problem on Chains of Regular Tetrahedra

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1.1 Introduction

There are five regular solids: tetrahedron, cube, octahedron, dodecahedron, and icosahedron. If I give you a set of identical cubes, and challenge you to join them so that faces meet exactly and they form a closed ring, it won't take you long to realise that you can do it with eight cubes. Just make a 3×3 square and remove the one in the middle. It's also possible to form a ring using octahedra, dodecahedra, and icosahedra, and again eight of them will work; see Figure 1.

That leaves only the simplest regular solid, the tetrahedron. Can you form a closed ring using identical tetrahedra? You might like to think about this before moving on.

What about Figure 2? It looks convincing, but if you do the calculations, it turns out that there has to be a very small gap between two apparently adjacent faces.

The Polish mathematician and educator Hugo Steinhaus (1887-1972) was a student of David Hilbert, and he helped to set up the Lwów School of Mathematics at the Jan Kazimierz University in Lwo´w (now Lviv, Ukraine) in the 1920s. His interests were wide-ranging. Among them is the curious puzzle of tetrahedral rings. It first saw the light of day when Steinhaus published two questions in the Problems section of *Colloquium Mathematicum* [10]. Translated from the French, they are:

P 175. The image of a regular tetrahedron T_1 (fixed in Euclidean space of 3 dimensions) under reflection in one of its faces gives the tetrahedron T_2. Iteration gives rise to a chain of pairwise congruent tetrahedra T_n. Supposing that each face serves as a mirror only once, demonstrate that:

1. $m \neq n$ implies $T_m \neq T_n$,

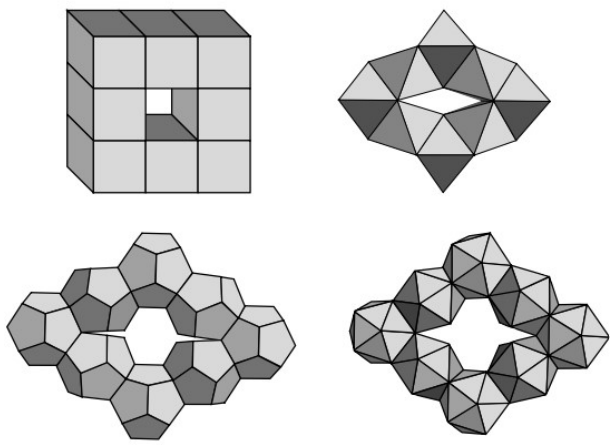

Figure 1: Closed chains of cubes, octahedra, dodecahedra, and icosahedra

2. Whatever the region R may be, there exists a chain of tetrahedra T_n such that the set of vertices is dense in R.

The problem's history involves the famous *Scottish Book*, a notebook of open mathematical problems compiled by regular visitors to the Scottish Café in what is now Lviv. An English version can be found at [16], and the original is available online as Volume 0 at [17]. When World War II ended, Steinhaus revived the book as the *New Scottish Book*, Volumes 1 and 2 at [17], and his problem is Problem 290 1.III.1956 from the *New Scottish Book*.

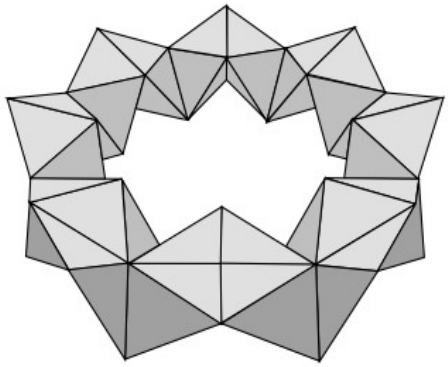

Figure 2: A closed chain of tetrahedra? No, there's a tiny gap.

1.1.1 Impossibility Proofs

Problem (2) is still open, but (1) was quickly answered by Stanisław Świerczkowski [12, 13]. It is equivalent to the statement that no nontrivial closed chain of regular tetrahedra, joined face to face, exists. This is striking, because Figure 1 shows that such chains exist for the other four regular polyhedra.

It turns out that the problem is 'really' one in group theory. Recall that a *group* is a set G with an operation that assigns to any two elements $g, h \in G$ a 'product' $gh \in G$, subject to some simple algebraic conditions. The product must be *associative*, meaning that $(gh)k = g(hk)$; there should be an *identity* 1 such that $1g = g1 = g$, and every element g should have an inverse g^{-1} such that $gg^{-1} = g^{-1}g = 1$. Examples are the *symmetry groups* of the regular solids, consisting of all rigid motions that leave the solid occupying the same space. The product operation is: Perform h and g in turn. The identity is 'leave the solid where it is'. The inverse of g is 'undo the rigid motion g'.

If we don't require the identity and inverses, we get a *semigroup*. All known approaches to Steinhaus's problem convert it into group theory. I want to tell you about a new approach, which converts the group-theoretic version of the problem into a question about a semigroup.

In [12] Świerczkowski wrote down two rotations in $\mathbb{R}^3$, and proved that they generate a free group on two generators. He noted that this result disproves the existence of a closed chain of regular tetrahedra, writing: 'This corollary gives a positive answer to a question of H. STEINHAUS…. However we shall not prove it here.' In [13] he explained the connection with chains of tetrahedra.

This group-theoretic result can also be used to prove the famous Banach-Tarski paradox [15]: A solid ball in R can be dissected into finitely many disjoint subsets, which can be fitted together via rigid motions to create two solid balls, each congruent to the original. (Doing this with a physical object would violate conservation laws, but the 'piece' concerned are infinitely complex and do not have well-defined volumes. So it's not possible to get rich by taking a gold sphere, cutting it up, and putting the bits together to get two of them, each the size of the original.)

Świerczkowski's two rotations are through the same angle, $\cos^{-1}(\frac{1}{3})$, and they have axes at right angles to each other. He defined them as 3×3 matrices acting on 3-dimensional space $\mathbb{R}^3$. By 'generate a free group' he meant that distinct sequences of these rotations and their inverses give distinct motions in space, except that a rotation followed by its inverse cancels out.

For example, if we write the rotations as R and S, then the sequence

$$RRSRS^{-1}R^{-1}RSSSR$$

cancels repeatedly to give

$$RRSRS^{-1}\mathbf{R^{-1}R}SSSR$$

$$RRSRS^{-1}SSSR$$

$$RRSR\mathbf{S^{-1}S}SSR$$

$$RRSRSSR$$

where the boldface symbols are those that cancel. No further cancellation is possible. Świerczkowski's result is that if two sequences, with all such cancellations removed, give the same rotation, they are the same sequence.

This would not be true, for example, if R were rotation through a right angle, because then $RRRRR$ would be the same rotation as R, but have a different sequence.

It is convenient to refer to sequences of symbols like those above as *words* in the *generators* R, S.

Several other authors have devised proofs of Świerczkowski's theorem, but all of them are complicated and not very transparent. Świerczkowski's proof uses induction on a sequence of integers determined by two matrices, and his main aim is to prove that these are not divisible by 3. Proofs by Dekker [4] and Mason [9] show that the group generated by reflections in the four planes through the origin parallel to the faces of the tetrahedron is isomorphic to a 'free product' $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$. This group consists of sequences of four rigid motions R_1, R_2, R_3, R_4 in which R_iR_i is replaced by the empty sequence (because doing the same reflection twice gives the identity transformation). Tomkowicz and Wagon [15] use 'barycentric coordinates' to represent the generating reflections as 4×4 matrices. The entries of products of these matrices are polynomials, evaluated at $\pm\frac{1}{3}, \pm\frac{2}{3}$. Again, the key step in an inductive proof involving divisibility by a power of 3.

In this article, I will sketch another proof. It also shows that the four reflections generate $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$. It uses ordinary Cartesian coordinates. The number 3 plays a key role, but the proof proceeds along different lines, and (aside from some routine calculations with 3×3 matrices) it is simple and easy to follow. It is based on a strange trick that replaces the group generated by the reflections with a closely related 'semigroup' — like a group but without the requirement that every element has an inverse.

1.2 Free Product Group Implies No Chain

It's fairly easy to see why the free product structure of the group implies that no closed chain of tetrahedra exists. Full details are in [11], and I'll omit some technical remarks.

Geometrically, the main interest is in an *embedded* chain, meaning that distinct tetrahedra don't overlap except where they meet at faces. The chains in Figure 1 are all embedded. This condition causes difficulties when trying to construct chains, but it's irrelevant to the impossibility proof. If no closed chain can exist, embedded or not, then no embedded chain can exist. So we can ignore this issue.

Świerczkowski's argument (modified to use reflections) is straightforward. Suppose, for a contradiction, that a closed chain exists. To add a new tetrahedron on one end, reflect the end tetrahedron in one of its faces. A sequence of such reflections (a word in the group) determines a unique chain of face-to-face tetrahedra, and a nontrivial sequence determines a nontrivial chain. If the chain joins up, the sequence must reduce to the identity transformation, but since the sequence is nontrivial, this is impossible.

1.2.1 Almost Closed Chains

In the absence of a closed chain, a natural question, also asked by Świerczkowski [14], arises: Can almost closed chains be formed, in the sense that the gap between the initial and final tetrahedra can be made as small as we please?

Elgersma and Wagon [5] show that the answer is 'yes' for non-embedded chains. The embedded case is more interesting. Elgersma and Wagon [6, 7] construct closed embedded chains with arbitrarily small gaps. They start with a Boerdijk-Coxeter helix [2, 3], which Fuller [8] calls a

tetrahelix. This is a chain of identical regular tetrahedra, whose vertices all lie on a fixed cylinder. They join four copies of a tetrahelix of suitable length into a diamond shape. If the length L is $q-1$ where p/q is a convergent of the continued fraction of $\theta = \cos^{-1}\frac{2}{3}/(2\pi)$, the resulting 'quadrahelix' is almost closed, and the size of the gap tends to zero as q increases. For example when $L = 601,944$ the gap has size 1.3×10^{-7}.

1.3 Reflections in Faces of the Tetrahedron

We represent the four reflections as matrices. It is convenient to translate the reflected tetrahedron so that the origin remains fixed. This does not affect the orientation of the tetrahedra in the chain: The translations form a 'normal subgroup' and can be 'factored out'; that is, ignored.

Choose a fixed reference tetrahedron $\Delta \subseteq \mathbb{R}^3$ with vertices

$$(1,1,1) \quad (1,-1,-1) \quad (-1,1,-1) \quad (-1,-1,1)$$

as in Figure 3.

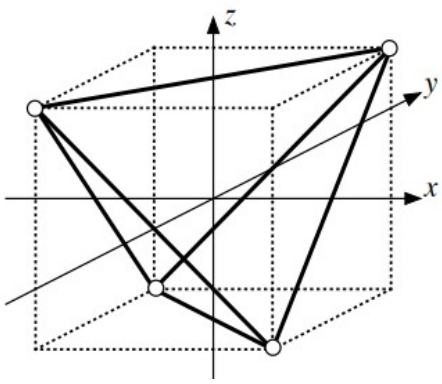

Figure 3: The cube $[-1,1]^3$ and the reference tetrahedron Δ.

1.3.1 Symmetries of the Tetrahedron

The eight points $(\pm 1, \pm 1, \pm 1)$ are the vertices of a cube. The symmetry group of the cube consists of all permutations of the coordinates (x, y, z) together with sign changes on any coordinate, so has order 48. The symmetry group $\mathrm{Sym}(\Delta)$ of the tetrahedron is the subgroup in which the number of minus signs is even, and has order 24. It is, of course, isomorphic to the symmetric group S_4, which permutes the four vertices. All matrices in the symmetry group $\mathrm{Sym}(\Delta)$ have integer entries.

1.3.2 The 3×3 Matrices

Next, we compute the four reflection matrices, which are:

$$R_0 = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix} \qquad R_1 = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

$$R_2 = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix} \qquad R_3 = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

1.4 Free Product Structure

We now give a simple, structural proof of the main result of Dekker [4] and Mason [9], which, as we have seen, implies that no closed tetrahedral chain exists.

Theorem 4.1 *The group $\mathcal{R}$ is isomorphic to the free product $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$.*

Our proof is based on a rather curious semigroup, and we discuss this first. To get rid of fractions, define $Q_i = 3R_i$. We can then reduce modulo 3, to get matrices

$$S_i = Q_i \pmod 3, \quad 0 \le i \le 3,$$

which are:

$$S_0 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \qquad S_1 = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$$

$$S_2 = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \qquad S_3 = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

Here the ± 1 lie in $\mathbb{Z}_3$, but in fact the calculations reported below also apply in $\mathbb{Z}$, except when the zero matrix arises and some entries may be multiples of 3.

The twelve products $S_iS_j (i \neq j)$ are:

$$S_0S_1 = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & -1 \end{bmatrix} \qquad S_0S_2 = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \qquad S_0S_3 = \begin{bmatrix} -1 & 1 & 1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

$$S_1S_0 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{bmatrix} \qquad S_1S_2 = \begin{bmatrix} -1 & 1 & -1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \qquad S_1S_3 = \begin{bmatrix} 1 & -1 & -1 \\ 1 & -1 & -1 \\ -1 & 1 & 1 \end{bmatrix}$$

$$S_2S_0 = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \qquad S_2S_1 = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix} \qquad S_2S_3 = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix}$$

$$S_3S_0 = \begin{bmatrix} -1 & -1 & -1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \qquad S_2S_1 = \begin{bmatrix} 1 & 1 & -1 \\ -1 & -1 & 1 \\ -1 & -1 & 1 \end{bmatrix} \qquad S_2S_3 = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ -1 & 1 & -1 \end{bmatrix}$$

Clearly $Q_1^2 = 9$. $R_i^2 = 9I$, so $S_i^2 = 0$. The sixteen matrices S_i and $S_iS_j (i \neq j)$ are distinct, and distinct from their negatives $-S_i$ and $-S_iS_j (i \neq j)$. Let $\mathcal{S}$ be the set of these 32 matrices. Case-by-case analysis shows that the S_i satisfy the following relations:

$$\begin{aligned} S_iS_jS_i &= S_i (i \neq j) \\ S_iS_jS_k &= -S_iS_k (i \neq j, i \neq k, j \neq k) \end{aligned} \tag{4.1}$$

(Using Sym(Δ)we can reduce this calculation to the special case $i = 0, j = 1, k = 2$.)

