

The Mathematics Consortium



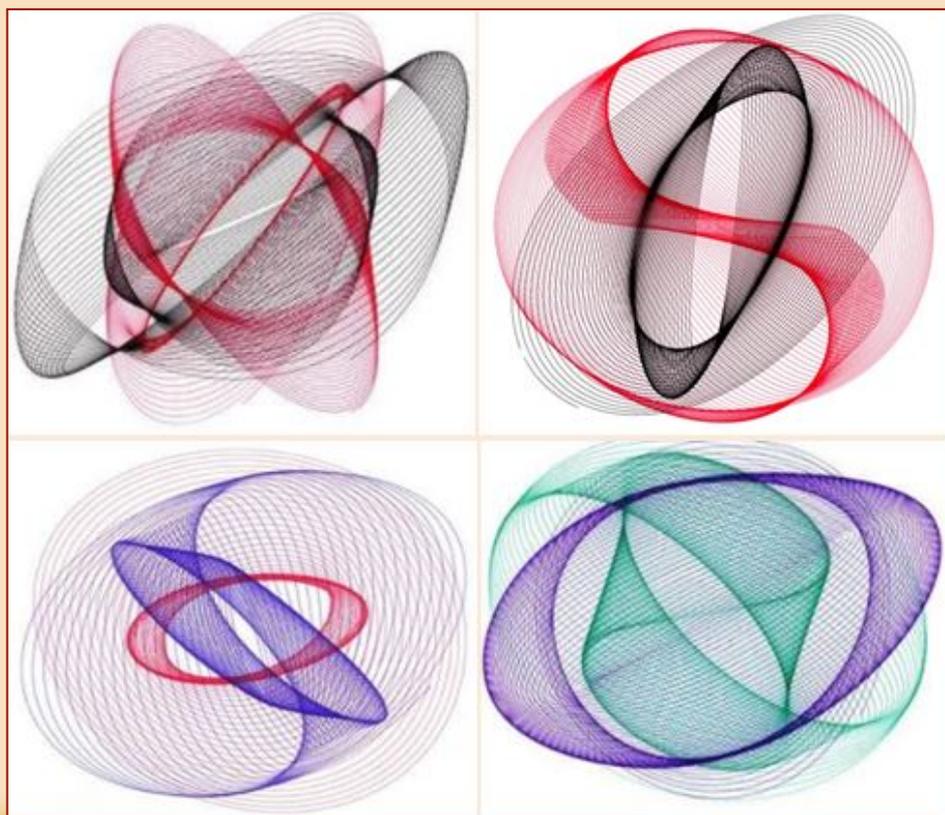
BULLETIN

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HARMONOGRAMS



Mathematics, Physics and Engineering come together to create artistic Geometrical designs

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About the cover-page: A harmonograph, a new exhibit at the Matthews-Fuller Health Sciences Library, as shown in the figure on the left, is a mechanical apparatus that employs pendulums to create a geometric image. The drawings it creates are called Lissajous curves or Harmonograms. Some such Harmonograms are depicted on the front cover.



The harmonograph was invented in 1844 by Hugh Blackburn, a professor of mathematics at the University of Glasgow. It employs three pendulums to control the movement of a pen relative to a drawing surface. Two linked pendulums move a pen in a circular motion along one axis and the third pendulum moves the drawing surface in a rotary motion along a perpendicular axis. By varying the frequency and phase of the pendulums relative to one another, different patterns are created. The Artist Don Fitzpatrick, the IT Specialist for the Dartmouth Biomedical Libraries, built the harmonograph used in this exhibit based on several different designs in order to make a machine that could be portable and multi functional.

Source: The Harmonograph: A New Exhibit at the Matthews-Fuller Health Sciences Library Posted on April 21, 2015 by Donald Fitzpatrick.

From the Editors' Desk

According to the new guidelines issued by AICTE in March 2021, Physics and Mathematics are no longer compulsory subjects in the 12th grade for admission to B. E. and B. Tech. programmes in AICTE approved Engineering Institutes.

Under the new norms, a candidate is eligible for getting admission in B.E. and B.Tech. programmes if he has scored at least 45% in any three subjects out of the list of 14 provided in the new handbook, which are Physics, Mathematics, Chemistry, Computer Science, Electronics, Information Technology, Biology, Agriculture, Informatics Practices, Biotechnology, Technical Vocational subject, Engineering Graphics, Business Studies and Entrepreneurship. This rule will come into effect from the academic year 2021-22 onwards.

However, if a student selects three subjects which do not have much connection with the science stream, whether he can study Engineering, is a Big question. AICTE suggests a remedy that Universities can offer suitable bridge courses in Mathematics for students coming from diverse backgrounds to achieve the desired learning outcome of the programme.

The academic community, however, believe that a bridge course is a remedial course and can only fill a gap in learning. It cannot be a foundational course. It is important to note that in standards 11 and 12 we introduce important concepts of limit, continuity, differentiability and integrability which form a basis for higher level Mathematics useful in Engineering and Technology. And it is not possible to cover all these topics in just a half semester / a semester bridge course and that too along with higher level topics taught in the first semester.

In fact, the growing spirit of interdisciplinary engineering education necessitates foundational study of Mathematics in high school even for isolated and specific engineering programmes like Textile Engineering or Biotechnology. As it is, the quality of engineers that are produced by most of the technical Institutes (barring a few elite Institutes) is not adequate enough to get them a job befitting to their qualifications. We believe that such a blanket dilution of criteria for admission to engineering courses will further worsen the situation. Hence, it is worthwhile to reconsider the new guidelines and / or the measures to ensure the desired learning outcomes.

In Article 2, Prof. Vijay D. Pathak discusses the role of the Mathematics departments in the Technical Institutes and suggest some logistic and qualitative measures to improve the quality of Mathematics education in Technical Institutes.

In the first article of the issue entitled "Pandemics and paradox: surprising math in the simplest models" Prof. Scott Sheffield from MIT, USA considers the low prevalence linearized SEIR model for the spread of an epidemic. The author shows that the best possible strategy is to alternate between high and low activity, for keeping the number of infections low. Similar results are also obtained for different notions of activity incorporating crowding effects.

Dr. D. V. Shah writes about important events which occurred in the Mathematics world during last six months. In the previous April issue of our bulletin, we paid special tributes to one of the distinguished Indian mathematicians Prof. C. S. Seshadri and within a short span we lost another distinguished mathematician Padma Bhushan Prof. M. S. Narasimhan who made fundamental contributions to diverse fields in mathematics. We pay our spontaneous tributes to Prof. Narasimhan and other mathematicians who left us in the recent past. We also look forward to bringing to our readers more material related to Prof. Narasimhan, in one of the forthcoming issues of the Bulletin, as our special tribute to him..

In Problem Corner, Dr. Udayan Prajapati presents a problem involving the greatest integer function along with its solution and poses a problem from graph theory for our readers. Dr. Ramesh Kasilingam gives a calendar of Academic events, planned during August, 2021 to October, 2021. Prof. Sudhir Ghorpade gives update on TMC-Distinguish Lecture Series.

We are very happy to bring out the first issue of Volume 3 in July 2021. We thank all the authors, all the editors, our designers Mrs. Prajakta Holkar and Dr. R. D. Holkar and all those who have directly or indirectly helped us in bringing out this issue on time.

Chief Editor, TMC bulletin.

1. Pandemics and paradox: surprising math in the simplest models

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Scott Sheffield is the Leighton Faculty Professor of Mathematics as of July 2017. He joined the MIT faculty as Professor in 2008, following a faculty appointment at the Courant Institute at NYU. He received a PhD in mathematics from Stanford University in 2003 under the supervision of Amir Dembo, and completed the AB and AM degrees in mathematics from Harvard in 1998. His interests include probability, game theory and mathematical physics, with a special focus on random objects with fractal properties, including curves, surfaces, growth processes, and trees. He is especially known for his work on models with conformal symmetries, including Schramm-Loewner evolutions, Gaussian free fields and Liouville quantum gravity. A Sloan fellow and NSF Faculty CAREER awardee, Sheffield received the 2006 Rollo Davidson award for work on spatial models of probability theory and especially their relationship to stochastic (Schramm) Loewner evolutions. He received the Presidential Early Career Award for Scientists and Engineers in 2009; Michel Loeve International Prize in Probability, awarded by U.C. Berkeley in 2011; a Simons Fellowship in Mathematics in 2014; the Clay Research Award (with Jason Miller, former Schramm Fellow) in 2017. He also received an Aisenstadt Chair at the Center for Mathematical Research, University of Montreal for September 2016. In 2021, Sheffield was elected Fellow of the American Academy of Arts and Sciences. He is nominated as a plenary speaker of the ICM 2022.

Abstract. We consider the low-prevalence linearized SEIR epidemic model for a society that has resolved to keep future infections low in anticipation of a vaccine. The society can vary its amount of potentially-infection-spreading activity over time, within a certain feasible range. Because the activity has social or economic value, the society aims to maximize activity overall subject to infection rate constraints.

We find that consistent policies are the *worst possible* in terms of activity, while the best policies alternate between high and low activity. In a variant involving multiple subpopulations, we find that the best policies are maximally *coordinated* (maintaining similar prevalence among subpopulations) but *oscillatory* (having growth rates that vary in time).

It turns out that linearized SEIR is mathematically equivalent to an idealized racecar model (with different subpopulations corresponding to different cars) and the amount of fuel used corresponds to the amount of activity. Using this analogy, steady V-shaped formations (in which one subpopulation “leads the way” with consistently higher prevalence and activity, while others follow behind with lower prevalence and activity) are especially problematic. These formations are *very effective* at minimizing fuel use, hence *very ineffective* at boosting activity. In an appendix, we obtain analogous results for alternative notions of activity, which incorporate crowding effects.