Let $\mathcal{S}$ be the set of all of the above 32 matrices together with the zero matrix, so that

$$\mathcal{S} = \left\{0, \pm S_i, \pm S_iS_j : 0 \le i, j \le 3, i \neq j\right\}.$$

Theorem 4.2 (a) *The set $\mathcal{S}$ is a semigroup.*
(b) *The product of two nonzero members of $\mathcal{S}$ is nonzero, except for the trivial cases*

$$S_iS_i = 0 \qquad S_i(S_iS_j) = 0 \qquad (S_iS_j)S_j = 0 \qquad (S_iS_j)(S_jS_k) = 0$$

and similar products involving minus signs.
Proof For (a) we must show that all products of nonzero elements of $\mathcal{S}$ lie in $\mathcal{S}$. For (b) we must also show these products are nonzero. Both follow from a case-by-case check.

For products S_iS_j this is clear. Products of the form $S_i(S_jS_k)$ and $(S_iS_j)S_k$ are taken care of directly by the relations (4.1). Those relations also imply that when $i \neq j, k \neq l$, we have

$$(S_iS_j).(S_kS_l) = \begin{cases} 0 & j = k \\ S_i & j \neq k, k \neq i \\ -S_iS_l & j \neq k, k \neq i \end{cases}$$

When $j = k$ the string $S_iS_jS_kS_l = S_iS_jS_jS_l$ is trivial.

In particular, when $i \neq j$,

$$(S_iS_j)^2 = (S_iS_jS_i)S_j = S_iS_j$$

which is nonzero, unlike squares of the S_i.

We are now ready to give the:
Proof of Theorem 4.1 The four free factors $\mathbb{Z}_2$ are generated respectively by R_0, R_1, R_2, R_3. We claim that the only relations between these generators are $R_i^2 = I$, where I is the identity.

Using the relations R_i^2 we can write any element $\gamma \epsilon \mathcal{R}$ in the form

$$\gamma = R_{i_k}R_{i_{k-1}} \dots R_{i_2}R_{i_1}$$

where $R_{i_j} \neq R_{i_{j+1}}$ for all $1 \leq j \leq k-1$. We claim this representation as a word is unique.

If not, some nontrivial word is equal to the identity I. The corresponding nontrivial word in the Q_i of length l is equal to $3^l I$. Modulo 3, this word becomes zero. Consider the corresponding word in the S_i, which is also nontrivial:

$$\omega = S_{i_k}S_{i_{k-1}} \dots S_{i_2}S_{i_1}$$

By Theorem 4.2, ω lies in $\mathcal{S} \setminus \{0\}$, so all of its entries are $\pm 1 \pmod 3$. Therefore $\omega \neq 0$, so no nontrivial word in the R_i can be the identity.
Remark 4.3 This trick has several curious features. First, we multiply the matrices by 3. Then we reduce modulo 3, which 'throws factors of 3 away'. Normally this would be pointless, reducing everything to zero, but here we start with fractions, not integers.

It is also unusual to study a problem in group theory by replacing the group by a semigroup. Usually, semigroups are less tractable, lacking inverses. Here the absence of inverses is crucial to the proof.

1.4.1 Non-Existence of a Closed Chain

As remarked earlier, Theorem 4.1 implies the non-existence of a nontrivial closed chain of regular tetrahedra. For completeness, we give details.
Theorem 4.4 *No nontrivial closed chain of tetrahedra exists.*
Proof Suppose, for a contradiction, that there is such a chain. Consider the corresponding product of reflection R_i in $\mathcal{R}$.

Because each reflection R_i fixes the origin, the construction of the chain of tetrahedra corresponding to a given element.

$$R_{j_k}R_{j_{k-1}} \dots R_{j_2}R_{j_1} \in \mathcal{S}$$

does not add successive tetrahedra to an otherwise stationary chain. Instead, the chain corresponding to $R_{j_{k-1}} \dots R_{j_2} R_{j_1}$ is reflected by R_{j_k}, and then translated by an appropriate amount so that it joins to the corresponding face of the reference tetrahedron Δ. Thus the chain at stage k has the structure

$$\Delta \to R_{j_k}\Delta \to R_{j_k}R_{j_{k-1}}\Delta \to R_{j_k}R_{j_{k-1}}R_{j_{k-2}}\Delta \to \cdots \to R_{j_k}R_{j_{k-1}} \dots R_{j_2}R_{j_1}\Delta$$

where the arrow indicates 'joins at a face'.

A necessary condition for the chain to close up is that some nontrivial product of reflections is a symmetry of the tetrahedron:

$$R_{j_k}R_{j_{k-1}} \dots R_{j_2}R_{j_1} = A$$

where $A \in Sym(\Delta)$. Then the corresponding nontrivial product in $\mathcal{S}$ satisfies

$$S_{j_k}S_{j_{k-1}} \dots S_{j_2}S_{j_1} = 3^{k+1}A = 0$$

since A has integer entries. This yields the same contradiction as in the proof of Theorem 4.1.

1.5 Related Problems

Świerczkowski's second question has so far resisted attack, but there are some simpler problems in this area that you might like to investigate. One is to form rings using other kinds of polyhedra, such as the *semiregular polyhedra*, whose faces can be different regular polygons. Another is to use more than one type of polyhedron. For example the tetrahedron, octahedron, and icosahedron all have triangular faces, which can be joined exactly. The semiregular polyhedra could be used here as well as the regular ones. You could also impose rules on which type of polyhedron joins to which. For example, can octahedra and icosahedra alternate round the ring?

Acknowledgements This article is based on the first part of [11]. I thank Stan Wagon and Staszek Janeczko for discussions about chains of tetrahedra and for helpful comments on a draft of that paper, and an anonymous reviewer for helpful comments.

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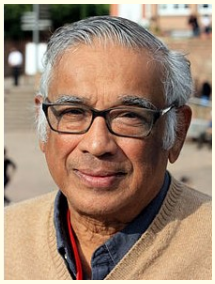
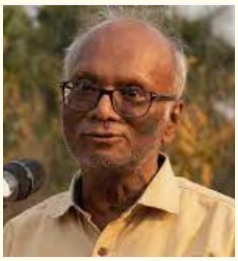
Padma Awardees -2023 from the Mathematics world



Padma Vibhushan awardee Sathamangalam Ranga Iyengar Srinivasa Varadhan FRS (born 1940) is an Indian American mathematician, currently a professor at the Courant Institute at New York University, USA. He is widely recognised as one of the most influential mathematicians of the 20th century. He is known for his fundamental contributions to probability theory and in particular for creating a unified theory of large deviations and to the theory of diffusion processes with an orientation towards the refinement and further development of Itô's stochastic calculus. In the year 2007, he became the first Asian to win the Abel Prize. He is also a recipient of the National Medal of Science, USA (2010), the Birkhoff Prize (1994), and the Leroy P Steele Prize for Seminal Contribution to Research (1996) from the American Mathematical Society.



Padma Bhushan awardee Deepak Dhar (born 1951) is an Indian theoretical physicist and is currently a distinguished professor at the Indian Institute of Science Education and Research, Pune. He is widely respected for his research on statistical physics and stochastic processes. In 2022, he became the first ever Indian scientist to be chosen for the Boltzmann Medal. He is also a winner of the TWAS prize (2002), and Shanti Swarup Bhatnagar Prize for Science and Technology (1991). He is also an elected fellow of The World Academy of Sciences. Dhar held various positions at TIFR for over three decades until his superannuation in 2016.

2. Prisoners and Hats Problem
Introducing the Binary Hamming Codes

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2.1 Prisoners and Hats Problem

The combinatorial game or puzzle that is presented in the article is a kind of mathematical game. This is somewhat in contrast with the usual idea of a mathematical game. Games like Nim (or even Fibonacci Nim) are totally deterministic. In fact, even the deep game of chess, is also a deterministic game. The game described in this article, despite being mathematical has a different flavour. The elegance of both, the game as also the solution, is that, despite being probabilistic in formulation, the framework in which a solution can be worked out is purely combinatorial (and deterministic)!

The problem is as follows. We have n prisoners each given to wear a hat of colour red or blue and a prisoner cannot see the colour of his own hat but can see the colour of every other hat. **All the 2^n possibilities are equally likely**.

At a given point of time, each prisoner has to either identify all the hat colours (including his own) correctly and has to declare himself as a **Volunteer** for doing so. If he does not volunteer, then he says **Pass**. All the prisoners are let off if **every volunteer is correct and there is at least one volunteer**. Otherwise, all the prisoners are simultaneously convicted. It is clear that no matter how best the strategy one adopts, there is no foolproof system (since the creator of the game can change colours) and hence a probability of $\frac{1}{2}$ is achievable.

As a simple observation (exercise) convince yourself that nothing better than a probability of half can be achieved if we have two prisoners. With 3 people a higher success rate is possible as the following argument shows. Really speaking, the volunteer (whosoever opts to be one) has to guess the colour of his hat correctly! There are now $2^3 = 8$ possibilities. By a simple pigeon-hole principle, two people must wear hats of the same colour say Red. Suppose that these are on the persons 2 and 3 presenting the first person (volunteer) the situation $|?|R|R|$. If he chooses $? = B$, then he fails only when the given sequence is |R|R|R|.

The Strategy then is as follows. Each prisoners looks at the other two hats. If they have different colours, he says **PASS**. If they have the same colour, he **Volunteers** and declares colour of his hat to be the opposite colour. We can then claim that this strategy will work with a higher percentage. In fact, if two persons are Red (respectively Blue) and the third is Blue (respectively Red), then exactly one person will volunteer (others will say pass) and he will also succeed!

When does the strategy fail? *The strategy will fail exactly when all the three hats are red (or all the three hats are blue).* Now every person will volunteer and will give a wrong answer! Fortunately, this can only arise in 2 out of 8 cases.

In conclusion that strategy has a success rate of $\frac{3}{4}$ which is higher than $\frac{1}{2}$!

What to do if $n \geq 4$? The strategy is simply to ask someone (many persons that are initially fixed) to keep shut and say PASS. This works with the same probability.

Now convert Red to 1 and Blue to 0. Then our strategy is something like the following. We look at the matrix H:

$$H = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \tag{2.1}$$

and every player tries to solve the equation H_x^t. Thus player 2 is trying to guess the question mark in x_1 ? x_3 based on his knowledge of x_1 and x_3.

2.2 Looking for a general solution and the binary world

Let $n = 7$. Then the problem of player number 5 is either to volunteer to guess a situation like

$$x_1x_2x_3x_4?x_6x_7$$

where if he does hazard volunteering then he replaces ? by either 1 or 0 (based on the knowledge of every x_i for $i \neq 5$). Of course, he can choose to PASS (but everyone should not PASS!).

Consider the following 3×7 binary matrix

$$H = H_3 = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} \tag{2.2}$$

We write our vectors (over the field $\mathbb{F}$ with 2 elements) as row vectors. This matrix has rank 3 hence has nullity $7 - 3 = 4$. Let C denote the null space of H. A vector

$$c = (c_1c_2c_3c_4c_5c_6c_7) \in C \Leftrightarrow Hc^t = 0 = (000)^t.$$

It follows at once that C is a linear subspace of $\mathbb{F}^7$ and C has dimension 4.

A (binary) word $v = (v_1v_2v_3v_4v_5v_6v_7)$ is an ordered 7-tuple (a binary sequence). The words that are in C are to be called codewords. Words can be added (all operations performed modulo 2). Also if x and y are two words then the (Hamming) distance between them is defined by:

$$d(x, y) = |\{i : x_i \neq y_i\}|$$

The Hamming distance d is a metric on the ser of all the words. Further, if we define the weight of a word x by:

$$w(x) = |\{i : x_i = 1\}|$$

then $d(x, y) = w(x + y)$. Clearly $w(x) = 0$ if and only if $x = 0$. The distance of C is the smallest non-zero distance among the codewords. Thanks to the fact that we have a linear code, this is also the (minimum) weight of a non-zero codeword, to be called weight of the code C. Notice that if $x \in \mathbb{F}^7$, then Hx^t is just a linear combination of the columns of the matrix H and in fact, it is the sum of precisely those columns i where $x_i = 1$. Since no columns of H is a zero column vector, it follows that C cannot have a codeword with weight 1. Can C have a codeword with weight 2? If there was such a codeword, then some two columns of H add to a zero column vector. But then those two columns must be identical which is not true. Finally, the first two columns add to give the third column and hence C has a codeword of weight 3.

We have thus established that the code C (called the Hamming Code) is a [7, 4, 3] code. Here, the length of C is 7, the dimension is 4 and the distance (or weight) is 3.

Let $c \in C$ be sent over a (noisy) channel and suppose y is received after transmission. For any vector $x \in \mathbb{F}^7$, *the syndrome* of y, denoted by S_y is defined to be the column vector H_y^t. The code C can correct upto a single error. There is a simple way of handling correction of upto one error. This works as follows. Suppose $c \in C$ was sent over the channel and was received as y. Then the error vector is $y - c = e$. Then $S_y = He^t$ since $Hc^t = 0$. Since the channel makes only upto one error, e has weight 0 when $y = c$ and if e has weight 1, then He^t is just a column of H, which can be identified and hence the error can be corrected. In fact correction here is really very easy: If H_y^t is the j-th column of H then the error has occured on the j-th coordinate and can simply be corrected by changing y_i (from 0 to 1 or 1 to 0).

The code has an additional property: Draw a ball of radius 1 at a codeword c say B_c. Then the number of words in B_c is $1 + \binom{7}{1} = 8 = 2^3$ and no two such balls can have a non-empty intersection: If they did, there would be a word whose distance from two codewords is at the most 1 and hence the two codewords themselves will have distance at the most 2, which is a contradiction. Since the

number of such balls is 2^4 with each ball containing 2^3 words, we see that $\mathbb{F}^7$ is a union of these balls that are pairwise disjoint (because $2^3 \times 2^4 = 2^7$).

A code C is perfect if the weight of C is an odd number d such that balls of radius e (that are pairwise disjoint) cover $\mathbb{F}^n$ (where $d = 2e+1$).

As we just saw the code $C = \mathbf{H}_3$ is a perfect code. It can correct upto a single error. As an example, observe that the sum of the first three columns of H is zero giving us the codeword $c = (1110000)$. If the word $y = (1111000)$ is received, then its distance from c is 1 and c is the unique codeword whose distance from y is at the most 1. If we write $y - c = y + c = e$ then the syndrome computation gives $S_y = S_c + S_e = S_e$ and since $e = (0001000)$ (error at the fourth place) we obtain $He^t = (001)^t$. Since this is the 4-th column of the matrix H we conclude that the error occurred at the 4-th place.

We are now ready to make a general definition of the Hamming Code $\mathbf{H}_r$ as follows.