1.1 INTRODUCTION

Non-pharmaceutical interventions (NPIs), including the prohibition or rescheduling of activities, are used to limit the damage caused by pandemics. When managing a pandemic, a government or population may decide that the cost of obtaining significant herd immunity through infection is unacceptably high, and that it is therefore necessary to maintain low disease prevalence through NPIs until vaccines are available.

One important and controversial question is the following: once disease prevalence is low (say, 50 confirmed cases per million per day), is it better to keep the effective reproductive rate as close to one as possible (aiming for consistency and sustainability) or to “suppress and resuppress” (i.e.,

drive the prevalence down even further, then relax restrictions, then restore restrictions if/when prevalence rates recover, etc.)?

In a recent paper, the author and others showed that strict but intermittent measures were better than consistently moderate measures at optimizing certain utility functions within the low-prevalence limit of SEIR/SEIS and its variants 3. In this follow-up note, we show that consistent strategies are actually the *least effective* when measured by a certain type of *activity*. We also work out the *most effective* strategies, including in settings where upper and lower bounds on the infection rate are imposed. We then explore settings with multiple subpopulations and find that, in terms of prevalence, it is much better to aim for *geographic consistency* and *temporal variability* than other way around.

The models in this paper are simplifications that omit important considerations. They are meant to generate hypotheses and inform judgment, not resolve real world questions on their own. Nonetheless, it is interesting that, within these models, some of the *intuitively-best-sounding* practices are actually the worst.

Acknowledgement: We thank Morris Ang, Minjae Park, Joshua Pfeffer, Pu Yu, and the co-authors of [3] for useful conversations. The author is partially supported by NSF award DMS 1712862.

1.2 METHODS

1.2.1 Setup and motivation: a linearized SEIR optimal control problem

During a pandemic, there may come a time when a state or country resolves to keep its total number of future infections small (perhaps less than 2 percent of the population) up until a later time T at which a vaccination program will commence. If we assume dynamics are given by a standard SEIR model, this means that S and R can be treated as (essentially) constant for the remaining duration. We choose our time unit so that the mean incubation time is 1, and the mean infectious time is the constant γ^{-1} . We then obtain the *linearized* ODE [1] given by

$$\dot{E}(t) = -E(t) + \beta(t)I(t), \quad \dot{I}(t) = E(t) - \gamma I(t), \quad (1.1)$$

where $E(t)$ represents the fraction of the population *exposed* (infected but not yet infectious), $I(t)$ represents the fraction that is *infectious*, and $\beta(t) \in [\beta_{\min}, \beta_{\max}]$ is a control parameter (a measurable function of t) describing the rate at which disease transmission occurs at time t .

We assume that there is some *flexible activity* (haircuts, conversations, surgeries, lessons, factory shifts, etc.) that has social/economic value but also carries transmission risk. By “flexible” we mean that its utility is not dependent on *when* it occurs. Our policy tool is deciding how much of this activity to schedule/allow and when to do so. For now, we assume that the disease transmission caused by flexible activity is primarily due to the activity itself (not ancillary crowding effects) so that disease transmission is linear in the amount of activity. (We will discuss alternatives in Appendix 1.6.) We interpret $\beta(t)$ as the *total* amount of transmission-inducing activity (flexible or otherwise) happening at time t . We interpret β_{\min} as the amount of transmission-inducing activity that occurs when no flexible activity is scheduled, so that $\beta(t) - \beta_{\min}$ is the amount of *flexible* activity at time t . We interpret $\beta_{\max} - \beta_{\min}$ as the maximal amount of flexible activity that can be scheduled at once (due to limitations on space or on the number of individuals available to be active). Define the *total activity* by $A = \int_0^T \beta(t)dt$. Our goal will be to find strategies that (subject to restrictions) maximize A .

The main assumption underlying this goal is that the social/economic utility derived from flexible activity depends only on A , not on how the flexible activity is temporally distributed. We *do not* assume that all flexible activity is of equal value. For example, we allow for the possibility that if A were small, only very important activity would be allowed, but if A were larger, more

discretionary activity would take place.¹ As long as social/economic utility is an increasing function of A (*not necessarily linear*) it is sensible for maximizing A to be an objective.

Alternative objective functions (accounting for activity that is *not* perfectly flexible, e.g. because people pursuing different activities at once might *crowd* each other in a way that increases transmission) will be discussed in Appendix 1.6. For now, we will focus only on maximizing A , not on minimizing total infections. (Effectively, we are assuming that prevalence is low enough that further minimizing infections is not the primary consideration.) But we will consider imposing upper and lower bounds on infection rates. See also [3] for further references, as well as some discussion of the probability distributions governing incubation and infectious periods, social networks, and other factors beyond the scope of this note.

We assume a vaccine will arrive at time T . We do not model the vaccination strategy, but we allow that the cost of controlling the disease during its implementation may depend on the terminal values $E(T)$ and $I(T)$. Thus it is of interest to find the optimal $\beta : [0, T] \rightarrow [\beta_{\min}, \beta_{\max}]$ yielding any particular choice of $E(T)$ and $I(T)$, and to find the amount of activity that approach yields. (Finding the optimal values for $E(T)$ and $I(T)$ would then be a second step, and would depend on the vaccination rollout model used.)

1.2.2 Problem statements

Problem 1.2.1. *Given $E(0)$, $I(0)$, $E(T)$ and $I(T)$, find the β that maximizes A .*

To simplify the presentation, we change coordinates to reduce Problem 1.2.1 to a one-dimensional problem. Define the *velocity* of the disease to be $V(t) := E(t)/I(t)$. The term “velocity” is motivated by the fact that

$$\frac{\partial}{\partial t} \log I(t) = \frac{\dot{I}(t)}{I(t)} = V(t) - \gamma, \quad \log \frac{I(T)}{I(0)} = \int_0^T V(t) dt - \gamma T, \quad (1.2)$$

so that $V(t)$ is (up to additive constant) the rate at which $\log I(t)$ is changing. Computing further we find

$$\begin{aligned} \dot{V}(t) &= \frac{\dot{E}(t)I(t) - \dot{I}(t)E(t)}{I(t)^2} \\ &= \frac{(-E(t) + \beta(t)I(t))I(t) - (E(t) - \gamma I(t))E(t)}{I(t)^2} \\ &= \frac{-E(t)I(t) + \beta(t)I(t)^2 - E(t)^2 + \gamma I(t)E(t)}{I(t)^2} \\ &= \beta(t) - V(t)^2 + (\gamma - 1)V(t) \\ &= \beta(t) - \phi(V(t)), \end{aligned} \quad (1.3)$$

where ϕ is the quadratic function defined by $\phi(x) := x^2 + (1 - \gamma)x$. This also implies $\beta(t) = \dot{V}(t) + \phi(V(t))$. We then have

$$A = \int_0^T \phi(V(t)) dt + V(T) - V(0) = \int_0^T V(t)^2 dt + \int_0^T (1 - \gamma)V(t) dt + V(T) - V(0). \quad (1.4)$$

The existence and uniqueness of solutions to (1.3) are immediate from the Carathéodory existence theorem and the corresponding uniqueness conditions, provided β is a measurable

¹For example, if a maskless conversation contributes twice as much disease transmission risk as a similar masked conversation, then the maskless conversation would count as twice as much “activity.” So “removing a mask during a close conversation” could be treated as a form of discretionary activity that might only occur in larger A scenarios. If an 8-person party involves more than twice as much transmission as a 4-person party, then it would count as more than twice as much activity.

function from $[0, T]$ to $[\beta_{\min}, \beta_{\max}]$. See [2, Theorem 5.3]hale1980ODE. If the reader prefers not to consider this much generality, it is fine to focus on the case that β is piecewise continuous (or even piecewise constant).

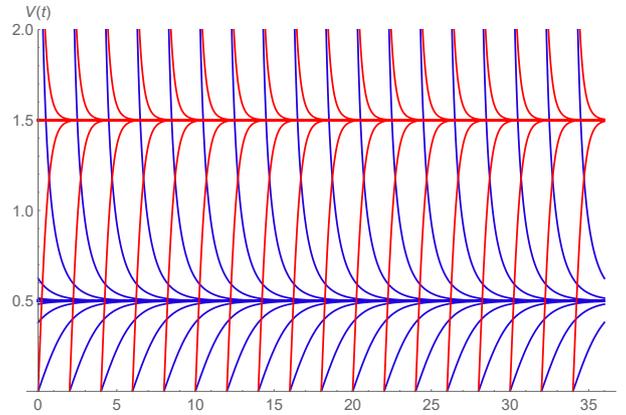


Figure 1.1: **Blue:** solutions to $\dot{V}(t) = \beta(t) - \phi(V(t))$ with $\beta(t) = \beta_{\min} = 1/4$. Here $\gamma = 1$ so $\phi(x) = x^2$. **Red:** solutions with $\beta(t) = \beta_{\max} = 9/4$. **Black:** possible solution with $\beta(t)$ varying within $[\beta_{\min}, \beta_{\max}]$. Regardless of the initial $V(0) > 0$, the (blue) solutions with $\beta(t) = \beta_{\min}$ for all time (no flexible activity) converges to $v_{\min} = \phi^{-1}(\beta_{\min}) = 1/2$. Similarly, the (red) solutions with $\beta(t) = \beta_{\max}$ for all time (maximal flexible activity) converge to $v_{\max} = \phi^{-1}(\beta_{\max}) = 3/2$. Generally, the $V(t)$ satisfying (1.5) are Lipschitz continuous curves such that $\dot{V}(t)$ (which is a.e. defined) always lies *between* the derivatives of the blue and red curves that pass through $(t, V(t))$. Other examples include **bang-bang alternators**, where $V(t)$ alternates between tracing red and blue curves as in Figure 1.3 and **constant functions** $V(t) = c$ for $c \in [v_{\min}, v_{\max}]$. The **special constant function** $V(t) = \gamma = 1$ is the curve for which $E(t)$, $I(t)$, and $\beta(t) = 1$ all remain constant. The time unit is the mean incubation period; if mean incubation is 4 days, then the 36 units above represent 144 days.