Parity Check matrix of the Hamming Code: H_r is a matrix with r rows and $n = 2^r - 1$ (binary) columns. *Every* non-zero column vector occurs as a column of H_r and in fact these column vectors can be arranged lexicographically. The code $C = \mathbf{H}_r$ has exactly those words $c = (c_1 c_2 \ldots c_n)$ that satisfy the linear constraint: $Hc^t = 0$.

The Code Parameters: The Hamming code $\mathbf{H}_r$ has dimension $n - r = 2^r - 1 - r$ and has (minimum) weight 3. It is thus an $[n, n-r, 3] = [2^r - 1, 2^r - 1 - r, 3]$ code.

This code can indeed correct upto a single error and is also perfect thanks to the following calculation. The total number of words is $|\mathbb{F}^n| = 2^n = 2^{2r-1}$ while a ball of radius 1 around a codeword has exactly $1 + n = 2^r$ words. Since these balls are disjoint, we have:

$$|C| \times n = (2^{2^r - 1 - r}) \times 2^r \leq 2^n$$

and we are done since equality holds!

2.3 Solving Prisoner's Hat Problem

Some Computation where prisoners know the Hamming Code!: Clearly the Jail Manager has fixed a binary vector $x = (x_1 x_2 \ldots x_n)$ (with each $x_i = 0$ or 1) and the information available to a person is of the form:

$$(x_1 x_2 \ldots x_{i-1} ? x_{i+1} \ldots x_n)$$

and he has to either correctly guess ? or declare a PASS. He calculates if any one of $? = 1$ or $? = 0$ will put x in C. Both cannot because then we indeed have two codewords that differ only at one place; impossible! If both $? = 1$ or $? = 0$ do not put x in C, then he says PASS. If the solution is $? = 1$ then his answer would be $? = 0$ and if the solution is $? = 0$ then his answer would be $? = 1$.

The Procedure Almost Works: The Jail Keeper (must) fix a word $x = (x_1 x_2 \ldots x_n)$. He cannot change his mind when the guess work is going on. Then there are exactly two possibilities. Let $x \notin C$. Here, we have a unique codeword c whose distance from x is 1 and hence a unique j such that $c = (c_1 c_2 \ldots c_n)$ with $x_i = c_i$ for all $i \neq j$ and $x_j \neq c_j$. Here the j-th prisoner (who is presented with $(x_1 x_2 \ldots x_{j-1} ? x_{j+1} \ldots x_n)$) will indeed know c_j (using the Syndrome computation) and will give the correct answer by taking $x_j \neq c_j$. **Others PASS!** Otherwise we will have two distinct codewords that differ from each other on at the most two positions!

The Case of Failure: Here $x \in C$. Prisoner j has been presented with the knowledge of every x_i except x_j . He correctly finds x_j so that $x \in C$ but gives the opposite answer. *Here, every person answers but a wrong one!* Strangely, this (wrong answer) will happen for every prisoner! Indeed, if we combine all the answers then we obtain a joint answer $y = (y_1 y_2 \ldots y_n)$ such that $y_i \neq x_i$ and this is the case of total failure.

The Success Rate: This is simply the ratio of codewords not in C to the total number of words. We may simply calculate the failure rate (size of the complementary set) which is

$$\frac{|C|}{|\mathbb{F}^n|} = \frac{2^k}{2^n} = \frac{1}{2^{n-k}}.$$

Here, the dimension of the code is $k = n - r = 2^r - 1 - r$ while $n = 2^r - 1$ and hence $n - k = r$ so that the failure rate is:

$$\frac{1}{2^{n-k}} = \frac{1}{2^r} \to 0$$

as $r \to \infty$.

2.4 The Hamming Code

For a linear $[n, k, d]$ code, the ratio $\frac{k}{n}$ is generally defined to be the rate of the code. A little strange situation is that the rate of the Hamming code $\mathbf{H}_r$ actually tends to 1 as $r \to \infty$ and hence $\mathbf{H}_r$ is a very good code for a single error correction. On the other hand, the Prisoner Hat problem has a fortunate situation when the Jail keeper is kind enough to choose the setting as a word not in the code! This is resolved through the following mathematics, where $r \to \infty$:

$$\frac{2^r - 1 - r}{2^r - 1} \to 1$$

but:

$$\frac{2^{2^r-1-r}}{2^{2^r-1}} = \frac{1}{2^r} \to 0.$$

Apart from the ease of correcting a single error through the syndrome computation, it is very easy to construct a parity check matrix H_{r+1} of $\mathbf{H}_{r+1}$ very easily. The matrix H_{r+1} is obtained by adding an extra row of all 0s (with $2^r - 1$ columns) followed by a column vector with one only on the last $r + 1$-th row and then writing H_r with last row of all 1's added:

$$H_{r+1} = \begin{bmatrix} H_r & 0 & H_r \\ 0 & 1 & 1 \end{bmatrix} \tag{2.3}$$

where 0s and 1s are of proper size.

The situation we discussed so far (keeping the Hamming codes in mind), was that of being able to pack as many balls of radius 1 around chosen codewords as we can satisfying the stipulation that the balls are disjoint in the ambient space or set $\mathbb{F}^n$. This is a maximization problem and Hamming codes, being perfect, are optimal in that sense. Thus we are given a fixed positive integer e (in the case of Hamming codes, $e = 1$) and we try to pack as many balls of radius e in the ambient space $\mathbb{F}^n$ as possible.

A problem that is exactly dual to this is finding the least number of balls with radius e that are required to cover the entire $\mathbb{F}^n$ as possible. Here, we don't bother about overlaps! This is called the Covering Problem. Putting it slightly differently, the packing radius of C is the largest e such that balls of radius e around codewords are disjoint, while the covering radius is the smallest e such that balls of radius e around codewords can cover the entire ambient space $\mathbb{F}^n$. Both the objectives are achieved by the Hamming code since the packing and covering radii are both equal to 1! This latter property (covering radius) of the Hamming codes is used in DATA COMPRESSION.

2.5 General framework of coding theory

Coding theory is a vast discipline that has intersections with electrical engineering as also computer science, besides mathematics. One of the easiest example of a perfect code is that of a repetition code. This just has two codewords (in the binary situation) and hence is very poor in terms of rate of the code. On the other hand, it has a very high error correcting ability! Coding theory initially began with its most obvious day to day use in the library book classification and the ISBN numbering. After the discovery of Hamming codes, there was an amount of lull in the field of error correcting codes. A major breakthrough was obtained in the discovery of the Bose Chaudhuri-Hocquenghem codes, called the BCH codes. At the most elementary level, BCH code can correct

two errors in an optimal manner. This is simply achieved by enlarging the parity check matrix. But the philosophy developed by the discovery of BCH codes paved way for the construction of a very large class of algebraic codes called the Cyclic codes.

2.6 Conclusion and further reading

As such coding theory is also intimately connected with other areas of discrete mathematics, called the design theory and graph theory. There are substantial and non-trivial contributions of Indian mathematicians to design theory and this field almost began with a seminal paper of R. C. Bose [1] in 1939.

A textbook by van Lint [5] is a good source for coding theory. The problem discussed here can raise both the following questions. *If n is an integer that is strictly between $2^r - 1$ and $2^{r+1} - 1$, what is the optimal strategy?* Clearly, a success rate which is at least as large as the situation when the number of prisoners is (exactly) $2^r - 1$, indeed, can be achieved. However, this is not the best and in many situations, and a precise answer is unknown. Secondly, the problem is considerably harder when the number of colours is three or more. These questions are partly connected with the covering radius of a code and answers to some of the questions may be found in the references [2, 3, 4].

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3. The Decimal Place-Value System Part I : The Vedic Roots

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In this multi-part article we discuss the historical development of the decimal place-value system for representation of numbers that is now universally adopted. This first part deals with the Vedic roots of the system.

The Vedas and the ancillary Vedic literature show great propensity for involving numbers in the discourse, including some quite large ones. What is more, the decimal place value system that we now use for representation of numbers seems rooted in a way in their early practices.

Thinking of numbers beyond a certain threshold became possible in various civilizations once people realized the possibility of grouping the members of a collection, or a heap, into bundles (or packets) of moderate sizes, and tracking the bundles, bundles of bundles, and so on. Choosing each bundle to have the same size (with members in one-one correspondence with each other), leaves a remainder smaller than the chosen bundle size. The bundles thus formed, other than the remainder lot, could be further bundled in a similar way (with the same bundle size), and so on; at some stage in this iteration the number of bundles would be smaller than the chosen bundle size (then the whole lot is the remainder at that stage), and the process stops. Then all that would be needed to comprehend the size of a collection would be (i) to keep track of the size of the remainders at each stage (sequentialy), and (ii) to have identifiers for the sizes appearing as remainders, varying between a single unit to one less than the chosen bundle size; the identifiers could be names (verbal representations) for each such entity, or notations (symbolic representations, with notation in script), or some combination of the two, the main point being that each of the sizes is understood, and can be communicated, in an unambiguous fashion.

The Babylonians (Mesopotamians), who are historically perhaps the earliest ones known for having adopted such a strategy, chose 60 as the bundle size (now referred to as the base for the system), thus introducing what is now called the sexagesimal system[1] While the Babylonians seem to have found it convenient to have 60 for the base, presumably on account of certain aspects of arithmetic, especially division by 3, the system has a disadvantage with regard to the second aspect involved in the process mentioned above; namely, it involves having identifiers for all sizes between 1 and 59 (in our current decimal notation); it would be cumbersome to have individual names for all these, and to remember and use them in practice. The Babylonians, whose needs perhaps were limited to having the identifiers only for written representation[2] got around this by having, to put it simply, 10 as a subbase and an intermediate grouping of those between 10 and 59 into subbundles of 10s.

The Vedic people, on the other hand, chose from the beginning (of their history as known to us) the bundle size to be 10, consistently at each stage. It is simpler to remember names for sizes one to nine, and this is a great convenience in a setting relying largely on verbal communication[3]. The systematization in this respect, meeting the twin aspects of the counting requirement seems to have been in place already by the time the first lot of the hymns of Ṛgveda came to be composed, dated to be around 1500 BCE[4]: the sizes between 1 and 9 had acquired[5] the names *eka, dvi, tri,*

[1]The Mayans also followed such a system with 20 as the base.

[2]From our present knowledge about them it is not clear whether they also had verbal communication as a supplement, and if so whether the individual sizes carried any names for the purpose.

[3]Still fewer, say 8, would have made it even simpler in this respect, but 10 affords a convenience in terms of our having 10 fingers. Too few would in any case not be convenient as the iterations invovled would become longer.

[4]Some of it could be from the era before arrival in India, which is suggested by the cognate names in other civilizations [8].

[5]It would be good to bear in mind here that such nomenclatures and practices emerge through a social process and

catur, pañca, ṣaṣ (or *ṣaṭ)*, *sapta, aṣṭa*, and *nava* respectively for our digits 1 to 9; for the various statements about the number-names made here, and later in the article, the reader is referred to [6]. Interestingly these names have remained unchanged through the long life of the Sanskrit language, notwithstanding significant changes having taken place in other respects; moreover we find their residues in many contemporary Indian languages, especially from north India.

For identifying the sizes of the collections one could at this stage rely simply on noting the sizes of the remainders sequentially; we can denote by say 35781, the size of a heap that left remainders of sizes corresponding to 1,8,7,5 and 3 respectively, sequentially along the iteration process; we could choose to write them in the reverse order (or any particular order that we adopt) so long as a definite convention is followed between the one communicating it and the recipient of the information.

However, here comes a crucial point. The Vedic tradition was largely an oral tradition, with very little recourse to writing, if any, until much later[6]. Of course they could still deal with the need as above by reciting the names of the remainders in order, say *tri - pañca - sapta - aṣṭa - eka* (or some particular order adopted by convention), for the number as above. A brief reflexion would show that, though this is feasible as far as meeting the basic objective is involved, it would be quite cumbersome in general. Also, choosing such a course misses the point that there is another major cognitive input contained in the sequence, namely the "magnitude" of the sizes at different stages of the iteration in the sequence. This suggests having names for the magnitudes of the various group sizes and using them in the verbal communications, instead of simply reciting the sequence. The remainder at the first step of the bundling are units, those at the second stage are 10s, at the next stage 10s of 10s, and so on. In the Ṛgveda 10 is called *daśa*, ten of them put together is called *śata*, and the next two successive multiples by 10 as *sahasra* (1000) and *ayuta* (10000). At that stage (Ṛgveda being the earliest source in this respect) they seem to have been content with those many names, as far as the length of the sequence of iterations is concerned[7]. Interestingly the term *sahasra* is apparently derived from *sahas* which in Sanskrit means "power", and the suffix *ra* signifyies "possessing", together meaning "powerful", and *ayuta* signifies the negative of *yuta* which means "yoked" or "bounded", thus *ayuta* means "unbounded". It may be noticed that both the terms manifest a certain awe for the sizes of the numbers involved.[8] The status in this respect was to change soon, by the time of Yajurveda, where we see much larger powers of 10 appearing in the discourse (see below).

With the preparation seen so far, the number in our example could now be stated as 3 *ayuta*, 5 *sahasra*, 7 *śata*, 8 *daśa*, ending with *eka*. Well, that is still *not* how they did it. After all, there is such thing as the grammar of a language (not just as developed by grammarians from time to time, but also as it evolves in the social process) bringing in simplicity and aesthetic features into the means of verbal communication which would otherwise be simply a drab reproduction of a logico-mechanical process.

The naming process is seen to avail of the operations of additions and multiplication. The numbers 11 to 19 got the names *ekādaśa, dvādaśa, trayodaśa, caturdaśa, pañcadaśa, ṣoḍaśa, sap-*

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not "introduced" by professionals such as mathematicians, linguists, grammarians etc., who generally have only a role in systematizing them based on the wisdom arising from their respective disciplines.

[6]The precise details about when writing was adopted, and to what extent, are hazy, and in any case we will not go into them here.

[7]Two more terms *niyuta* and *arbuda* which in later literature stand for 10^5 and 10^7 occur in the Ṛgveda but it is believed that they do not represent numbers, but rather objects or beings such as demons [2]. Also, the term *ayuta*, though it makes unmistakable appearances at various places with the sense as 10000, the number itself is also often mentioned as *daśa sahasra*, a nomenclature that prevailed in the later times, extending to our times, putting together the terms for ten and thousand.