Now note that (1.2) and the definition of V imply that fixing the quadruple $(E(0), I(0), E(T), I(T))$ modulo multiplicative constant is equivalent to fixing the triple $(V(0), V(T), \int_0^T V(t)dt)$. So we rephrase Problem 1.2.1 as an equivalent problem of maximizing (1.4) given this triple. Since the latter three RHS terms are determined by the triple, this is equivalent to maximizing the first term $\int_0^T V(t)^2 dt$, hence equivalent to the following:

Problem 1.2.2. Given prescribed values for $V(0)$, $V(T)$, and $\int_0^T V(t)dt$, find a Lipschitz function V that maximizes $\int_0^T V(t)^2 dt$ subject to the constraint $\dot{V}(t) + \phi(V(t)) \in [\beta_{\min}, \beta_{\max}]$ for all t , or equivalently

$$\beta_{\min} - \phi(V(t)) \leq \dot{V}(t) \leq \beta_{\max} - \phi(V(t)). \tag{1.5}$$

Alternative phrasing: let t be a random variable chosen uniformly from $[0, T]$ and choose V to maximize the variance of $V(t)$ given (1.5) and a prescribed value for the expectation of $V(t)$, as well as $V(0)$ and $V(T)$.

See Figure 1.1 for an intuitive picture of what the $V(t)$ satisfying (1.5) are like. For any $b > 0$ we write $\phi^{-1}(b)$ for the unique positive solution x to $\phi(x) = b$. By the quadratic formula, $\phi^{-1}(b) = \frac{(\gamma-1) + \sqrt{(\gamma-1)^2 + 4b}}{2}$. It is easily seen from (1.3) that if we set $\beta(t) = b$ for all t , then $V(t)$ converges to $\phi^{-1}(b)$ as $t \rightarrow \infty$ (regardless of the initial value $V(0)$). For short, write $v_{\min} = \phi^{-1}(\beta_{\min})$ and $v_{\max} = \phi^{-1}(\beta_{\max})$. Note that if $V(0) \in [v_{\min}, v_{\max}]$ then $V(t) \in [v_{\min}, v_{\max}]$ for all time t , regardless of β .

We also consider a constrained version:

Problem 1.2.3. Solve Problem 1.2.1 with the added constraint that $C_1 \leq \log I(t) \leq C_2$ for all $t \in [0, T]$.

As motivation, note that imposing an upper bound on $\log I(t)$ is a way to ensure that a health care system is not overwhelmed and to limit the daily risk assumed by individual workers. It is also a crude way to ensure that the overall number of infections does not become too large: perhaps $I(t) = e^{C_2}$ is about the level at which the price of infection becomes unacceptable. On the other side, if neighboring states and countries have not eliminated the disease, and are maintaining steady levels of infection, then cases may be reintroduced from those localities at some small but steady rate, which would effectively impose a lower bound on $\log I(t)$.²

1.2.3 Physics analogy

As an instructive metaphor, interpret $X(t) := \log I(t) + \gamma t$ as the *position* of a rocket-powered car along a frictionless street, the derivative $\dot{X}(t) = V(t)$ as the *velocity*, and the second derivative $\ddot{X}(t) = \beta(t) - \phi(V(t))$ as the *acceleration*. Interpret $\beta(t)$ as an *internal force* applied via the gas pedal and $-\phi(V(t))$ as an *external force* which is a quadratic function of the velocity, accounting for wind resistance and/or gravity.³ Here β_{\min} corresponds to the gas pedal not being pressed and β_{\max} corresponds to a fully pressed pedal (there are no brakes). We can interpret A as the total amount of fuel used⁴ and imagine we are trying to waste as much fuel as possible subject to given boundary conditions. When we impose the constraints $C_1 \leq \log I(t) \leq C_2$ we interpret them as bounding the car between two trucks moving at constant speed. See Figures 1.2 and 1.3.

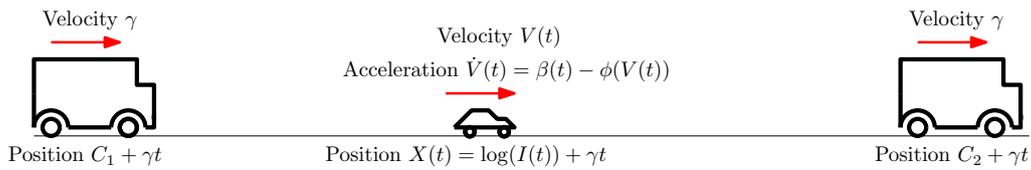


Figure 1.2: The objective is to maximize $\int \beta(t)dt$ given the car’s initial and final position and velocity. Because the external force $-\phi(V(t))$ is quadratic in the velocity $V(t)$ (and the mean velocity and acceleration are determined by the initial and final position and velocity) this is equivalent to maximizing $\int \phi(V(t))dt$, which in turn is equivalent to maximizing the variance of the car’s velocity. Note that in Figure 1.1, a red curve started at height $v_{\min} = .5$ gets almost to $v_{\max} = 1.5$ quickly (over a couple time units, i.e. a couple multiples of the mean incubation time); and a blue curve started at $v_{\max} = 1.5$ gets almost to $v_{\min} = .5$ quickly. This means the car can quickly change speeds from (roughly) v_{\min} to (roughly) v_{\max} and back. It makes intuitive sense that, in order to maximize the car’s velocity variance over a long time period, one might want to alternate between $\beta(t) = \beta_{\max}$ (holding down the pedal until the car is close to the leading truck) and $\beta(t) = \beta_{\min}$ (releasing the pedal until the car is close to the trailing truck) as in Figure 1.3.

²The upward drift on $\log I(t)$ caused by this influx, which is more pronounced when $\log I(t)$ is low, might be offset to some degree by contact tracing that is more effective when $\log I(t)$ is low. Other low-prevalence considerations (randomness, possible periods with no disease, large jumps due to superspreaders, etc.) are beyond the scope of this note. An alternative to the rigid lower bound is to add an extra fixed-prevalence subpopulation, in the language of Section 1.3.2, that is only weakly connected to the other subpopulations. An alternative to the rigid upper bound is to subtract a multiple of $\int_0^T I(t)dt$ from the objective function, which would heavily penalize larger $\log I(t)$ values but would not matter much for smaller $\log I(t)$ values.

³If $\gamma = 1$, then $-\phi(V(t)) = -V(t)^2$ is the standard *quadratic drag* used to model wind resistance. If $\gamma \neq 1$ then $\phi(V(t)) = (V(t) - \frac{\gamma-1}{2})^2 - (\frac{\gamma-1}{2})^2$, which corresponds to a prevailing wind of speed $\frac{\gamma-1}{2}$ and a street sloped to yield a velocity-independent gravitational force of $(\frac{\gamma-1}{2})^2$. The metaphor breaks down if $V(t) < \frac{\gamma-1}{2}$ (i.e., if the windspeed is forward but the car is moving slower than the wind) since in this case the force from the wind is in the wrong direction.

⁴Assume fuel weight is small compared to car weight, so overall car weight does not change.

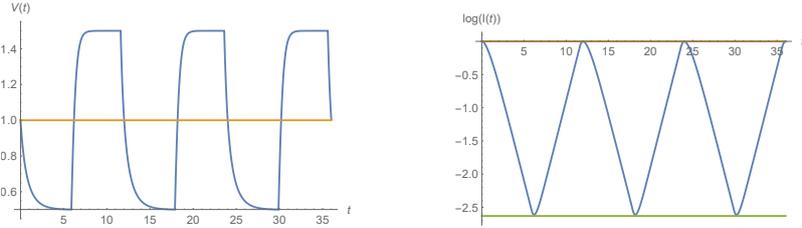


Figure 1.3: Example $V(t)$ in the setting of Figure 1.1 (left) and the corresponding $\log I(t)$ (right) obtained by integrating $V(t) - \gamma$. Here $\gamma = 1$, $V(0) = \gamma$ and the function $V(t)$ makes six excursions away from γ (three below and three above) which correspond to the six alternating intervals on which $\log I(t)$ decreases or increases. In this example, β alternates between β_{\min} and β_{\max} and $\log I(t)$ oscillates between $C_2 = 0$ (orange line) and $C_1 \approx -2.6$ (green line) so that $I(t)$ changes by a factor of about $e^{2.6} \approx 13.5$. Each 12-unit period has 5.712 units of $\beta(t) = 2.25$ and 6.288 units of $\beta(t) = .25$. The activity per 12-unit period is $6.288 \cdot .25 + 5.712 \cdot 2.25 = 14.424$ which is 20 percent higher than 12. That is, the alternating β scenario (blue curve) allows 20 percent more activity than the constant $\beta(t) = 1$ scenario (orange curve).

1.3 RESULTS

1.3.1 Activity minimizers and maximizers

The following is immediate from the statement of Problem 1.2.2 and the fact that the variance of a constant random variable is zero.