[8]The plural form *"sahasrāṇi"*, meaning "thousands", is seen in some instances to signify numerous or even innumerable (see [2], p. 561-562, for further details and more examples along these lines); this may be likened to similar practices seen today.

tadaśa, aṣṭādaśa and *navadaśa*[9] respectively, following an additive principle, denoting the aspect of 10 being added to the first remainder digit involved in the process; one can readily recognize the identity of the digital remainder in the first part of the composite word - the conjoining process manifests some variations, involving issues of grammar, that we shall not go into. The multiples of 10, from 20 to 90, were called *viṃśati, triṃśat, catvāriṃśat, pañcaśat, ṣaṣṭi, saptati, aśīti,* and *navati* respectively. As may be seen, the names assigned reflect the respective digit involved,[10] together with a form of conjugation signifying multiplication by 10;[11] we note that since pairing of each digit with 10 is by now involved in two ways, through addition and through multiplication, it becomes important to distinguish the two, through devices of grammar; the specific details involve issues of Sanskrit - Vedic - grammar and are beyond the scope of this article, it being beyond the author's expertise.[12] The numbers from 21 to 99 are then obtained by similar compounding (called *dvandva*), using the additive principle; thus *aṣṭā-viṃsati* is 28, *trayas-triṃśat* is 33, *navacatvariṃśat* is 49, *navaṣaṣṭi* 69, *navāśīti* is 89, *pañcanavati* is 95, etc.[13]

In expressing larger numbers a greater degree of freedom is seen to have been availed, depending on the context. This suggests a special role accorded to 100 in their reckoning, presumably arising out of their practical relevance. While larger numbers have a significant presence, they nevertheless seem to constitute a small sample, not adqueate to be able to discern any definite pattern in respect of their use or representation. Some examples of larger numbers include *ekaśatam* [101], *trimśacchatam* [130], *dve śate* [200] ; *triṇi śatā tri sahasraṇi trimśacca nava ca* [3,339] (appearing in 3.9.9 and 10.52.6, in identical words - note the appearance of the hundreds before the thousands); *saṣtiṃ sahasrā navatiṃnava* [60,099] (in 1.53.9), *navatirnava sahasrā* [99,000] (in 10.98.10); (the reference numbers stand for the numbers of the manḍala, sūkta and mantra, respectively, separated by dots.)[14]

The aura around the largeness of *sahasra* and *ayuta* is seen to have disappeared by the time of Yajurveda and we see the sequence of decuple terms (powers of ten) extended well beyond the above (those being retained, with their respective values):

> *niyuta* [10^5], *prayuta* [10^6], *arbuda* [10^7], *nyarbuda* [10^8], *samudra* [10^9], *madhya* [10^{10}], *anta* [10^{11}], *parārdha* [10^{12}].

This is known as Medhātithi's list after the associated Vedic seer (cf. [4], [5]). It appears in the *Vājasaneyī* recension of the Yajurveda, the *Taittirīya-saṃhitā* (IV. 4.11.4) ([7], p.342), the

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[9] *navadaśa*, 19, expressed here through addivive composition was also expressed as *ekānnaviṃśati*, following a subtractive composition; *viṃśati* stands for 20 (see below), and the composite signifies $20 - 1 = 19$. During the sūtra period *ekānna* changed to *ekona*, which is the surviving form of the representation ([1], p. 53). Analogous comment applies also to representations for 29, 39, … described below.

[10] This point, signifying simple multiplication, in close conformity with the basic theme as indicated above, is notable in contrast with usages such as "soixante dix" (literally "sixty ten") for 70 in French, which manifests an influence of the sexagesimal system.

[11] Actually one may wonder whether the aspect of multiplication by ten is tacitly involved in this, or whether it is a matter of simply keeping track of the number of bundles of ten involved; it is notable in this respect that the term for 10 does not feature as an ingredient in any of the names. Such a distinction seems relevant in the context of the later developments concerning repersentation of numbers in the Brāhmi and Kharoṣṭi numerals (recalled in a later part of this article). We shall however not go further into this quest here, and adopt the interpretation in terms of multiplication, as in [2].

[12] As a matter of fact, there are some ambiguities involved, which have been subject of interest of scholars - see [2], p. 575, but that is beside the point.

[13] The issues with regard to generating suitable nomenclature for the numbers, as seen here, had to be grappled with by various present day languages as well, including English, that base themselves on the decimal system. The case of Vedic Sanskrit is of special interest as it is the earliest known instance of adoption of such a process, and may have served historically as a prototype for others.

[14] On the other hand, numbers also appesr at various places expressed in other, more complicated forms, such as *tribhih ekādaśaiḥ* to mean 33 (in 1.34.11) or *ṣaṭtrimsat ca caturah* to mean 40(=36+4) (in 10.114.6), presumably for reasons of poetical convenience or prefereńcė; thanks are due to Dr. Medha Limaye for pointing this out.

Śatapatha Brāhmaṇa (IX. 1.2.16) ([7], p.342), and also in various other ancient texts ([3], Part I, p. 9-10).[15]

The choices of some of the names seem interesting, as well as curious in some ways. Thus, *samudra* means sea, or ocean, and the choice of the name for a billion is evidently inspired by the vastness of the oceans to the human eye. *anta* means the end, and presumably the name signifies that at some point in time the sequence of decuple terms was meant to be terminated there, and *madhya* which means "middle", for ten billion was a convenient 'middle' between 'the ocean' and 'the end'. *parardha* is an interesting addition, as it means "half-way to the other world".

Unlike the first few names from the Ṛgveda, these names have not survived, not with their respective values at any rate. Indeed the exercise of naming of higher and higher powers of 10 flourished for centuries together, in a competitive spirit it would seem[16], going to enormous lengths (see [4]), some of which we shall see in a later part; a standardisation seems to have eluded the community, until the time of Bhāskarācārya in the 12th century, who popularized the list largely due to Śridhara from the 8th century. Śrīdhara's list has *lakṣa* (or *lakṣya*) for 10^5, *prayuta* for 10^6, *koṭi* for 10^7, ... and it ends with *parārdha*, which by then acquired the value 10^{17}, rather than its Yajurvedic value of 10^{12}; the intermediate decuple terms after *koṭi* are *arbuda, abja* (or *abda*), *kharva, nikharva, mahāsaroja, śaṅku* (or *śaṅkha*), *saritāṃ-pati, antya,* and *madhya*; in Bhāskarācārya's list the terms for 10^{12} and 10^{14} are replaced by *mahāpadma* and *jaladhi*, which are synonymous with the original terms. The reader is referred to [4] for various further details in this respect. It is not quite clear how far the decuple terms after *koṭi* (or *daśa koṭi*, which is a composite name) from the list got used, but they were eventually forgotten - manifesting that they were not needed in practice.[17]

The decuple terms of high orders that came to be introduced are not known to have significant practical role, and were more like creatures from fantasyland with little functionality. However, their being a part of the repertoire brought out the vastness and the ordered nature of the numberland, and benefited the process of eventual emergence of the place-value system of representing numbers through symbols, as we use today.

In retrospect it may seem that with such elaborate paraphernelia in place, when writing came into vogue it would have been but a natural step to write numbers in terms of the digits corresponding to the relevant decuple terms, in a sequence in their "respective places". It is one of the oddities of history that it was not quite to be so. Several centuries are seen to have passed before the full-fledged written form of the decimal place value system came into use, as we shall see in the second part.

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[15] It is mentioned in [4], quoting some references, that according to some authors Medhātithi's list extends further, with names for 7 more decuple terms, going upto 10^{19}, the next seven terms being given as *uṣas, vyuṣṭi, udeṣyat (or deśyat), udyat; udita, suvarga, and loka.* It may seem puzzling to find an ambiguity on such an issue; in this respect it may be noted that there is a place (Yajurveda 7.2.20 - cf. [8], p.14) where the powers upto *parārdha* mentioned above occur in succession, and afford unambigous interpretation, while on the other hand for other terms found scattered elsewhere in the composition, the specific interpretation could be an issue.

[16] Unfortunately the competitors in the naming game seem to have missed the point that there is no need to name the decuple terms beyond a stage (dictated by practical usage), and that instead one could have a notation like the exponent that we now use, or even nomenclature involving the power of 10 sought to be indicated, along the lines of the names found in the śulbasūtras for roots of different orders *dvikaraṇī, trikaraṇī* etc. - the latter possibility has been pointed out in [2], p. 557.

[17] Interestingly, in the ancient Jaina tradition it was more common to use *'koti'* - or rather the ardhamagadhi equivalent *'kodi'* - repeatedly to convey some of the large numbers; *"koti koti koti koti"* stood for 10^{28}. Surprisingly the practice however does not seem to have been adhered to by the later Jaina authors, who in various other respects were following their ancient tradition.

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Padma Awardees -2023 from the Mathematics world

Padma Shri awardee Radha Charan Gupta (born 1935) is an Indian historian of mathematics, formerly a professor of Mathematics at Birla Institute of Technology, Mesra, until mandatory retirement in 1995. In 2009 he became the first Indian to get the Kenneth O. May Prize in History of Mathematics. Earlier he was elected a Fellow of the National Academy of Sciences, India (1991) and corresponding member of the International Academy of the History of Science (1995). He was Editor of Ganita Bharati, the Bulletin of the Indian Society of History of Mathematics, since its inception in 1979 until 2005. He has written over 500 papers and a number of books on History of Mathematics.

4. What is happening in the Mathematical world?

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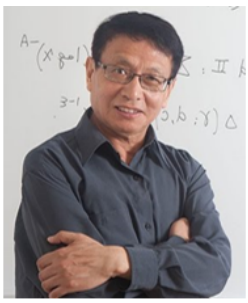
4.1 Yitang Zhang claims to have solved the Landau-Siegel Zero Conjecture



After shocking the mathematics community in 2013 with a major result of deciphering a centuries-old question about prime numbers, number theorist *Prof. Yitang Zhang* now claims to have solved a weaker version of the Landau-Siegel Zero Conjecture, an analogue of the celebrated Riemann hypothesis, which is considered as one of the biggest unsolved problems in mathematics. Yitang Zhang, who works at the University of California at Santa Barbara, posted his proposed solution - a 111-page preprint on the arXiv preprint server on November 4, 2022. The paper, "Discrete mean estimates and the Landau-Siegel zero", has not been peer reviewed yet, but if verified, it could be a historic breakthrough for number theory and mathematics in general. But other claims, including one by Zhang (in 2007), have turned out to be faulty, and it will take a while for researchers to examine Zhang's argument to see if it is correct.

The Landau-Siegel zeros conjecture - named after mathematicians Edmund Landau and Carl Siegel - is similar to and less challenging than the Riemann hypothesis. This conjecture concerns the possible existence of zero points of a type of L-functions, a modified version of the zeta function, known today as the *Dirichlet L-function*. It is effectively a generalised form of the zeta function. It looks like this: $L(s,\chi) = \sum_n \frac{\chi(n)}{n^s}$, where, s is a complex number and χ is a function from N to C. The generalized Riemann hypothesis (GRH) is the conjecture that all zeros of the L-function $L(s,\chi)$ corresponding to a real character χ, that lie in the critical strip $(0 < Re(s) < 1)$ should also lie on the critical line, $Re(s) = 1/2$. A Landau-Siegel zero of the function $L(s,\chi)$ is its real zero that lies between $1/2$ and 1. Obviously, proving that there are no Landau-Siegel zeroes would be a weak form of proving the GRH. It wouldn't allow us to claim that all non-trivial zeroes of the generalized function lie on the critical line - but it would allow us to say that some non-trivial zeroes definitely *don't lie outside the line*.

Although it has been known for millennia that there are infinitely many prime numbers, there is no way to predict whether a given number will be prime; only the probability that it will be, given its size. Solving either the Riemann or Landau-Siegel problems would mean that the distribution of prime numbers does not have huge statistical fluctuations.

Disproving the existence of Landau-Siegel zeroes requires mathematicians to prove that $L(1,\chi)$ is much greater than $(\log\ D)^{-1}$, for $D \in \mathbb{Z} \in D \in Z$, a fundamental discriminant (i.e., the discriminant of a quadratic number field corresponding to a real character χ). In 2007, Zhang had posted a preprint claiming that he had proved that $L(1,\chi)$ was much greater than $(\log\ D)^{-17}(\log(\log\ D))^{-1}$. But his proof turned out to be wrong when mathematicians noticed the incorrectness of a few key ideas involved in that paper. In his new paper, Zhang claims to have proved that $\mathrm{L}(1,\chi)$ is much greater than $(\log\ D)^{-2022}$.

Zhang's first big breakthrough came in 2013, when he proved that there are infinitely many pairs of prime numbers separated by an even number that is lower than 70,000,000.This was the first big step towards solving a major question in number theory - whether there are infinitely many pairs of primes that differ by just 2, such as the primes 5 and 7 or 11 and 13. (Number theorist James Maynard at the University of Oxford, UK, won a Fields Medal in July for improving on Zhang's result by lowering the bound to just 246, among other achievements.)

If Zhang's paper holds up to scrutiny, it could be the mathematician's second landmark contribution to the understanding of prime numbers.

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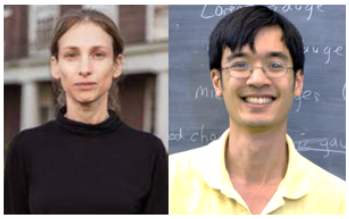
4.2 Geometry breaks decades-old Tiling Conjecture

Since ancient times, artists and geometers have wondered how shapes can tile the entire plane without gaps or overlaps. Not much had been known until fairly recent times. Mathematicians predicted that if they imposed enough restrictions on how a shape might tile space, they could force a periodic pattern to emerge. But they were wrong.

It is easy to cover a floor with copies of squares, triangles or hexagons. In the 1960s, mathematicians found strange sets of tiles that can completely cover the plane, but only in ways that never repeat. Mathematicians tried to understand the structure of such tiling. The first such non-repeating, or aperiodic, pattern relied on a set of 20,426 different tiles. Mathematicians wanted to know if they could drive that number down. By the mid-1970s, Roger Penrose (who won the 2020 Nobel Prize in Physics for work on black holes) proved that a simple set of just two tiles, dubbed "kites" and"darts", served the purpose.

Some years ago, the mathematician Siddhartha Bhattacharya (from TIFR, Mumbai) proved that no matter how complicated or subtle a tile design you come up with, if you are only able to use shifts, or translations, of a single tile, then it's impossible to devise a tile that can cover the whole plane aperiodically. Mathematicians conjectured that Bhattacharya's two-dimensional result would also hold in higher-dimensional spaces. Just as no aperiodic two-dimensional tile exists, they supposed that no suitable three-dimensional block (or more complicated tile) exists, and so on in an arbitrarily large number of dimensions. This hypothesis was dubbed the periodic tiling conjecture.