Proposition 1. *In the setting of Problem 1.2.2, if one fixes $V(0) = V(T) \in [v_{\min}, v_{\max}]$ and sets $\int_0^T V(t)dt = T \cdot V(0)$ then the minimal activity solution is the constant-velocity solution with $V(t) = V(0)$ for all t , which corresponds to $\beta(t) = \phi(V(0))$ for all t . In other words, if one aims to maximize A , constant velocity strategies are the worst possible.*

If $V(0) \neq V(T)$, then it is not possible to make the velocity variance exactly zero; but the worst possible strategy is still the one that makes this variance as small as possible. Glancing at Figure 1.1, it appears that to *maximize* the variance of $V(t)$, given its mean and initial/final values, one would ideally want $V(t)$ to spend most of its time near $v_{\min} = .5$ or near $v_{\max} = 1.5$, with as little time as possible spent transitioning between the two sides. One might guess that if $V(0) \neq V(T)$ the optimal strategy would be this: first move $V(t)$ as quickly as possible toward one side of $[v_{\min}, v_{\max}]$ (the one accessible without crossing $V(T)$), then at some point move $V(t)$ as quickly as possible toward the other side, and then at some point move $V(t)$ as quickly as possible toward $V(T)$. This is correct and we formalize this as follows.

Proposition 2. *In the setting of Problem 1.2.2, if $V(T) > V(0)$, then any optimal solution has $\beta(t) = \beta_{\max}$ on a single interval, with $\beta(t) = \beta_{\min}$ before and after that. If $V(T) < V(0)$, then the optimal solution has $\beta(t) = \beta_{\min}$ on a single interval, with $\beta(t) = \beta_{\max}$ before and after that. If $V(T) = V(0)$ then there is an optimal solution of each of the two types mentioned above.*

This is proved by showing that if V does *not* have the form described then one can modify it in a way that increases $\int V(t)^2 dt$ while keeping $\int V(t)dt$ the same. See Appendix 1.5 for details. A similar argument is made for more general objective functions in Appendix 1.6.

Proposition 3. *In the setting of Problem 1.2.3, with $\gamma \in (v_{\min}, v_{\max})$, assume $\log I(0)$ and $\log I(T)$ are fixed values in $\{C_1, C_2\}$ and $E(0)$ and $E(T)$ are fixed so that $V(0) = V(T) = \gamma$, as in Figure 1.3. Then in any optimal solution, $\beta(t)$ alternates between β_{\min} and β_{\max} finitely many times, and the $V(t)$ graph (like the one in Figure 1.3) has finitely many excursions away from γ , which alternatively go above γ (β_{\max} on the way up, β_{\min} on the way down) or below γ (β_{\min} on the way down, β_{\max} on the way up).*

Each excursion maximizes or minimizes the “area” $\int_{s_1}^{s_2} (V(t) - \gamma) dt = \log(I(s_2)) - \log(I(s_1))$ (where s_1, s_2 are endpoints) — so $\log I(t)$ crosses from C_1 to C_2 or back and the area is $\pm(C_2 - C_1)$ — except possibly for two smaller excursions (which together correspond to $\log I(T)$ crossing from one of $\{C_1, C_2\}$ to an intermediate value and back). For more general boundary conditions, if an optimal solution V hits $\{C_1, C_2\}$ at least once, and t_1 and t_2 are the first and last times this happens, then the restriction of V to $[t_1, t_2]$ behaves as described above, while the restrictions to $[0, t_1]$ or $[t_2, T]$ each have the form described in Proposition 2. If V never hits a wall, then it must be of the form described in Proposition 2.

Proposition 3 formalizes the notion, suggested by Figures 1.2 and 1.3, that when T is large, the optimal long-term strategy is to alternate between maximal forward acceleration and maximal reverse acceleration, timing the accelerations so that the car’s velocity reaches γ exactly as it reaches each truck. On the other hand, one may have to break the Figure 1.3 pattern at a couple of turn-around points to ensure that the boundary conditions are satisfied. The proof is similar to the proof of Proposition 2. One checks that if $V(t)$ does not have the asserted form then it is possible to make modifications to increase $\int V(t)^2 dt$ while keeping $\int V(t) dt$ the same and keeping $\log I(t)$ within bounds. See Appendix 1.5.

1.3.2 Multiple subpopulations

Suppose there are several *disjoint* subpopulations (factory workers in Town A, factory workers in Town B, students/teachers in Town A, students/teachers in Town B, etc.) each of which has some amount of flexible activity that can be set independently. Suppose further that there is some interaction between members of different groups that does not depend on the level of flexible activity (e.g., because a student and a factory worker live in the same household). To formalize this, for $1 \leq k \leq n$ write

$$\dot{E}_k(t) = -E_k(t) + \beta_k(t)I_k(t) + \sum_{j \neq k} \alpha_{j,k} I_j(t), \quad \dot{I}_k(t) = E_k(t) - \gamma I_k(t), \quad (1.6)$$

where for each k the process $\beta_k : [0, T] \rightarrow [\beta_{\min}, \beta_{\max}]$ is a control parameter (a measurable function of t). Letting $V_k(t) := E_k(t)/I_k(t)$ and $X_k(t) := \log I_k(t) + \gamma t$, we find

$$\begin{aligned} \dot{V}_k(t) &= \frac{\dot{E}_k(t)I_k(t) - \dot{I}_k(t)E_k(t)}{I_k(t)^2} \\ &= \frac{(-E_k(t) + \beta_k(t)I_k(t) + \sum_{j \neq k} \alpha_{j,k} I_j(t))I_k(t) - (E_k(t) - \gamma I_k(t))E_k(t)}{I_k(t)^2} \\ &= \frac{-E_k(t)I_k(t) + \beta_k(t)I_k(t)^2 - E_k(t)^2 + \gamma I_k(t)E_k(t) + \sum_{j \neq k} \alpha_{j,k} I_j(t)I_k(t)}{I_k(t)^2} \\ &= \beta_k(t) - V_k(t)^2 + (\gamma - 1)V_k(t) + \sum_{j \neq k} \alpha_{j,k} I_j(t)/I_k(t) \\ &= \beta_k(t) - \phi(V_k(t)) + \sum_{j \neq k} \alpha_{j,k} e^{X_j(t) - X_k(t)}, \end{aligned} \quad (1.7)$$

where again ϕ is the quadratic function defined by $\phi(x) := x^2 + (1 - \gamma)x$. This also implies

$$\beta_k(t) = \dot{V}_k(t) + \phi(V_k(t)) - \sum_{j \neq k} \alpha_{j,k} e^{X_j(t) - X_k(t)}. \quad (1.8)$$

We then have

$$\begin{aligned} A &= \sum_{k=1}^n \int_0^T \left(\phi(V_k(t)) - \sum_{j \neq k} \alpha_{j,k} e^{X_j(t) - X_k(t)} \right) dt + V_k(T) - V_k(0) \\ &= \sum_{k=1}^n \left(\int_0^T V_k(t)^2 dt + \int_0^T (1 - \gamma)V_k(t) dt + V_k(T) - V_k(0) - \int_0^T \sum_{j \neq k} \alpha_{j,k} e^{X_j(t) - X_k(t)} dt \right). \end{aligned} \quad (1.9)$$

Removing the terms that depend only on the given boundary values, the objective becomes

$$\sum_{k=1}^n \left(\int_0^T V_k(t)^2 dt \right) - \sum_{j \neq k} \left(\int_0^T \alpha_{j,k} e^{X_j(t) - X_k(t)} dt \right), \tag{1.10}$$

subject to the bounds on $\beta_k(t)$ and whatever initial and final values for the V_k and X_k are assumed. If we assume further that $\alpha_{j,k} = \alpha_{k,j}$ (which might make sense if the subpopulations are of similar size) then we find that A is a constant plus

$$\sum_{k=1}^n \left(\int_0^T V_k(t)^2 dt \right) - \sum_{j \neq k} \left(\int_0^T \alpha_{j,k} \cosh(X_j(t) - X_k(t)) dt \right). \tag{1.11}$$

The first term is a measure of policy oscillation: it is large if the velocities $V_k(t)$ have large swings. The latter term is a measure of policy coordination: it is largest if prevalence does not differ too much from one subpopulation to another. The fact that A is equal to (a constant plus) (1.11) can be summarized in English as follows: **once boundary values are fixed, activity is largest when policies are coordinated but oscillatory**. On the other side, one might expect that when T is large, activity-minimizing policies could involve traveling for long stretches in “migrating bird” patterns like the one in Figure 1.4, where only one subpopulation (or a small number of them) is substantially active, and others acquire small amounts of infection at a steady rate from the active subpopulations, despite being themselves relatively inactive.

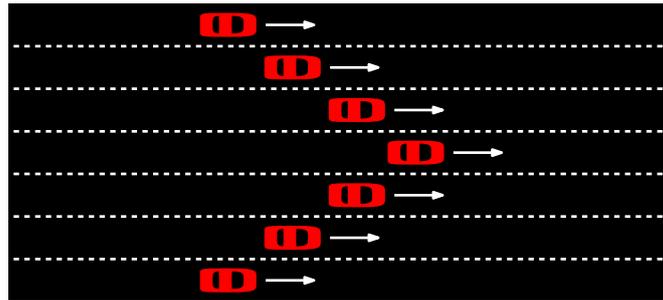


Figure 1.4: Adapting the analogy in Figure 1.2, let $X_k(t)$ and $V_k(t)$ denote the rightward position and velocity of the k th car from the top. Let $\beta_k(t)$ be the corresponding internally generated force, describing how much fuel the k th car is using. Assume that $\alpha_{j,k} = 1$ if $|j - k| = 1$ and 0 otherwise (so cars only influence their neighbors). Per (1.10), if $|j - k| = 1$, then the amount of forward drafting force the j th car induces on the k th car is $e^{X_j(t) - X_k(t)}$. This force is small (but positive) if the j th car is behind the k th car, but it becomes large as the j th car gets far ahead of the k th car (which prevents the cars from getting too far apart). Assume that all cars have velocity γ but the middle car uses a lot of fuel (so $\beta_4 \gg \beta_{\min}$) while the others use very little thanks to drafting forces (so that $\beta_k \approx \beta_{\min}$ for $k \neq 4$). If a pattern like this is the *best possible* for saving fuel, then it is the *worst possible* for boosting activity. In Figure 1.3 the activity gap between best and worst policy was 20 percent. In this figure, 6 of the 7 cars have very low $\beta(t)$ and (depending on the parameters) keeping all X_k equal might allow for *several times* as much activity.

1.4 DISCUSSION

There are many factors we have not considered: implementation costs, inoculum size, contact tracing, herd immunity effects (which may be significant in subpopulations even if overall prevalence is low), unpredictable super-spreader events at low prevalence (perhaps quickly boosting cases from 1 per million to 100 per million), regional disease-elimination opportunities, subpopulation differences, etc. But at least within the models presented here, coordinated suppression (followed

by relaxation and resuppression as needed) appears superior to temporal consistency and subpopulation variability. As noted in 3, the benefits of staggering flexible activity are smaller when there is less flexible activity to stagger—but larger when more realistic incubation/infectious period distributions are incorporated into the model.

These findings recall the cliché that if everyone perfectly distanced for three weeks the disease would disappear. The cliché does not take into account that some contact cannot be eliminated, some infections last unusually long, etc. But this paper shows that within simple models that do account for these things, the same principle applies: people enjoy more contact overall when their activity is coordinated.

It may be *especially inefficient* for some subpopulations to tightly close for the long term while others remain open enough to maintain a steady disease prevalence, as in the V-shaped pattern from Figure 1.4. We remark that one can imagine this type of pattern arising with no government action at all — e.g., if individuals voluntarily reduce activity once prevalence nears a threshold, but that threshold differs among subpopulations. It could also arise if subpopulations pull in opposite directions due to differing preferences or needs; perhaps β_{\min} and β_{\max} differ from group to group, or perhaps some prefer, all things considered, to acquire substantial herd immunity through infection, while others prefer to keep prevalence low. There are many social, political and game theoretic issues we won't discuss.

Instead, we conclude by reiterating our main point: within the simple SEIR-based models discussed here, the *amount* of activity a society enjoys is higher when the activity is *oscillatory* and *coordinated*

1.5 PROOFS

Proof of Proposition 2. It is not hard to see that the differentiable functions $V : [0, T] \rightarrow (0, \infty)$ that satisfy (1.5) are precisely those that (in the language of Figure 1.1) never cross a blue curve from above to below and never cross a red curve from below to above. More generally (without assuming differentiability of V) one may take this as a formal definition of what it *means* to satisfy (1.5). Such functions are Lipschitz and hence differentiable outside of a set of Lebesgue measure zero by Rademacher's theorem, but V may have points of non-differentiability, since β may have discontinuities.

Taking this view, it is clear that the set of V that satisfy these constraints—and have the given values of $V(0)$, $V(T)$, and $\int_0^T V(t)dt$ — is compact w.r.t. the L^∞ norm (precompactness follows from Arzelà-Ascoli and the constraint is clearly preserved under L^∞ limits).

Suppose the given values for $V(0)$, $V(T)$, and $\int_0^T V(t)dt$ are such that there exists at least one V with these values that satisfies (1.5). (The proposition statement holds trivially otherwise.) Then the existence of an *optimal* $V(t)$ follows from the continuity of $\int_0^T V(t)^2 dt$ w.r.t. the L^∞ norm and the above-mentioned compactness. We aim to show that any such V has the form described in the proposition statement.

Given any V satisfying (1.5), we define a point $t \in (0, T)$ to be *taut* if either $\beta(t) = \beta_{\max}$ a.e. in a neighborhood of t or $\beta(t) = \beta_{\min}$ a.e. in a neighborhood of t . In other words, t is taut if $V(t)$ traces a blue curve or a red curve in a neighborhood of t . We say that t is a *sharp peak* if $\beta(t) = \beta_{\max}$ a.e. in (t_1, t) and $\beta(t) = \beta_{\min}$ a.e. in (t, t_2) for some $t_1 < t < t_2$. In other words, $V(t)$ traces a red curve to the left of t and a blue curve to the right. Define a *sharp valley* analogously. Call t *upward-flexible* if it is neither taut nor a sharp peak. If t is upward-flexible then one can modify $V(t)$ (shifting it “upward”) in any small neighborhood of t in a way that increases $\int_0^T V(t)dt$ by any sufficiently small amount while respecting (1.5). This can be done for example by replacing V with the supremum of V and a function that is taut except for a sharp peak that lies just above V near t . The analogous statement holds if t is *downward-flexible*, i.e., neither taut nor a sharp valley. Call a point *doubly flexible* if it is both upward and downward flexible.

Note that if $V(s) > V(t)$, and s is upward-flexible and t is downward-flexible, then one can

increase V in a neighborhood of s and compensate by decreasing V in a neighborhood of t in a way that increases $\int_0^T V(t)^2 dt$ while keeping $\int_0^T V(t) dt$ the same. We conclude from this that if V is optimal, the supremum of $V(s)$ over upward-flexible s is at most the infimum of $V(s)$ over downward flexible s . In other words, there exists a v such that all points on the graph of V below height v are either taut or sharp valleys, and all points on the graph of V above height v are either taut or sharp peaks. Similar arguments show that $V(t)$ cannot be locally constant at v unless $v \in \{v_{min}, v_{max}\}$ (in which case the height v is crossed only once).

If V is optimal and *monotone non-increasing* (i.e., in a region where the red and blue curves are downward—this can happen in Figure 1.1 if $V(0)$ and $V(T)$ are both greater than v_{max}) then the above implies that there can be at most one sharp peak above v (where $\beta(t)$ changes from β_{max} to β_{min}) and at most one sharp valley below v (where $\beta(t)$ changes from β_{min} to β_{max}) and no doubly flexible points, which implies the proposition statement in this special case (similarly if V is monotone non-decreasing).

If V is not monotone, it still follows from the above that any excursions away from the horizontal line of height v are either V -shaped (blue curve going down, red curve going up) or Λ -shaped (red curve going up, blue curve coming down). Define an *occupation measure* ν by letting $\nu(S)$ denote the Lebesgue measure of $\{t \in [a, b] : V(t) \in S\}$. Clearly the occupation measure determines $\int_0^T V(t) dt$ and $\int_0^T V(t)^2 dt$. So if V is optimal then any W with the same occupation measure, which also satisfies (1.5), must be optimal as well.

Now we will argue that if V crosses any given horizontal line (i.e., passes from above to below or vice versa) more than twice then V is not optimal. The idea is explained in Figure 1.5. Suppose (for sake of getting a contradiction) that there exists an interval $(t_1, t_2) \subset (0, T)$ such that $V(t_1) = V(t_2)$ and $V(t) > V(t_1)$ for $t \in (t_1, t_2)$. Suppose that there also exists another interval $(t_3, t_4) \subset (0, T)$, of positive distance from (t_1, t_2) , such that either $V(t_3) = V(t_1)$ or $V(t_4) = V(t_1)$ and $V(t) > V(t_1)$ for $t \in (t_3, t_4)$. Then, as explained in Figure 1.5, one can “rearrange” V to create a W with the same occupation measure but with doubly flexible points of different heights. The first steps in the figure should be self-explanatory, but the final “flattening the N ” step requires explanation. Suppose generally that there is an interval $[a, b]$ such on this interval V achieves its minimum at a and its maximum at b but V is not monotone on $[a, b]$. We can then construct the unique continuous function $W(t)$ that agrees with V outside of (a, b) , that has the same occupation measure as V , and that is monotone non-decreasing. In other words, W “spends the same amount of time at each vertical height” as V but W is non-decreasing (so it visits the heights strictly in increasing order). It is then clear that W satisfies (1.5) and also that the slope is non-extremal at points of different heights, so that W (and hence V) is suboptimal, which is the desired contradiction.

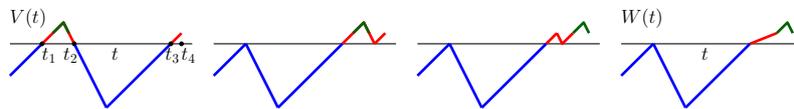


Figure 1.5: First figure transformed into second figure by “swapping” red/green peak and blue valley. Second transformed into third by swapping green peak and red valley. Third transformed into fourth by “flattening” the red N in a way that preserves the occupation measure but produces a curve of non-extremal slope.