Now, *Rachel Greenfeld* (left), a mathematician at the Institute for Advanced Study in Princeton, New Jersey, along with *Terence Tao* (right) of the University of California, Los Angeles, finally settled the conjecture - but not in the way mathematicians had anticipated. They constructed a tile that can aperiodically fill a high-dimensional space but cannot do so periodically, thus disproving the conjecture. The work marks a new way to construct aperiodic tiles - one that Greenfeld and Tao now think could be applied to disproving other tiling-related conjectures.

Source: *https://www.quantamagazine.org/nasty-geometry-breaks-decades-old-tiling- conjecture - 20221215/*

4.3 Patterson's conjecture on Cubic Gauss sums is proved

Caltech mathematicians *Alex Dunn* and *Maksym Radziwill* finally proved a 175 years old perplexing feature of numbers which first stumbled the German mathematician Ernst Kummer. At one point the 1950s, this unusual feature was thought to have been wrong, but then, decades later, mathematicians found hints that it may in fact be true. Now, after several twists and turns, two Caltech mathematicians *Alexander (Alex) Dunn (left)*, a postdoc at Caltech and his advisor *Maksym Radziwill* (right), Professor of mathematics, have at last found a proof that Kummer was right all along.

The problem is related with quadratic Gauss sums, named after the 18th century mathematician Carl Friedrich Gauss. Gauss would choose some prime number p, then sum up numbers of the form $e^{2i\pi n^2/p}$. Since their inception, quadratic Gauss sums have proved invaluable for tasks like counting solutions to certain types of equations.
This summing activity involves a type of mathematics known as modular arithmetic. In the 19th century, Kummer was interested in looking at the distribution of cubic Gauss sums, where n^2 in the exponent is replaced by n^3, for nontrivial primes (primes that have a remainder of 1 when you divide by 3), or in a modulo p system. He did this by hand for the first 45 nontrivial primes. When he plotted the answers (after being "normalized" - divided by a suitable constant) one by one on a number line, he discovered that they were distributed in an odd way. Kummer laid out his observations, along with a conjecture: If you somehow managed to plot all of the infinitely many cubic Gauss sums, you'd see most of them between $1/2$ and 1; fewer between $-1/2$ and $1/2$; and still fewer between -1 and $-1/2$.
When dealing with the distribution of natural objects in number theory, the simple expectation is that one has an equal distribution, and if not, there should be a very convincing reason. That is why it was so shocking that Kummer claimed that this was not the case in this instance.
Later, in the 1950s, Selberg, von Neumann and Goldstine set out to test this on their early computer. Selberg programmed it to calculate the cubic Gauss sums for all the non-trivial primes less than 10,000 - around 600 sums in all. They discovered that as the primes got bigger, the normalized sums became less inclined to cluster near 1. With convincing evidence that Kummer's conjecture was wrong. The solutions seemed to have a random distribution.
Then a graduate student Samuel Patterson of the University of Cambridge proposed a solution in 1978, which is now referred to as Patterson's conjecture. His conjecture was motivated by what happens when numbers are randomly placed anywhere between -1 and 1. If you add up N of these random numbers, the typical size of the sum will be $N^{1/2}$ (it could be positive or negative). Likewise, if cubic Gauss sums were scattered evenly from -1 to 1, you'd expect N of them to add up to roughly $N^{1/2}$. With this in mind, Patterson added up N cubic Gauss sums, ignoring the requirement to stick to the prime numbers. He found that the sum was around $N^{5/6}$ - bigger than $N^{1/2}$, but less than N. This value implied that the sums behaved like random numbers but with a weak force pressuring them toward positive values, called a bias. As N got larger and larger, the randomness would start to overwhelm the bias, and so if you somehow looked at all of the infinitely many cubic Gauss sums at once, they'd appear evenly distributed. But Patterson wasn't able to do the same calculations for prime numbers, so in 1978, he officially wrote it down as a conjecture: If you add up the cubic Gauss sums for prime numbers, you should get the same $N^{5/6}$ behavior.
In the late 1970s, Roger Heath-Brown and Patterson teamed up to work on the problem, and then, in 2000, Heath-Brown developed a tool known as a cubic large sieve to help prove Patterson's conjecture. He got close but the complete solution remained out of reach. Now, about two years ago, Caltech researchers Dunn and Radziwill decided to work together to try to crack the problem of Patterson's conjecture. They cracked the problem when they realized that the sieve was not working properly, or had a "barrier" that they were able to remove.

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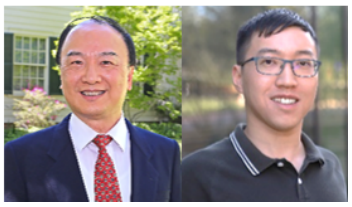
4.4 Centuries-old fluid equations problem now solved using computer

For centuries, mathematicians have sought to understand and model the motion of fluids. Perhaps the oldest and most prominent of these models, is represented by an equation formulated by Leonhard Euler more than 250 years ago, describe the flow of an ideal, incompressible fluid: a

fluid with no viscosity, or internal friction. Yet much remains unknown about the Euler equations - including whether they're always an accurate model of ideal fluid flow. One of the central problems in fluid dynamics is to figure out if the equations ever fail, leading to nonsensical values that render them unable to predict a fluid's future states.



In 2013, Thomas Hou, from the California Institute of Technology, and Guo Luo, now at the Hang Seng University of Hong Kong, proposed a scenario in which the Euler equations would lead to a singularity. They developed a computer simulation of a fluid in a cylinder whose top half swirled clockwise while its bottom half swirled counterclockwise. As they ran the simulation, more complicated currents started to move up and down. That, in turn, led to strange behavior along the boundary of the cylinder where opposing flows met. The fluid's vorticity - a measure of rotation - grew so fast that it seemed poised to blowup.

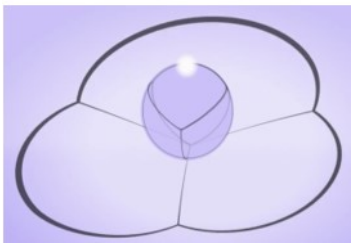
Hou and Luo's work was suggestive, but not a true proof. That is because it is impossible for a computer to calculate infinite values. It can get very close to seeing a singularity, but it can't actually reach it - meaning that the solution might be very accurate, but it is still an approximation. Recently, *Thomas Hou* (left) and his former graduate student *Jiajie Chen* (right) have finally succeeded in proving the result. In a 177-page proof posted online recently, they have shown that a particular version of the Euler equations does indeed sometimes fail. The proof marks a major breakthrough - and while it does not completely solve the problem for the more general version of the equations, it offers hope that such a solution is finally within reach.

This proof makes significant use of computers. This arguably makes it difficult for other mathematicians to verify it.

Source: *https://www.quantamagazine.org/computer-helps-prove-long-sought-fluid-equation- singularity 20221116/*

4.5 'Monumental' proof solves triple bubble problem and more

The decades-old Sullivan's conjecture, about the best way to minimize the surface area of a bubble cluster, was thought to be out of reach for three bubbles and up - until a new breakthrough result. The bubble problem is simple enough to state: You start with a list of numbers for the volumes, and then ask how to separately enclose those volumes of air using the least surface area. To solve this problem, mathematicians must consider a wide range of different possible shapes for the bubble walls. And if the assignment is to enclose, say, five volumes, we do not even have the luxury of limiting our attention to clusters of five bubbles - perhaps the best way to minimize surface area involves splitting one of the volumes across multiple bubbles. Even in the simpler setting of the two-dimensional plane, no one knows the best way to enclose, say, nine or 10 areas. As the number of bubbles grows, one cannot really even get any conceivable conjecture.



But more than a quarter century ago, John Sullivan, now of the Technical University of Berlin, realized that in certain cases, there could be a guiding conjecture. Bubble problems make sense in any dimension, and Sullivan found that as long as the number of volumes enclosed is at most one greater than the dimension, there is a particular way to enclose the volumes, that is, a sort of shadow of a perfectly symmetric bubble cluster on a sphere. This shadow cluster, he conjectured, should be the one that minimizes surface area. Sullivan's three-bubble cluster in the flat plane is shown in the figure 1.

Over the decade that followed, a series of groundbreaking papers appeared, proving Sullivan's conjecture for enclosing only two volumes. Here, the solution is familiar, double bubble made of two spherical pieces with a flat or spherical wall between them depending on whether the two bubbles have the same or different volumes. But Sullivan's conjecture for three volumes was not proved so far.

Now, in a paper posted online, *Emanuel Milman* of the Technion in Haifa, Israel, and *Joe Neeman*, of the University of Texas, Austin, have proved Sullivan's conjecture for triple bubbles in dimensions three and up and quadruple bubbles in dimensions four and up, with a follow-up paper on quintuple bubbles in dimensions five and up in the works.

When it comes to six or more bubbles, Milman and Neeman have shown that the best cluster must have many of the key attributes of Sullivan's conjecture, potentially starting mathematicians on the road to proving the conjecture for these cases too. Their central theorem is monumental. It is a brilliant accomplishment with lots of new ideas. Their work is a whole new approach rather than an extension of previous methods.

Source: *https://www.quantamagazine.org/monumental-math-proof-solves-triple-bubble-problem-and-more-20221006/?mc_cid=8ad570b961&mc_eid=df6d259cb7*

4.6 Teenager Daniel Larsen solves stubborn riddle about prime number look-alikes

At the age of 17, a high school student *Daniel Larsen* has proved a key theorem about Carmichael numbers - strange entities that mimic the primes. It would be a paper that any mathematician would be really proud to have written.

Carl Friedrich Gauss posed a very important problem as to "how to distinguish a prime number from a composite number". For hundreds of years, mathematicians searched for an efficient way to do so. The problem has also become relevant in the context of modern cryptography, as some of today's most widely used cryptosystems involve doing arithmetic with enormous primes.

Over a century ago, in the search for a fast and powerful primality test, mathematicians stumbled on a type of numbers that fool tests into thinking they are prime, even though they are not. These pseudoprimes, known as Carmichael numbers, have been difficult to find. It was only in the mid-1990s, that mathematicians proved there are infinitely many of them. Being able to say something more about how they are distributed along the number line has posed an even greater challenge.

In the mid-17th century, Pierre de Fermat wrote a letter to his friend Frénicle de Bessy, in which he stated what would later be known as "Fermat's little theorem". If N is a prime number, then $b^N - b$ is always a multiple of N, no matter what b is. Mathematicians saw the potential for a perfect test of whether a given number is prime or composite. They knew that if N is prime, $b^N - b$ is always a multiple of N. What if the reverse was also true? That is, if $b^N - b$ is a multiple of N for all values of b, must N be prime?

Unfortunately, it turned out that in very rare cases, N can satisfy this condition and still be composite. The smallest such number is 561: For any integer b, $b^{561} - b$ is always a multiple of 561, even though 561 is not prime. Numbers like these were named after the mathematician Robert Carmichael, who is often credited with publishing the first example in 1910.

Though mathematicians suspected that there must be infinitely many Carmichael numbers, they are relatively few compared to the primes, which made them difficult to pin down. Then in 1994, Red Alford, Andrew Granville and Carl Pomerance published a breakthrough paper in which they finally proved that there are indeed infinitely many of these pseudoprimes.

The key to Larsen's interest in Carmichael numbers in the first place was the results by Maynard and Tao on prime gaps. Larsen wanted to understand some of the mathematics underlying Maynard and Tao's work. He kept on searching results in this direction, until finally, in February 2021, Larsen came across a paper on Carmichael numbers, which he found both beautiful and comprehensible. He used the known fact that if you take certain sequences of enough numbers, some subset of those numbers would consist of primes. Finally, in November 2021, he came up

with a proof of the result regarding gaps in Carmichael numbers. In fact, Larsen's argument did not just allow him to show that a Carmichael number must always appear between X and $2X$, his proof works for much smaller intervals as well. Mathematicians now hope it will also help reveal other aspects of the behavior of these strange numbers.
Source: *https://www.quantamagazine.org/teenager-solves-stubborn-riddle-about-prime-number - look-alikes-20221013/*

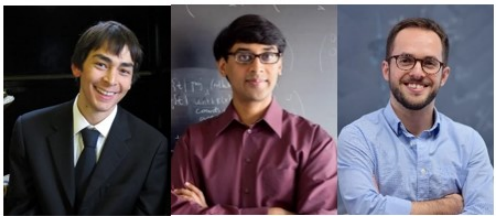
4.7 Groundbreaking mathematical formula discovered

A groundbreaking mathematical formula has been discovered, which could transform medical procedures, natural gas extraction, and plastic packaging production in the future.
The new equation, developed by scientists at the University of Bristol, indicates that diffusive movement through porous material can be modelled exactly. It comes a century after world-leading physicists Albert Einstein and Marian von Smoluchowski derived the first diffusion equation and marks important progress in representing motion for a wide range of entities, from microscopic particles and natural organisms, / to man-made devices.

Until now, scientists looking at particle motion through porous materials such as biological tissues, polymers, various rocks and sponges, have had to rely on approximations or incomplete perspectives. Various theoretical approaches to study diffusion through permeable interfaces have been proposed in the past: Green's functions in discrete and continuous space, spectral decompositions and scattering techniques. These techniques, while valuable, have been limited in their scope as they either demand spatial symmetries, e.g., analytical Green's functions or employ a coarse-grained representation of the heterogeneities, e.g., effective medium approximations. In addition, these various approaches have failed to construct a unified framework capable of representing the diffusive dynamics with both permeable and reactive heterogeneities. Given the wide-spread occurrence of permeable membranes, the above limitations call for the development of a fundamental theory of diffusion through permeable interfaces.
The new findings, published in the journal Physical Review Research, provide a novel technique presenting exciting opportunities in a diverse range of settings including health, energy, and the food industry. Lead author Toby Kay, who is completing a Ph.D. in Engineering Mathematics, revolutionized the modelling of diffusing entities through complex media of all scales, from cellular components and geological compounds to environmental habitats. Such a model allows to derive an inhomogeneous diffusion equation (DE) and the Smoluchowski equation (SE) to account for the presence of a porous barrier.
Previously, mathematical attempts to represent movement through environments scattered with objects that hinder motion, known as porous barriers, have been limited. By solving this problem, Kay paved the way for exciting advances in many different sectors because porous barriers are routinely encountered by animals, cellular organisms and humans.
Further research is needed to apply this mathematical tool to experimental applications, which could improve products and services. This latest discovery is a further significant step forward in improving our understanding of motion in all its shapes and forms - collectively termed the mathematics of movement - which has many exciting potential applications.
Source: *https://journals.aps.org/prresearch/abstract/10.1103/PhysRevResearch.4.L032039*

4.8 Mathematical trio advances centuries-old number theory problem

The work - on the proportion of whole numbers that can be written as the sum of two cubed fractions - makes a major headway on a problem that mathematicians have been thinking about for centuries.