Now suppose $V(T) > V(0)$. Since V crosses every horizontal line at most twice, V must be monotone non-decreasing between $V(0)$ and the first time t_1 at which V reaches its maximum, monotone non-increasing between t_1 and the first time t_2 at which V reaches its minimum, and monotone non-increasing between t_2 and T . (We allow for the degenerate possibility that $0 = t_1$ or $t_2 = T$.) Since V has at most one height at which doubly flexible points occur, the above conditions (and the fact that V cannot be locally constant) imply that V cannot have any doubly flexible points, and indeed must have the form required in the proposition statement. A similar argument applies

if $V(T) < V(0)$. If $V(T) = V(0)$, the same argument shows that V must have one upward and one downward excursion away from its initial position—but these excursions can be made in either order, so there are two equally optimal solutions in this case. (The one that puts the downward excursion first would yield fewer *infections* but these are not included in the definition of A .) \square

Proof of Proposition 3. The arguments in the proof of Proposition 2 imply that each excursion of V above γ must obtain the maximal possible slope (with $\beta(t) = \beta_{\max}$) on an initial interval and the minimal possible slope (with $\beta(t) = \beta_{\min}$) for the rest of the excursion; in other words, in the language of Figure 1.1, its graph is a concatenation of a red curve and a blue curve. If this were not the case, we could apply the procedures in the proof of Proposition 2 to this interval and produce a W with higher second moment. (Note that if we apply these procedures to an interval on which $V(t) > \gamma$, they do not change the fact that $\log I(t)$ is increasing over the course of that interval and they do not change the amount of increase, so they cannot lead to a violation of the $C_1 \leq \log I(t) \leq C_2$ constraint.) Similarly, each excursion of V below γ consists of a maximally downward segment (blue) followed by a maximally upward segment (red).

Let us take the same argument a bit further. Suppose $V(t)$ is optimal and suppose that $\log I(t)$ does not hit C_1 or C_2 during an interval $[s_1, s_2]$. Then we claim that every local maximum (minimum) of V in (s_1, s_2) must be a global maximum (minimum). First observe that V must have derivative in $\{\beta_{\min}, \beta_{\max}\}$ almost everywhere within (s_1, s_2) , since otherwise (sufficiently small) perturbations like those described in the proof of Proposition 2 would increase $\int_0^T V(t)^2 dt$ without violating the C_1 and C_2 conditions. Next, suppose that a local (but not global) maximum is obtained at s . Then (since V is not locally constant) by choosing arbitrarily small ϵ , we can arrange so that the component of $\{t : V(t) > V(s) - \epsilon\}$ containing s is arbitrarily small but non-empty. We can then “redistribute” the local time corresponding to that component elsewhere, as in Figure 1.5, to produce a W with the same occupation measure as V , and if ϵ is small enough this redistribution will not change the fact that C_1 and C_2 fail to be hit, but it will also produce a positive mass of places where $\beta(t) \notin \{\beta_{\min}, \beta_{\max}\}$, which enables an improvement to $\int_0^T V(t)^2 dt$, which is a contradiction. We conclude from this that within $[s_1, s_2]$, the function V must take the form described in Proposition 2. By taking limits, we find that the same is true if either or both of $\log I(s_1)$ and $\log I(s_2)$ lie in $\{C_1, C_2\}$ but $V(s) \notin \{C_1, C_2\}$ for $s \in (s_1, s_2)$.

If $\log I(s_1) = \log I(s_2) = C_1$ then we must have $V(s_1) = V(s_2) = \gamma$ and in between s_1 and s_2 (recalling the statement of Proposition 2) V makes one upward and one downward excursion away from γ , and the areas between these curves and the horizontal line at height γ both have equal area as in Figure 1.3. This area must be strictly less than $C_2 - C_1$ if C_2 is never hit in (s_1, s_2) . In this case (and the analogous case with the roles of C_1 and C_2 reversed) we refer to (s_1, s_2) as a “single wall excursion” (since the same element of $\{C_1, C_2\}$ is hit at both endpoints). Similarly, if $\log I(s_1) = C_1$ and $\log I(s_2) = C_2$ (but these two endpoints are avoided for (s_1, s_2)) then between s_1 and s_2 the function V must make a single positive excursion enclosing the maximal possible area above γ , namely area $C_2 - C_1$, precisely as in Figure 1.3. In this case we refer to (s_1, s_2) as a “double wall excursion.”

Next, we will argue that if there are *two* single wall excursions, then one can make one of the excursions bigger and the other one smaller in a way that does not change $\int_0^t V(t) dt$ and $\int_0^t V(t)^2 dt$ but that makes it so that $\int_0^t V(t)^2 dt$ is no longer maximal (a contradiction). One of the two single wall excursions (call it “smaller”) must have size less than or equal to that of the other (larger) one. As we have done before (in Figure 1.5) we can then “move mass” from near the tips of the corresponding upper/lower excursions of V (away from γ) in the smaller one to the upper/lower excursions of V (away from γ) in the larger one in a way that produces a new function that is suboptimal, and this yields the contradiction.

We conclude that there is at most one single wall excursion, and the rest of the proposition follows. \square

1.6 MORE GENERAL UTILITY FUNCTIONS

1.6.1 Setup and motivation

We now generalize our original setup to account for crowding effects. In this setting, we consider two kinds of activity: first, $\mu(t)$ is the amount of *useful activity* taking place at time t . Informally, think of $\mu(t)$ as encoding the (risk-weighted) number of conversations, restaurant meals, haircuts, etc. Second, $\beta(t)$ is the amount of *transmission activity* that drives the ODE (1.1). Instead of making these quantities equal as before, we now assume they are related by $\mu(t) = u(\beta(t))$ where u is increasing and continuous but potentially *non-linear*. We then denote the *total useful activity* by $U = \int_0^T \mu(t) dt = \int_0^T u(\beta(t)) dt$ and assume that our goal is to maximize U , instead of maximizing $A = \int_0^T \beta(t) dt$.⁵

One way to motivate this is to suppose that some fraction of the disease transmission $\beta(t)$ comes from *deliberate close interaction* (i.e., is spread between friends or coworkers choosing to engage in a valuable activity together) and that the rest comes from infectious air lingering in public places (hallways, subway cars, etc.) The former might be linear in $\mu(t)$ (twice as many conversations means twice as many chances for spread) but the latter might be quadratic in $\mu(t)$ (if the density of infectious particles in the air and the number of people inhaling them are both linear in $\mu(t)$). Combining the two effects, we might find $\beta(t) = \psi(\mu(t)) := a_1\mu(t) + a_2\mu(t)^2$ where a_1 and a_2 are positive constants, and taking the positive inverse,

$$u(x) = \psi^{-1}(x) = \frac{-a_1 + \sqrt{a_1^2 + 4a_2x}}{2a_2}. \quad (1.12)$$

As a concrete example, suppose $a_1 = 3/4$ and $a_2 = 1/4$. If $\beta(t) = \mu(t) = 1$ then 25 percent of the transmission comes from “lingering air” (the quadratic term). If $\beta(t) = 2.25$ then (after solving for $\mu(t)$) about 38 percent of the transmission comes from lingering air; if $\beta(t) = .25$ it is only about 9 percent. In this example, “crowding effects” play a larger role when activity is higher.⁶

We stress that $\mu(t)$ is a measure of the *amount* of useful activity—defined in a way that ensures $\beta(t) = u^{-1}(\mu(t))$. It is *not* a measure of the *value* of the activity. If a more-valued activity causes the same amount of disease transmission as a less-valued activity, then it will make the same contribution to $\mu(t)$. We allow for the possibility that in scenarios where U is kept small, only very important activity will be allowed, but in scenarios where U is large, more discretionary activity will occur. We only assume that utility is an *increasing function* of U so that maximizing U is a reasonable objective.

We can generalize Problem 1.2.1 by replacing A with

$$U = \int_0^T u(\beta(t)) dt = \int_0^T u(\dot{V}(t) + \phi(V(t))) dt, \quad (1.13)$$

where u is some fixed twice-differentiable function, and as before we write $\phi(x) = x^2 + (1 - \gamma)x$. The setup in Problem 1.2.1 amounts to taking $u(b) = b$ for $b \in [\beta_{\min}, \beta_{\max}]$.

Now let us express (1.13) a different way. Write

$$G_y(x) := u(x + \phi(y)) - u(\phi(y)) - u'(\phi(y))x. \quad (1.14)$$

Observe that for all y we have $G_y(0) = G'_y(0) = 0$. If u is (strictly) concave then (for fixed y) G_y is (strictly) concave, and has a maximum at 0. To ensure that (1.14) makes sense for relevant inputs,

⁵In the car analogy, $\mu(t)$ is fuel use, $\beta(t)$ is force, but the relationship is non-linear. Maximizing fuel use is again the goal.

⁶The assumption that $\mu(t)$ and $\beta(t)$ determine one another via a single function u is a simplification. In principle, the relationship between $\mu(t)$ and $\beta(t)$ could vary in time. For example, perhaps deliberate interaction is a larger factor for weekend activities and lingering air is a larger factor on weekdays, so that the same u cannot be used for both.

it will be convenient for us to extend the definition of u beyond $[\beta_{\min}, \beta_{\max}]$ to the full range of ϕ (which is all of $[0, \infty)$ when $\gamma = 1$) in such a way that u remains concave. It does not matter exactly how we do this, but one natural approach is to assume u is differentiable everywhere but affine—or strictly concave but nearly affine—outside of $[\beta_{\min}, \beta_{\max}]$. Now we can rewrite (1.14) with $x = \dot{V}(t)$ and $y = V(t)$ to obtain

$$u\left(\dot{V}(t) + \phi(V(t))\right) = G_{V(t)}(\dot{V}(t)) + u(\phi(V(t))) + u'(\phi(V(t)))\dot{V}(t) \quad (1.15)$$

and substituting this into (1.13) yields

$$U = \int_0^T u\left(\dot{V}(t) + \phi(V(t))\right) dt = \int_0^T \left(G_{V(t)}(\dot{V}(t)) + u(\phi(V(t))) + u'(\phi(V(t)))\dot{V}(t) \right) dt. \quad (1.16)$$

The final RHS term can be written $u'(\phi(V(t)))\dot{V}(t) = \frac{\partial}{\partial t} r(V(t)) = r'(V(t))\dot{V}(t)$ where $r'(x) = u'(\phi(x))$, i.e., $r(a) := \int_0^a u'(\phi(x)) dx$. Thus the final RHS term integrates to a quantity that depends only on $V(T)$ and $V(0)$. If we remove that, the objective becomes

$$\int_0^T u(\phi(V(t))) dt + \int_0^T G_{V(t)}(\dot{V}(t)) dt \quad (1.17)$$

Writing it this way, we have separated the objective into two pieces: the first term ascribes different benefits to different velocities via the function $u \circ \phi$. The second (non-positive) term ascribes costs to non-zero acceleration rates (in a manner that also depends on velocity).

We can now formulate the problem in the language of Problem 1.2.2 as follows:

Problem 1.6.1. *Given $V(0)$, $V(T)$, and $\int_0^T V(t) dt$, find a V that maximizes (1.17) subject to (1.5).*

Now suppose that $u(x)$ is concave in x but $u(\phi(x))$ is strictly convex. The latter holds if $u(x) = x$ but fails if u is “too concave.” To illustrate what this means, consider the $\gamma = 1$ case.⁷ In this case $\phi(x) = x^2$ so $u(\phi(x)) = u(x^2)$ is strictly convex if $u(x) = x^\alpha$ for $\alpha \in (1/2, 1]$, but not if $u(x) = x^\alpha$ for $\alpha \leq 1/2$. Note that $u(x^2)$ is also strictly convex if u is as given in (1.12), for any positive a_1 and a_2 .

Since the mean of V is fixed, the first term of (1.17) is the worst possible if V is constant (by Jensen’s inequality). But the second term penalizes fluctuation (i.e., one pays a price for non-zero derivative) so these two factors work against each other. On the other hand, if V varies slowly, the second term should not matter very much. Proposition 4 below is a simple illustration of that point.

1.6.2 Best and worst policies

If u is concave, then a rapidly fluctuating β may yield a lower U than a constant β with the same mean so that (in contrast to Proposition 1) constant strategies are not the worst possible. On the other hand, Proposition 4 states that constant $V(t)$ are the worst possible (given the corresponding boundary data) among functions that *vary slowly* in the sense of having no Fourier modes of short wavelength.

⁷In a probabilistic formulation of the SEIR model, setting $\gamma = 1$ corresponds to assuming that the incubation time and the infectious time are independent exponential random variables with the same rate 1. If $f(x) = xe^{-x}$ is the density function for the *sum* of these positive random variables then $\int_a^b f(t)\beta(t)dt$ is the expected number of people infected between time a and b by a person infected at time 0. If γ is either very small or very large, then the corresponding f is approximately exponential, and the model is effectively more like an SIR model; taking γ close to one ensures that f is more concentrated (a smaller standard deviation relative to its mean). If the true f is *actually much more* concentrated than xe^{-x} then the models of this paper are inadequate, and a different approach is needed (such as Erlang SEIR with a higher Erlang parameter, see 3). Still, $\gamma = 1$ might be the best approximation *within* the framework of this paper.

Proposition 4. *Suppose that $u(x)$ is smooth and concave in x but $u(\phi(x))$ is smooth and strictly convex. Then there exists a $C > 0$ (independent of T or the boundary data) such that constant- $V(t)$ strategies are the worst possible (i.e, U -minimizing, given constraints from Problem 1.6.1) among all differentiable $V(t)$ whose Fourier series decompositions on the interval $[0, T]$ include no mode with wave length less than C .*

Note that this proposition holds trivially if $T < C$, and hence provides no information in that case. It does not rule out the possibility that constant strategies are optimal over very short time periods.

Proof. Write v for the fixed value of $V(0) = V(T) = T^{-1} \int_0^T V(t)dt$. Let L be the affine function tangent to $u \circ \phi$ at v , and write $\tilde{u} = u \circ \phi - L$. Then we can write the first term of (1.17) as $\int_0^T L(V(t))dt + \int_0^T \tilde{u}(V(t))dt$. Since $\int_0^T L(V(t))dt$ is fixed by the boundary data, we can ignore that term, so the objective becomes

$$\int_0^T \tilde{u}(V(t))dt + \int_0^T G_{V(t)}(\dot{V}(t))dt \tag{1.18}$$

We can assume that $V(0) = V(T) \in (v_{min}, v_{max})$ (since otherwise there would only be one or zero possible solutions with the same boundary values for V and with $\int_0^T V(t) = TV(0)$, and the proposition statement would be trivially true). Thus, in (1.18) we need only to evaluate $\tilde{u}(v)$ for $v \in (v_{min}, v_{max})$ and $G_v(x)$ for $v \in (v_{min}, v_{max})$ and x in the bounded range of $\dot{V}(t)$ values possible when $V(t) \in [v_{min}, v_{max}]$.

Within this range, because of the convexity and smoothness assumptions, there exists a $c_1 > 0$ such that $\tilde{u}(x) \geq c_1(x - v)^2$ and there exists a c_2 such that $0 \geq G_v(x) \geq -c_2x^2$. Thus

$$\int_0^T \tilde{u}(V(t))dt + \int_0^T G_{V(t)}(\dot{V}(t))dt \geq c_1 \int_0^T (V(t) - v)^2dt - c_2 \int_0^T \dot{V}(t)^2dt \tag{1.19}$$

If we write $V_k(t) = e^{k2\pi it/T}$ for the k th Fourier mode, then $\int_0^T |\dot{V}_k(t)|^2dt = (2\pi k/T)^2 \int_0^T |V_k(t)|^2dt$. As long as $(2\pi k/T)^2 \leq c_1/c_2$, (1.18) will be negative if we set $V(t) = v + V_k$, and by orthogonality of the Fourier series, the same applies to any linear combination of Fourier modes V_k such that $(2\pi k/T)^2 \leq c_1/c_2$, or equivalently $k/T \leq \sqrt{c_1/c_2}/(2\pi)$, which means that the wavelength T/k satisfies $T/k \geq 2\pi\sqrt{c_2/c_1}$. \square

Proposition 5. *In the context of Proposition 4, the best possible (U -maximizing) $V(t)$ cannot cross any horizontal line more than twice in $(0, T)$. If $V(0) < V(T)$ and we write $m := \inf_{t \in [0, T]} V(t)$ and $M := \sup_{t \in [0, T]} V(t)$, then V must be monotone non-increasing between time 0 and the first time it hits m , then monotone non-decreasing until the first time it hits M , then monotone non-increasing again until time T . (Similar statements hold if $V(0) > V(T)$ or $V(0) = V(T)$, c.f. Proposition 2.)*

Proof. The argument in Figure 1.5 works exactly the same way in this setting; the only difference is that for the fourth “flattening” step, one can use Jensen’s inequality to show that utility is strictly larger for the flattened curve than for the original. The flattening does not change the first term (1.18), but it makes the second term strictly larger. That is, we claim that if W is a curve produced by flattening V on $[s_1, s_2]$ then

$$\int_0^T G_{V(t)}(\dot{V}(t))dt < \int_{s_1}^{s_2} G_{W(t)}(\dot{W}(t))dt. \tag{1.20}$$

To see this, note that the fundamental theorem of calculus and the construction of W imply

$$\int_{t \in [s_1, s_2]: V(t) \in (a, b)} \dot{V}(t)dt = \int_{t \in [s_1, s_2]: W(t) \in (a, b)} \dot{W}(t)dt. \tag{1.21}$$

If t is sampled uniformly from $[s_1, s_2]$ then there is an F such that $\mathbb{E}[\dot{V}(t)|V(t)] = F(V(t))$ and since (1.21) holds for any (a, b) this implies $E[\dot{W}(t)|W(t)] = F(W(t))$. On the other hand, since

$\dot{V}(t)$ assumes negative values with positive probability and $\dot{W}(t)$ is conditionally deterministic given $W(t)$, we have a strict inequality on conditional variance:

$$\mathbb{E}[\text{Var}[\dot{V}(t)|V(t)]] > \mathbb{E}[\text{Var}[\dot{W}(t)|W(t)]].$$

Since there is (on the range inputs possible here) a negative upper bound on the second derivative of $G_{V(t)}$ we deduce that

$$\mathbb{E}[\mathbb{E}[G_{V(t)}(\dot{V}(t))|V(t)]] < \mathbb{E}[\mathbb{E}[G_{W(t)}(\dot{W}(t))|W(t)]],$$

which implies (1.20).

This argument shows that V cannot cross any horizontal line more than twice. The rest of the proposition statement easily follows from this. \square

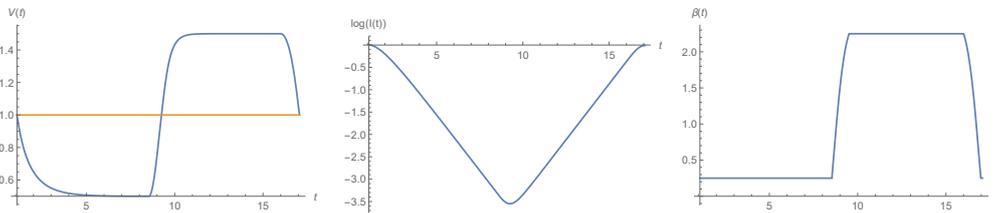


Figure 1.6: Analog of (one period of) Figure 1.3 with $u(x) = x - x^2/10$. Similar to Figure 1.3 except that the β transitions (right) from β_{\min} to β_{\max} and back take place gradually over a couple of time units instead of instantly. Here the corners in the graph of $V(T)$ are less sharp than in Figure 1.3, and the turnarounds in $\log I(t)$ are smoother. One might expect these effects to be more pronounced if u were more concave. Up to affine transformation, the u here is very close to the example in (1.12) with $a_1 = 3/4$ and $a_2 = 1/4$.