Levent Alpöge (left) of Harvard University, *Manjul Bhargava* (middle) of Princeton University and *Ari Shnidman* (right) of the Hebrew University of Jerusalem considered one of the oldest questions in number theory: How many integers can be written as the sum of two cubed rational numbers?

For example, 6 can be written as $\left(\frac{17}{21}\right)^3 + \left(\frac{37}{21}\right)^3$, while $13 = \left(\frac{7}{3}\right)^3 + \left(\frac{2}{3}\right)^3$.

Mathematicians have suspected for decades that half of all integers can be written this way. Just as with odd and even numbers, this property appears to divide whole numbers into two equal camps: those that are the sum of two cubes, and those that are not. But it had not been possible to prove this, or even give any bound on the proportion of whole numbers that fall into each camp. Mathematicians have calculated that, if something called the Birch and Swinnerton-Dyer conjecture is true (as is widely believed), about 59% of numbers up to 10 million are the sum of two rational cubes. There is no test for determining which numbers belong in which camp that is known to work for all numbers.

Now, in a paper posted online in late October 2022, Alpöge, Bhargava and Shnidman have shown that at least 2/21 (about 9.5%) and at most 5/6 (about 83%) of whole numbers can be written as the sum of two cubed fractions. The new work builds on a set of tools Bhargava has developed over the past 20 years, to explore the full family of elliptic curves. Understanding sums of two cubes means analyzing a much smaller family, and the smaller the family, the harder the problem. Proving the full conjecture - that exactly half of all integers are the sum of two cubes - will require eventually attacking the set of numbers that includes both numbers that are the sum of two cubes and ones that are not. Handling such numbers may require completely new ideas.

Source: *https://www.quantamagazine.org/mathematical-trio-advances-centuries-old-number- theory -problem-20221129/*

4.9 Decades-old theorem cracks an encryption algorithm in less than one hour

One of the greatest difficulties in the digital age is securing data from hacker attacks, a problem that experts, governments, and businesses throughout the globe are working hard to solve. While there are plans to create a connected and secure future, there is still plenty that can be learned from the past. One of the most promising post-quantum encryption methods developed by a group of prominent researchers has just been cracked by a decade-old curved mathematics theorem in less than an hour.

Researchers were requested to find flaws in a set of encryption algorithms that the US National Institute of Standards and Technology (NIST) believed would survive the power of quantum computers. A $50,000 reward was also given by Microsoft to anyone who could break the encryption of the SIKE algorithm, as several of their researchers were present on the SIKE team. To tackle the problem, a researcher at Katholieke Universiteit Leuven used a single conventional computer and a 1997 mathematical theorem made by Dr. Ernst Kani of Queen's University in Canada to defeat SIKE. Dr. Kani's idea uses arithmetic to calculate when and where a procedure can fail by joining two elliptical curves. The 1997 work that described these weaknesses served as the basis for the successful attack on the SIKE algorithm by researchers Wouter Castryck and Thomas Decru from KU Leuven.

Although cryptographers and computing engineers are not always well-versed in all the high-powered techniques of mathematics, many different skills and forms of knowledge can be combined to advance the way data is stored and transmitted. Cryptography uses a lot of sophisticated mathematics, especially arithmetic geometry. Computing experts and mathematics experts have to work together to advance this field.

Sources:

1. *https://wonderfulengineering.com/this-decades-old-maths- theorem-has-cracked-u-s- government -encryption/*
2. *https://www.gadgetany.com/news/an-old-math-theorem-helps-researchers-break-us- govt- encryption -algorithm-in-an-hour/*

4.10 Awards

4.10.1 Computer Expert Prof. Daniel Spielman awarded Breakthrough Prize in Mathematics

Sterling Professor of Computer Science and Statistics and Data Science *Daniel Spielman* will be recognized by the Breakthrough Prize in Mathematics for his pioneering research in algorithms and mathematics at the 2023 ceremony. The Breakthrough Prizes were established in 2012 by a group of entrepreneurs and philanthropists including Priscilla Chan and Mark Zuckerberg. Laureates in the three categories - mathematics, fundamental physics and life sciences - each receives a $3 million award.

Spielman was recognized for his contributions to theoretical computer science and mathematics, including in spectral graph theory, the Kadison-Singer problem, numerical linear algebra, optimization and coding theory. In addition to solving long-standing mathematical mysteries, his work has led to significant and very practical benefits in the fields of computing, signal processing, and engineering. His is the kind of work that reaffirms the fundamental value of abstract and foundational research, demonstrating how it can lay the groundwork for impactful and real-world benefits in numerous facets of life, in ways that we cannot always predict.

Spielman's research has led to countless applications, from better medical imaging to improving the design of clinical trials. His work has also helped revolutionize the field of error-correcting codes, which allows communication devices to transmit information even if part of it is corrupted. That work has made communication faster and more reliable and has been used for broadcasting high-definition television.

The Breakthrough Prize is just the latest of the many honors that Spielman's work has earned. In 2013, he was named a MacArthur Fellow. He had also won the Rolf Nevanlinna Prize, twice won the Gödel Prize, and George Pólya Prize in 2014.

Sources:

1. *https://indiaeducationdiary.in/yales-spielman-wins-3-million-breakthrough-prize/*
2. *https://yaledailynews.com/blog/2022/10/06/178448/*

4.10.2 New Horizons in Mathematics Prizes 2023 awarded to 3 mathematicians

Ana Caraiani (left), *Ronen Eldan* (middle) and *James Maynard* (right) are the winners of $1,00,000 worth New Horizons in Mathematics Prizes 2023 for having an early, but substantial impact on their fields.

The Breakthrough Prize founders acclaimed this year's winners for their outstanding contributions to basic science. Their incredible discoveries will pave the way for scientific discovery and spur innovation. These laureates and early-career scientists are pushing the boundaries of what is possible in research and science.

Ana Caraiani (born 1985) is a Romanian-American mathematician, who is a Royal Society University Research Fellow and Professor of Pure Mathematics at Imperial College London and University of Bonn. Her research interests include algebraic number theory and the Langlands

program. She is awarded the prize for diverse transformative contributions to the Langlands program, and in particular for work with Peter Scholze on the Hodge-Tate period map for Shimura varieties and its applications.

Ronen Eldan is an Israeli mathematician. Eldan is a professor at the Weizmann Institute of Science and Microsoft Research, working on probability theory, mathematical analysis, theoretical computer science and the theory of machine learning. He received the 2018 Erdős Prize, the 2022 Blavatnik Award for Young Scientists and was a speaker at the 2022 International Congress of Mathematicians. He was awarded the prize for the creation of the stochastic localization method, that has led to significant progress in several open problems in high-dimensional geometry and probability, including Jean Bourgain's slicing problem and the KLS conjecture.

James Alexander Maynard (born 10 June 1987) is an English mathematician of Oxford University and Institute for Advanced Study working in analytic number theory and in particular the theory of prime numbers. In 2017, he was appointed Research Professor at Oxford. Maynard is a fellow of St John's College, Oxford. He was awarded the Fields Medal in 2022. He was awarded the prize for multiple contributions to analytic number theory, and in particular to the distribution of prime numbers.

Source: *https://www.forbes.com/sites/michaeltnietzel/2022/09/23/the-2023-breakthrough-prizes-are-announced-1575-million-awarded-to-the-winners/?sh=674dd2ad6d10*

4.10.3 Maryam Mirzakhani New Frontiers Prizes 2023 awarded to 3 women mathematicians

Stanford University mathematicians *Maggie Miller* (left), *Jinyoung Park* (middle) and *Vera Traub* (right) are winners of the Maryam Mirzakhani New Frontiers Prize 2023 given by the Breakthrough Prize Foundation, which recognizes female mathematicians for early-career achievements.

The $50,000 prize is awarded to outstanding early-career women in the field of mathematics. The award is named after Iranian mathematician, Fields Medalist, and Stanford Professor Maryam Mirzakhani, who passed away from breast cancer in 2017.

Maggie Miller is a visiting Clay Research Fellow at Stanford and a Stanford Science Fellow. Miller works on geometric topology - the study of properties that are preserved in a shape as it deforms, stretches, or twists - and is interested in understanding surfaces in 4-dimensions. Miller was awarded the Prize for "work on fibered ribbon knots and surfaces in 4-dimensional manifolds". This recognizes Miller's PhD thesis project from Princeton University.

Jinyoung Park is the Szegö Assistant Professor in the School of Humanities and Sciences. Her research interests fall within the field of discrete mathematics, which covers mathematical structures that are distinct and separable, and often countable. Park's specific interests are extremal and probabilistic combinatorics, asymptotic enumeration, and graph theory. Park was awarded the Prize for "contributions to the resolution of several major conjectures on thresholds and selector processes".

Verra Traub is a junior professor at the Research Institute for Discrete Mathematics at the University of Bonn and a member of the Hausdorff Center for Mathematics. Her research interests are mostly in the areas of combinatorial optimization and approximation algorithms. Some specific topics I have worked on recently are the traveling salesman problem, vehicle routing, and connectivity augmentation. She was awarded the prize for advances in approximation results in classical combinatorial optimization problems, including the traveling salesman problem and network design.

Source: *https://www.forbes.com/sites/michaeltnietzel/2022/09/23/the-2023-breakthrough-prizes-are-announced-1575-million-awarded-to-the-winners/?sh=674dd2ad6d10*

4.10.4 Prof. Peter Kronheimer and Prof. Tomasz Mrowka win Steele Prize 2023

The American Mathematical Society (AMS) awarded Harvard University Professor *Peter Kronheimer* (left) and MIT Professor *Tomasz Mrowka* (right) the 2023 Steele Prize for seminal contribution to research for their 1993 paper "Gauge theory for embedded surfaces, I". Kronheimer and Mrowka's work introduced new notions and developed sophisticated technology that continues to play a central role in gauge theory and low-dimensional topology.

The authors and other researchers have developed the ideas in the cited paper to define new invariants in low-dimensional topology. The subject has grown to include relations with a wide array of topics, from knots to graph-colorings. The Steele Prize is awarded according to a six-year rotation of subject areas, the 2023 area being geometry and topology.

Source: *https://www.math.harvard.edu/peter-kronheimer-and-tomasz-mrowka-to-receive- 2023- steele - prize-for-seminal-contribution-to-research/*

4.11 Obituary

4.11.1 Prof. Steve Zelditch, a pioneer of the field of quantum chaos passes away at the age of 68

Steven *"Steve" Zelditch*, Professor of Mathematics at Northwestern University, died on 11 September, 2022 at the age of 68 after a battle with cancer. Zelditch was considered one of the founders and a giant in the field of quantum chaos, which relates chaotic motion of the point particles of classical mechanics to the smearing-out of the corresponding quantum mechanical states. In the 1980s, he made a breakthrough on quantum ergodicity for Riemann surfaces, and his subsequent work made progress in and drew ideas from probability, string theory, general relativity, complex geometry, partial differential equations and spectral and scattering theory.

Through his work, Zelditch sought to answer complicated questions. For example, when you listen to music, you can hear the difference between the low sound of a big bass drum and the higher sound of a small snare drum. But can you hear the shape of the drum? He found that if you assume the head of the drum is convex, has no holes, has at least one mirror symmetry and a few other technical assumptions, then you can reconstruct the shape of the drum by listening to it. This discovery was one of the many impacts Zelditch had on mathematics.

Steve Zelditch is the one who made 'quantum ergodicity' a true field of mathematical research. He gave the first complete proof of the theorem, then generalized it within all sorts of geometric contexts.

Zelditch earned his bachelor's degree from Harvard and his Ph.D. degree from the University of California, Berkeley. After postdoctoral positions at Columbia University, the University of California, Berkeley, and Massachusetts Institute of Technology (MIT), he joined the faculty at Johns Hopkins University, where he taught for 25 years before joining the department of mathematics at Northwestern in 2010.

He won the 2013 Stefan Bergman Prize for finding "deep and diverse relations between the Bergman kernel and many other areas, including complex geometry, probability and mathematical physics". He was an invited speaker at the International Congress of Mathematicians in Beijing in 2002. In 2013 he was named a fellow of the American Mathematical Society. He also authored more than 180 publications and served on the editorial boards of numerous journals.

Source: *https://news.northwestern.edu/stories/2022/09/mathematics-pioneer-steve-zelditch-dies-at-68/*

4.11.2 Prof. Narahari Umanath Prabhu passes away at the age of 98

Narahari Umanath Prabhu, a professor emeritus who helped make Cornell's School of Operations Research and Information Engineering (ORIE) truly international, died in Ithaca on Oct. 14, 2022 at the age of 98.
Known to most as Uma, Prabhu was born in Calicut (now Kozhikode), India, in 1924 and earned his B.A. in mathematics at Loyola College in Madras, India, an M.A. in statistics from the University of Bombay and an M.S. in mathematics from the University of Manchester in England - all at the top of his class.
Prof. Prabhu joined the ORIE faculty in 1965 at the age of 41. He had been living in Perth, Australia, and teaching at the University of Western Australia. His work focused on stochastic processes and queueing theory and he co-founded the journal of Stochastic Processes and their Applications, which he co-edited from 1973 to 1984. He was also editor of the Queueing Systems Journal from 1986 to 1994.
Prof. Prabhu was a prolific letter writer and contacted mathematicians and researchers across the globe to ask questions and discuss ideas. Many of these researchers then sent their students to Ithaca to study with him at Cornell. In this way, he expanded the diversity of the school and helped strengthen its international reputation. Prof. Prabhu retired from teaching in 1994 but remained committed to the Cornell and Ithaca communities and continued to play an active role in the life of the school.
Source: *https://news.cornell.edu/stories/2022/11/narahari-umanath-prabhu-professor-emeritus-dies-98*

□□□

Padma Awardees -2023 from the Mathematics world

Padma Shri awardee Sujatha Ramdorai (born 1962) is an algebraic number theorist and a professor of mathematics and Canada Research Chair at University of British Columbia, Canada. Prof. Sujatha initially worked in the areas of algebraic theory of quadratic forms and arithmetic geometry of elliptic curves and is known for her work on Iwasawa theory. She was previously a professor at Tata Institute of Fundamental Research, Mumbai. She won the Shanti Swarup Bhatnagar Prize in 2004 and in 2006 became the first Indian to win the prestigious ICTP Ramanujan Prize. In 2020 she was awarded the Krieger-Nelson Prize.

5. Problem Corner

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In the October 2022 issue of TMC Bulletin, we posed a problem from Geometry for our readers. We have received three solutions to that problem, by Prof. M. R. Modak from Pune, Mr. Shrestha Suraiya, a student from Bhaskaracharya Pratishthana, Pune, and Prof. J. N. Salunke from Latur. We present these solutions here.

Also, in this issue we pose a problem from Combinatorics for our readers. Readers are invited to email their solutions to Udayan Prajapati (udayan64@yahoo.com), Coordinator, Problem Corner, before 15th March, 2023. Most innovative solution will be published in the subsequent issue of the bulletin.

Problem posed in the previous issue: Let ABCD be a quadrilateral. Let the diagonals AC and BD meet at P. Let O_1 and O_2 be the circumcentres of triangles APD and BPC. Let M, N and O be the mid points AC, BD and O_1O_2. Then, show that $OM = ON$.

Solution 1: (Solved by Dr. Madhav R. Modak, Formerly of Department of Mathematics, Sir Parashurambhau College, Pune, India.)

Case 1: First let the quadrilateral ABCD be convex. See Figure 1. Let r_1, r_2 be the radii of the circumcircles of triangle APD and BPC respectively. Let

$\angle PAD = \alpha, \angle PCB = \beta$

$\angle PDA = \gamma, \angle PBC = \delta$, and $O_1O_2 = 2d$.

Note that NO is the median through the vertex N of ΔO_1NO_2 and MO is the median through the vertex M of ΔO_1MO_2. Hence, as $OO_1 = OO_2 = d$, we have $NO_1^2 + NO_2^2 = 2ON^2 + 2OO_1^2$ or

$$ON^2 = \frac{1}{2}(NO_1^2 + NO_2^2 - 2d^2) - - - - - - - - - (1)$$

and similarly,

$$OM^2 = \frac{1}{2}(MO_1^2 + MO_2^2 - 2d^2) - - - - - - - - - (2)$$

By (1) and (2), we see that, to show that $OM = ON$, it is enough to show that

$$NO_1^2 + NO_2^2 = MO_1^2 + MO_2^2. - - - - - - - - - - (3)$$

Now, PD is a chord of the circumcircle of ΔAPD and so $PD = 2r_1 \sin\alpha$, and similarly from $\Delta BPC, BP = 2r_2 \sin\beta$. Thus, $BD = 2ND = BP + PD = 2(r_1 \sin\alpha + r_2 \sin\beta)$.

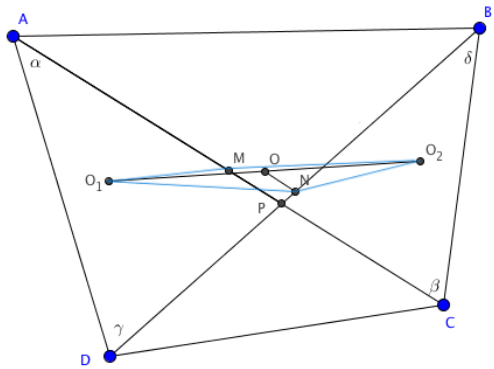

Figure 1

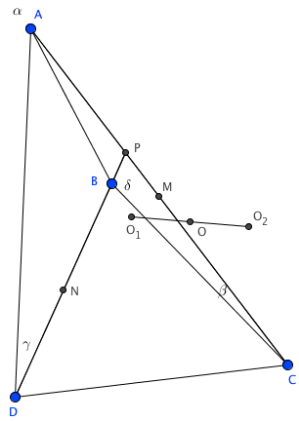

Figure 2

Next, since $\angle NDO_1 = 90^\circ - \alpha$, cosine rule for ΔO_1DN gives $NO_1^2 = O_1D^2 + ND^2 - NO_2^2 = r_2^2 + ND^2 - 2r_2.\ ND\cos(90^\circ - \alpha) = r_1^2 + ND^2 - 2r_1.\ ND\sin\alpha$. Similarly, ΔO_2NB gives $NO_2^2 = r_2^2 + ND^2 - 2r_2.\ ND\sin\beta$, as $BN = ND$. Thus

$$NO_1^2 = r_1^2 + r_2^2 + 2ND^2 - 2ND(r_1\sin\alpha + r_2\sin\beta) = r_1^2 + r_2^2, - - - - - - - - - - (4)$$

since $r_1 \sin\alpha + r_2 \sin\beta = ND$. Similarly, considering ΔO_1AM, since $\angle O_1AM = 90°$, we get by cosine rule, $O_1M^2 = r_1^2 + AM^2 - 2r_1AM\sin\gamma$, and ΔO_2MC gives $MO_2^2 = r_2^2 + CM^2 - 2r_2 . CM\sin\delta$. Since $AM = CM$, we get $O_1M^2 + O_2M^2 = r_1^2 + r_2^2$. Hence by (4), (3) follows.

Case 2: Now, let the quadrilateral ABCD be concave. See Figure 2. Then again, we only need to prove (3) and we get (4) as before. Next, as given in the figure, the rotation γ is taken to be negative. Hence, $AP = 2r_1\sin\gamma$ and $PC = -2r_2\sin\delta$. Thus, $AC = AP + PC = 2r_1\sin\gamma - 2r_2\sin\delta$. Also, as before, ΔO_1AM gives $MO_1^2 = r_2^2 + AM^2 - 2r_2AM\sin\gamma$. Finally, in ΔO_2CM, $\angle O_2CM = 180° - (90° - \delta)$. So, by cosine rule, $MO_2^2 = r_2^2 + CM^2 + 2r_2\sin\delta$, and $AM = CM$. Hence, $MO_1^2 + MO_2^2 = r_1^2 + r_2^2 + 2AM^2 - 2AM(r_1\sin\gamma - r_2\sin\delta) = r_1^2 + r_2^2 + 2AM^2 - 2AM(AM) = r_1^2 + r_2^2$. So, (3) holds in thid case also.

Solution 2: (Solved by Mr. Shresth Suraiya, Standard 9, Dhirubhai Ambani International School, Mumbai and Bhaskaracharya Pratishthana, Pune)

We prove a stronger result: $OP = OM = ON$. First, we prove $OP = OM$.
Let P_1 be the mid-point of $\overline{AP}$ and P_2 be the mid-point of $\overline{PC}$, so $AP_1 = \frac{AP}{2} = P_1P$ and $P_2C = PP_2 = \frac{PC}{2}$. Since M is the mid-point of $\overline{AC}$, $AM = MC = \frac{AC}{2} = \frac{AP+PC}{2} = P_1P + PP_2$. Now, $AP_2 = AP_1 + P_1P_2 = AM + MP_2$. Then $AP_1 = MP_2$(as$AM = P_1P_2$). This gives $P_1P = MP_2$.

Let P_3 be the mid-point of $\overline{P_1P_2}$, so $P_1P_3 = P_3P_2$. Let k be the perpendicular bisector of P_1P_2. So, $k \perp \overline{P_1P_2}$, so $k \parallel O_1P_2 \parallel O_2P_2$ (all are $\perp$ to $\overline{AC}$). Let k intersect $\overline{O_1O_2}$ at O_3. Since $P_1P_3 = P_3P_2$, we can use Equal Intercept Theorem to conclude that $O_1O_3 = O_3O_2$. Thus O_3 is the mid-point of $\overline{O_1O_2}$. However, O is also the mid-point of $\overline{O_1O_2}$. Since a segment has exactly one mid-point, O_3 and O must coincide. Hence, the perpendicular bisector k of P_1P_2 passes through O.

We had $P_1P = MP_2 = P_1P_3 - PP_3 = P_3P_2 - P_3M$. Since $P_1P_3 = P_3P_2$, we conclude that $PP_3 = P_3M$. So, P_3 is also the mid-point of PM. Hence k is also the perpendicular bisector of PM. Since O lies on k, we conclude that $OP = OM$ as a point on the perpendicular bisector of a line segment is equidistant from the two ends of that line segment.

Similarly, if we consider $\overline{BD}$, we will eventually get $OP = ON$. Combining, we get $OP = OM = ON$.

Solution 3: (By Prof. J. N. Salunke from Latur)
Let $ABCD$ be a quadrilateral in XY plane whose diagonals AC and BD meet at P and O_1 and O_2 be the circumcenters of triangles APD and BPC respectively.

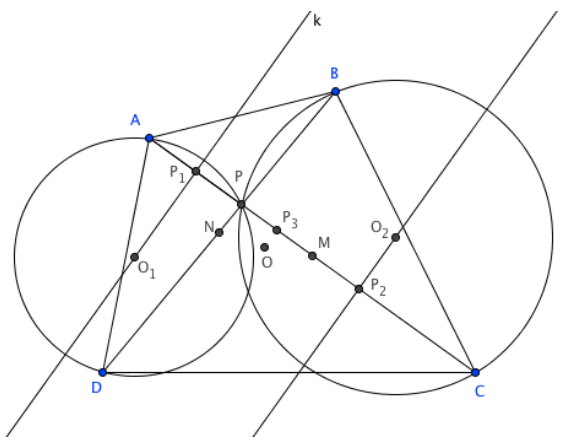

Figure 3

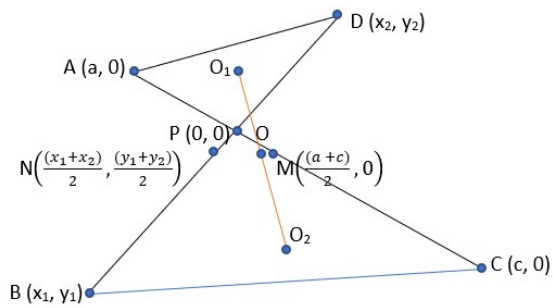

Figure 4

For simplicity in calculation, without loss of generality, we take $P \equiv (0,0)$ as origin and X-axis along AC. Then, $A \equiv (a,0)$ and $C \equiv (c,0)$ for some $a < 0$ and $c > 0$. Then, $M \equiv \left(\frac{a+c}{2}, 0\right)$. Let $B \equiv (x_1, y_1)$ and $D \equiv (x_2, y_2)$. Then, $N \equiv (\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2})$.
Slope of BD (i.e., of BP or PD) is $\frac{y_1}{x_2} = \frac{y_2}{x_2}$. $- - - - - - - - - - - - - - - (1)$
Equation of the circle circumscribing triangle APD (passing through the origin $P(0,0)$) is

$$x^2 + y^2 + 2gx + 2fy = 0. - - - - - - - - - - - - - (2)$$

It passes through $A(a,0)$ and $D(x_2,y_2)$. So, we have

$$a^2+2ga=0, \text{i.e., } g=\frac{-a}{2} \text{ and } x_2^2+y_2^2+2gx_2+2fy_2=0.$$

This gives $f=\frac{-x_2^2-y_2^2-2-(\frac{a}{2})x_2}{2y_2}=\frac{ax_2-x_2^2-y_2^2}{2y_2}$.
Then, $O_1(-g,-f)=\left(\frac{a}{2},\frac{x_2^2+y_2^2-ax_2}{2y_2}\right)$ is the circumcenter of the triangle APD.
Similarly, the circumcenter of the triangle BPC is, $O_2=\left(\frac{c}{2},\frac{x_1^2+y_1^2-cx_1}{2y_1}\right)$ and the midpoint of O_1O_2 is $O'=\left(\frac{a+c}{4},\frac{x_1^2+y_1^2-cx_1}{4y_1}+\frac{x_2^2+y_2^2-ax_2}{4y_2}\right)$. (We have replaced O by O' to avoid confusion of it with the origin (0, 0)).
By the distance formula,

$$\begin{aligned}O'M^2&=\left(\frac{a+c}{2}-\frac{a+c}{4}\right)^2+\left(\frac{x_1^2+y_1^2-cx_1}{4y_1}+\frac{x_2^2+y_2^2-ax_2}{4y_2}\right)^2\\&=\left(\frac{a+c}{4}\right)^2+\left(\frac{x_1^2+y_1^2-cx_1}{4y_1}+\frac{x_2^2+y_2^2-ax_2}{4y_2}\right)\text{----------}(3)\end{aligned}$$

$$\begin{aligned}O'N^2&=\left(\frac{x_1+x_2}{2}-\frac{a+c}{4}\right)^2+\left(\frac{y_1+y_2}{2}-\frac{x_1^2+y_1^2-cx_1}{4y_1}-\frac{x_2^2+y_2^2-ax_2}{4y_2}\right)^2\\&=\left(\frac{a+c}{4}\right)^2+\left(\frac{x_1+x_2}{2}\right)^2-\frac{(x_1+x_2)(a+c)}{4}+\left(\frac{x_1^2+y_1^2-cx_1}{4y_1}+\frac{x_2^2+y_2^2-ax_2}{4y_2}\right)^2+\\&\quad\left(\frac{y_1+y_2}{2}\right)^2-(y_1+y_2)\left(\frac{x_1^2+y_1^2-cx_1}{4y_1}+\frac{x_2^2+y_2^2-ax_2}{4y_2}\right)\\&=O'M^2+\left(\frac{x_1+x_2}{2}\right)^2-\frac{(x_1+x_2)(a+c)}{4}+\left(\frac{y_1+y_2}{2}\right)^2-(y_1+y_2)\\&\quad\left(\frac{x_1^2+y_1^2-cx_1}{4y_1}+\frac{x_2^2+y_2^2-ax_2}{4y_2}\right)\ (by\ (3))\\&=O'M^2+\frac{x_1^2+x_2^2}{4}+\frac{x_1x_2}{2}-\frac{a}{4}(x_1+x_2)-\frac{c}{4}(x_1+x_2)+\frac{y_1^2+y_2^2}{4}+\frac{y_1y_2}{2}\\&\quad-\frac{x_1^2+y_1^2-cx_1}{4}-\frac{y_2x_1^2}{4y_1}-\frac{y_1y_2}{4}+\frac{cx_1y_2}{4y_1}-\frac{y_1x_2^2}{4y_2}-\frac{y_1y_2}{4}+\frac{ay_1x_2}{4y_2}-\frac{x_2^2+y_2^2-ax_2}{4}\\&=O'M^2+\frac{x_1x_2}{2}-\frac{a}{4}(x_1+x_2)-\frac{c}{4}(x_1+x_2)+\frac{cx_1}{4}-\frac{y_2x_1^2}{4y_1}+\frac{cx_1y_2}{4y_1}-\frac{y_1x_2^2}{4y_2}+\frac{ay_1x_2}{4y_2}+\frac{ax_2}{4}\\&=O'M^2+\frac{x_1x_2}{2}-\frac{ax_1}{4}-\frac{cx_2}{4}-\frac{y_2x_1^2}{4y_1}+\frac{cx_1y_2}{4y_1}-\frac{y_1x_2^2}{4y_2}+\frac{ay_1x_2}{4y_2}\\&=O'M^2+\frac{x_1x_2}{2}-\frac{ax_1}{4}-\frac{cx_2}{4}-\frac{y_2x_1x_2}{4y_2}+\frac{cx_2y_2}{4y_2}-\frac{y_1x_2x_1}{4y_1}+\frac{ay_1x_1}{4y_1}\quad(by\ (1))\\&=O'M^2\end{aligned}$$

Hence, $O'N=O'M$, i.e., $ON=OM$.
(The solution by Dr. M. R. Modak is for both convex and concave quadrilaterals. The solution by Mr. Shresth is for the convex case. The solution by Prof. Salunke uses co-ordinate geometry.
- Editors)

Problem for this issue

Show that there is no positive integer n such that (1000^n-1) divides (2023^n-1).

□□□

6. International Calendar of Mathematics Events

Ramesh Kasilingam
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Email: rameshk@iitm.ac.in

February 2023

- February 11-12, 2023, The South Central Topology Conference II, Texas State University San Marcos, Texas. *https://www.hiroleetanaka.com/sctc/*
- February 4-5, 2023, Combinatorial Number Theory and Connected Topics - II (CONTACT - II), an online conference. *https://sites.google.com/view/contact-ii/home*
- February 13-15, 2023, Stuttgart, Germany: workshop Young perspectives on ihs manifolds, at the University of Stuttgart, Germany. *https://www.idsr.uni-stuttgart.de/ihs2023/*

March 2023

- March 6-8, 2023, Zeta Functions and Their Representations, a RIMS conference to promote women's participation in mathematics research, Kyoto University RIMS, Kyoto, Japan. *https://sites.google.com/view/rims2022zetafunctions-en/top*
- March 11-12, 2023, Southern Regional Number Theory Conference (SRNTC 2023), Louisiana State University, Baton Rouge, LA, USA. text*https://www.math.lsu.edu/ srntc/nt2023/*
- March 20-24, 2023, Nonlocal Operators and Markov Processes Będlewo, Poland. *https://nomp. pwr.edu.p*
- March 20-24, 2023, Homotopy theory in honor of Paul Goerss @ Northwestern. *https://antieau. github.io/workshops/202301-goerss.html*
- March 24-25, 2023, International Conference on Mathematics and its Application in Technology 2023 (ICMAT2023), Srinivasa Ramanujan Centre, SASTRA Deemed to be University, Kumbakonam-612001, Tamil Nadu, India. *www.sastra.edu/icmat2023*
- March 24-25, 2023, Artificial Intelligence Seminar, 2023 ANA Crowne Plaza Narita, Japan Address: 68 Horinouchi, Narita, Chiba 286-0107, Japan. *https://artificialintelligence.averconfe- rences.com*
- March 24-26, 2023, International Conference on Mathematical Science and Applications (ICMSA-2023) and 24th Annual Conference of Vijnana Parishad of India Department of Mathematics (Centre of Excellence, U.P. State Govt.) Dr. Bhimrao Ambedkar University, Agra-India. *http://dbrau.ac.in/2023/01/09/icmsa-2023/*
- March 29-31, 2023 International Conference on Recent Developments in Mathematical Modelling in Biology North-West University, Potchefstroom, South Africa. *https://natural-sciences. nwu.ac.za/paa/3MC-Conference-BM*
- March 31, 2023, April 2, 2023, Redbud Topology Conference, University of Oklahoma Norman, Oklahoma, USA. *https://redbud.math.ou.edu/spring-2023/*

April 2023

- April 1-2, 2023, Spring Eastern Sectional Meeting, Virtual Meeting virtually, hosted by the American Mathematical Society. *www.ams.org/meetings/sectional/2305_program.html*
- April 3-6, 2023, Number Theory Workshop - BMC 2023, University of Bath, Bath, UK. *https://sites.google.com/view/bmc-2023/workshops/number-theory*
- April 17-20, 2023, WINGS (Women in Number Theory and Geometry), a conference aimed at underrepresented genders in Number Theory and Geometry, 2023, Ringwood Hall, Chesterfield, Derbyshire, UK. *https://www.ucl.ac.uk/ ucahbar/wings.html*
- April 24-28, 2023, Trends in Computational Discrete Optimization ICERM Brown University, Providence, Rhode Island, USA. *https://icerm.brown.edu/programs/sp-s23/w3/*
- April 26-28, 2023, "Modern Problems of Mathematics and Mechanics -2023" Institute of Mathematics and Mechanics, Baku, Azerbaijan. *http://mpmm.imm.az/en/*

- April 27-28, 2023, 4th IMA and OR Society Conference on Mathematics of Operational Research Conference Aston, Birmingham, United Kingdom.
  https://ima.org.uk/20140/4thmath sofor/

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7. The 10th Heidelberg Laureate Forum (HLF)

T. R. Ramadas
(Deadline of applications: 11 February 2023)
Professor Emeritus, Chennai Mathematical, Kelambakkam 603103
Email: ramdas@cmi.ac.in

Heidelberg Laureate Forum: The 10th Heidelberg Laureate Forum (HLF) will take place in Heidelberg, Germany between 24-29 September 2023.
The HLF serves as a great platform for interaction between the masters in the fields of mathematics and computer science and young talents. Over the course of the week-long conference, young researchers will be given the exclusive possibility to profoundly connect with their scientific role models and find out how the laureates made it to the top of their fields.
The previous HLFs have been an exceptional success. As described by a young researcher, "Meeting the humans behind some of the most formidable inventions and discoveries of our time was phenomenal. What was unexpected, perhaps, was how warm and approachable these individuals turned out to be. I am grateful to have been given this opportunity, and to have shared it with peers who I hope to soon call colleagues, collaborators and friends".
Participation: At the HLF, all winners of the Fields Medal, the Abel Prize, the ACM A.M. Turing Award, the Nevanlinna Prize / IMU Abacus Medal, and the ACM Prize in Computing are invited to attend. In addition, young and talented computer scientists and mathematicians are invited to apply for participation.
Applications for participation at the 10th HLF are open in three categories: Undergraduate/Pre-Master, Graduate PhD, and Post Docs.
The IMU Adhering Organizations and national mathematical societies can also nominate young researchers. Nominated persons get "priority treatment", but, since there may be too many nominations, they have no acceptance guarantee. You may use the code IMU62851 when registering.
See the webpage *https://application.heidelberg-laureate-forum.org* for the online application and nomination forms. The deadline for application is 11 February 2023.
For questions regarding requirements and the application process, please contact Young Researchers Relations at: *yr@heidelberg-laureate-forum.org*.
Important Dates: The application period for the 10th HLF runs from 11 November 2022 until 11 February 2023. Young researchers at all phases of their careers (Undergraduate/Pre-Master, Graduate PhD, or Post Docs) are encouraged to complete and submit their applications by 11 February (midnight CET).
Selection Process: All applications that are completed and submitted by the deadline are meticulously reviewed by an international committee of experts to ensure that only the most qualified candidates are invited. There are 100 spaces available for each discipline of mathematics and computer science. All applicants will be notified by the end of April 2023 whether or not they are invited.
For more information, please visit: *www.heidelberg-laureate-forum.org*

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8. TMC Activities

S. A. Katre
Lokmanya Tilak Chair, S. P. Pune University, Pune, Email: sakatre@gmail.com

1. **One week workshop on "Mathematics and its Applications"** was organised by Department of Mathematics, University of Kashmir-South Campus, Fatehgarh, Anantnag J&K-192101, during September 16 - 22, 2022, in Collaboration with The Mathematical Consortium (TMC) and Kashmir Mathematical Society (KMS).
Speakers: 16th M. A. Sofi (Integral & Distance in finite & infinite dimensional spaces), A. P. Singh (Analytic functions - 2 talks). 17th V. K. Kukreja (perturbed Differential Equations), V. P. Saxena (BVP & I-function), Saket Choudhry (Uncertainty, Entropy, Information). 19th P. G. Siddheshwar (Fluid Dynamics), Kanchana C. (LaTeX, MatLab - 2 talks), M. A. Khanday (Human Physiology & Biothermal modelling). 20th M. A. Mir (Quaternion functions), M. H. Gulzar (Umer Khayam), B. A. Zargar (Mathematics as Art), Rajneesh Kumar (Tensors in Biomechanics). 21st S. M. Mallikarjunaiah (Finite Element Method), Nikhil Kumar Rajput (Machine Learning), Subuhi Khan (D.E. of orthogonal polynomials), Aftab Ahmad Shah (Semigroups). 22nd S. Pirzada (Fascinating Mathematics). Two discussion sessions were arranged on 17th and 22nd.
Patron: Dr. Mukhtar Ahmad Khanday, Director, UOK, South Campus;
Convener: Dr. Faroz Ahmad Bhat

2. **A Seminar entitled "Importance of Statistics in Data Science"** was organised by the Bhaskara's Association, Dept. of Mathematics, Faculty of Science, SGT Univ., Gurugram (Gurgaon), Hariyana- 122505, under The Mathematical Consortium (TMC) Regional Programmes on 20th October 2022 during 11:00 AM-1:00 PM.
Speakers: (i) Prof. S. C. Malik, Former Head, Dept. of Statistics, MD University Rohtak, Haryana Topic: Why Statistics for Data Analysis? He covered topics such as data types, quantitative tools, statistical subdisciplines, and hypothesis testing, type of distributions, different aspects of use of statistics in data science.
(ii) Dr Himanshu Choudhary, Asst. Vice President at Marke Topper Securities Pvt. Ltd., Gurugram Topic: "Importance of Statistical Tools in Data Analysis". The speaker discussed on the theme with the help of PYTHON and R-software, gave live demonstrations on the use of statistics in data analysis of Indian stock market, described different available statistical tools that can be applied to improve the performance of one's investment portfolio, explained planning, creating, and carrying out R&D through use of tools like R-software, Python, and SQL.
More than 125 participants, including UG/PG students from different programmes including Mathematics, Physics, Chemistry, Forensic, Nursing, EVS and non-medical, also research scholars, and teachers from several SGTU faculties, attended the event
Convener: Dr. Pradeep Malik, Head, Dept. of Math., FOSC, SGT Univ.

3. **A 2-day Workshop for College Teachers** was organised by the Department of Mathematics, IISER-TVM jointly with TMC, Bhaskaracharya Pratishthana and Ganit Mandal, Pune, during 10-11 December 2022. It was attended by 19 teachers and in addition local students. There were talks of 1 and ½ hour duration as follows:
10th Dec.: Prof. Jugal Verma (IIT Bombay), An introduction to Erhart's theory of lattice points, Prof. Sheetal Dharmatti (IISER-TVM), Continuous but nowhere differentiable functions, Prof. S. A. Katre (S. P. Pune Univ.), Ancient Indian Mathematics: From Bodhayana to Madhavacharya, Prof. Ramiz Reza (IISER-TVM), Wold-von Neumann decomposion of isometries and Wold-type decomposition for 2-isometries.
11th Dec.: Prof. K. N. Raghavan (IMSc, Chennai), Counting of orbits and its applications, Prof. Viji Thomas (IISER-TVM), Archimedes Challenge. On the 2nd day, in the afternoon session, a discussion session was conducted with participant teachers related to activities in Mathematics in various places in India, educational journals in Mathematics, Mathematics competitions in India, job opportunities in Mathematics etc.
Organizer: Prof. Viji Thomas, Head, Dept. of Math., IISER-TVM.

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Some TMC Activities - 2022

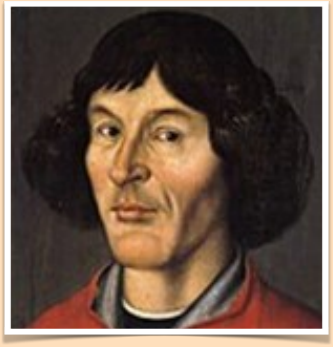
Workshop at University of Kashmir-South Campus.
From the left: Profs. S. Pirzada, Irshad A. Nawchoo, Mukhtar A. Khanday, Director

Prof. S. K. Malik, M. D. Univ. Rohtak lecturing at SGT Univ., Gurugram, Hariyana
In Audience: Dr. Himanshu Choudhary, Prof. R. C. Sharma & Prof. Lakhwinder Singh

Prof. S. A. Katre Lecturing at IISER, TVM.
Also seen, Prof. J. K. Verma & Dr. Viji Thomas

Leonid Vitaliyevich Kantorovich (19 Jan. 1912 - 07 April 1986)

A Soviet mathematician and economist. Developed theory and techniques for the optimal allocation of resources. He is regarded as the founder of linear programming. He was the winner of the Stalin Prize in 1949 and the Nobel Memorial Prize in Economic Sciences in 1975.



Nicolaus Copernicus (19 Feb. 1473 - 24 May 1543)

A Polish astronomer, physician, physicist, jurist, economist, mathematician, diplomat, translator, artist, philosopher, who formulated a model of the universe that placed the Sun at the center of the universe. Derived the quantitative theory of money and formulated an economic principle that later came to be called Gresham's law.

Jean-Baptiste Joseph Fourier (21 Mar. 1768 - 16 May 1830)

A French mathematician and physicist. Known for his investigation of Fourier series, which eventually led to the development of Fourier analysis and harmonic analysis, and their applications to problems of heat transfer and vibrations. Fourier is also generally credited with discovery of the greenhouse effect.

Publisher

The Mathematics Consortium (India),
(Reg. no. MAHA/562/2016 /Pune dated 01/04/2016),
43/16 Gunadhar Bungalow, Erandawane, Pune 411004, India.
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AUM Copy Point, G-26, Saffron Complex, Nr. Maharana Pratap Chowk,
Fatehgunj, Vadodara-390001; Phone: 0265 2786005;
Email: aumcopypoint.ap@gmail.com

Annual Subscription for 4 issues (Hard copies)

Individual	: TMC members: Rs. 700;	Others: Rs. 1000.
Societies/Institutions	: TMC members: Rs. 1400;	Others: Rs. 2000.
Outside India	: USD ($) 50.	

The amount should be deposited to the account of
"The Mathematics Consortium", Kotak Mahindra Bank, East Street Branch,
Pune, Maharashtra 411001, INDIA.

Account Number: 9412331450, IFSC Code: KKBK0000721