1.6.3 Euler-Lagrange solutions

In principle, one can find an optimal V explicitly using Euler-Lagrange theory. We briefly sketch the idea here. Consider an interval $(s, s + \Delta)$ on which V is known to increase monotonically from x_1 to x_2 and assume $\int_s^{s+\Delta} V(t)dt = \Lambda$. Once Λ and Δ are fixed, one can fix any constants a and b and aim to maximize

$$\int_s^{s+\Delta} w(V(t))dt + \int_s^T G_{V(t)}(\dot{V}(t))dt \tag{1.22}$$

where $w(v) := u \circ \phi(v) + av + b$. The extra $av + b$ terms do not affect the optimal solution, since the amount that they add to (1.22) is determined by Λ and Δ . However, one can also let Λ and Δ be *variable* parameters and then try to tune a and b so that the optimizer to (1.22) obtains the desired values.

During the interval $(t, t + \Delta)$ we interpret $h(x) := \dot{V}(V^{-1}(x))$ as the “speed” at which x is passed through, for $x \in (x_1, x_2)$, so that $1/h(x)$ is the density function for the occupation measure at x , and the goal becomes to maximize $\int_{x_1}^{x_2} \frac{G_x(h(x)) + w(x)}{h(x)} dx$. We can then use calculus to find (for each x) the $h(x)$ that optimizes the integrand, recalling the constraints on $h(x)$ from (1.5). If the optimizer is unique for each x , this determines the function h . Once h is known, solving the ODE $\dot{V}(t) = h(V(t))$ allows us to produce analogs of the red curves in Figure 1.1 that dictate the way V evolves during its upward trajectories. We can treat decreasing intervals similarly, obtaining analogs of the blue curves in Figure 1.1.

In general, finding a and b is a tricky optimization problem; however, if we *assume* or *guess* that the optimal V has an interval on which $V(t)$ is close to v_{\min} (and ergo $\dot{V}(t) \approx 0$) and an interval on which $V(t)$ is close to v_{\max} , then we can deduce that w must be close to zero at v_{\min} and v_{\max} (since

otherwise one could increase (1.22) by either prolonging or condensing these intervals) which determines approximately what a and b must be. (It is not hard to see that—if the mean of V and its endpoints in (v_{min}, v_{max}) are held fixed—this assumption is correct if T is large enough, but incorrect for smaller T .) Once a and b are known—and analogs of the blue and red curves in Figure 1.1 are drawn—the problem of figuring out where the “turnarounds” occur is essentially the same here as in Proposition 2. This approach was used to produce Figure 1.6.

Although we will not give details, we expect the arguments in the proof of Proposition 3 to work in a general u version of Proposition 5, enabling one to show that the long-term optimal $\log I$ oscillates between C_1 and C_2 in a similar fashion (with the blue and red curves coming from *some* choice of a and b). If the a and b are different from the ones guessed in producing Figure 1.3, then the shape of the turnarounds might be different as well.

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2. Mathematics Education in Engineering Disciplines

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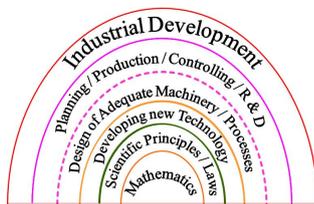
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2.1 PREAMBLE

Teaching of Mathematics at any level and to the students with varied interest has always been a challenging task and a matter of great concern for all those who are associated with the Mathematics education. Many efforts have been continuously going on for improving Mathematics education at school level. Also, various government bodies, universities/ academic institutions organize a number of short-time training programs for students majoring in Mathematics as well as for Mathematics faculties in the Institutes of Higher Education. However, in my personal opinion, barring a few elite technical institutes, yet there is much scope to improve the quality of the Mathematics education in engineering disciplines and in other allied disciplines, in general. Also, it is necessary to review the practices adopted by these institutions in dealing with such issues. Looking at the increasing number of Engineering and technological institutes and number of students seeking admissions in such institutes, this is a matter of great concern. Through this article, I seek to initiate a discussion on this issue and invite the readers to give their feedback, expressing their opinions in this regard.

2.2 WHY SUCH A STUDY?

Overall there are about 10,500 technical Institutes (offering Degree/Diploma courses in Engineering, Pharmacy, Biomedical, Architecture / MCA / MBA etc.) approved by AICTE with an intake capacity of about 33 lakhs. Against this intake capacity, in 2018-19, only about 18,30,000 enrolments took place and 7,65,000 recorded placements were reported (i. e. about 4%). Thus, more than 50% of the candidates were either unemployed or did not get a job befitting to their qualification or joined / initiated a family business. If we concentrate on only top AICTE Approved Engineering Colleges, then there are about 1350 institutes with total intake of about 4,40,000. In this article we would like to concentrate on Mathematics education in such Institutes excluding a few elite institutes (like IITs, IISERs, top ranking NITs etc.). In the current situation, when we are striving for self reliance (as a nation) such a study is of vital importance as these graduates will provide the core strength to developing Indigenous technologies in all the sectors. One may raise a question why the emphasis on Mathematics. My conceptualization of the answer to this question is depicted by the following Figure-1. It clearly indicates that Scientific principles and Mathematical formulations underlie any meaningful Industrial development.



We surveyed about 200 Engineering colleges (from their available websites) and looked at the status of the Mathematics departments / Faculties and Mathematics curriculum from some of these colleges. Out of these, websites of 50 colleges either could not be opened or did not give an adequate information. The remaining 150 colleges offer degree Engineering/Technology programs in 5 or more branches. Based on this survey and the model curriculum developed by AICTE, we discuss

here the various factors likely to influence the quality of Mathematics education in Engineering disciplines and focus on unified and cafeteria approach in development of Mathematics curriculum in various engineering disciplines.

2.3 CURRICULUM DEVELOPMENT AND CAFETERIA APPROACH

Systematic procedure for curriculum development, adequate curriculum and its implementation are some of the most important factors in Mathematics education in Engineering. AICTE has done a commendable job in designing a model curriculum (in two volumes) for various engineering degree programs in 2018. However, it seems that a unified approach for designing Mathematics curriculum in different branches is missing. This is evident if one observes that: a topic which is common to Mathematics papers in different branches is allotted different number of lectures; Mathematics I and Mathematics II are considered as common courses for all branches in the beginning of Volume I, however, in discipline-wise curriculum the contents of Mathematics I in Electrical engineering are different. This is probably because academic boards for different branches of Engineering have independently decided about the Mathematics component required for their students as well as the number of hours required to teach a specific topic.

There is no doubt that the faculty board in a specific branch can have better idea as to what topics in Mathematics are required for students of their branch. However, number of lectures and prerequisites required to teach a specific mathematics topic is better decided by the Mathematics experts. In order to resolve this dilemma, I am proposing here a cafeteria approach for designing Mathematics curriculum for various engineering degree programs.

As per AICTE model curriculum, the total credits allotted to the Mathematics courses are 10 to 12 or in some branches 14 to 16, if some special courses are required. We do not intend to suggest any changes to this weightage of the Mathematics courses in Engineering but we wish to suggest some changes in the methodology of designing Mathematics curricula and reorganize the structure of Mathematics courses. If we assume that minimum teaching days are 90 (15 weeks) then for a 4 credit course we may get about 60 hours (50 lectures + 10 tutorials) and for a 2 credit course 30 hours (25 lectures + 5 tutorials).

The first step in the cafeteria approach is that a board of Mathematics can prepare an exhaustive list of different streams based on various topics which are relevant for Mathematics education in engineering disciplines. Each stream can be divided into a number of modules indicating levels of depth in the topics included in the stream. This will help to include the modules as per a desired level of topics in that stream required in different branches. For each module detailed contents, total number of hours and the prerequisites required and stream wise list of reference / text books,

Table-1: Probable Streams and modules

Sr. No.	Streams	Prerequisites	Modules	hours
1.	Calculus (Functions of single variables)		1. Geometric applications & Mean-value theorems 2. Sequences and Series 3. Expansion of functions and indeterminate forms 4. Improper Integrals & Gamma, Beta functions 5. Error functions, Elliptic functions & Elliptic Integrals	6 6 4 4 4
2.	Analytical Geometry		2.1 Various coordinate systems 2.2 Cone, Cylinder & Central conicoids 2.3 Curve tracing in Cartesian and polar coordinates	3 6 4
3.	Multivariate Calculus	1.1 2.1	3.1 Multivariate Differential Calculus 3.2 Multiple integrals	8 6
4.	Vector Calculus	3.1 3.2	4.1 Vector differential Calculus 4.2 Vector Integral Calculus	6 6
5.	Linear Algebra		5.1 Matrix Algebra & systems of Linear equations. 5.2 Eigenvalues & Eigenvectors 5.3 Linear spaces and Linear transformations 5.4 Inner product spaces & orthogonality	5 5 6 4
6.	Ordinary Differential equations		6.1 Differential equations basics & First order ODE 6.2 Higher order ODEs with constant coefficient 6.3 Systems of ODEs 6.4 Higher order ODEs with variable coefficients 6.5 Bessel's functions & Legendre polynomials	4 5 4 4 4
7.	Integral transforms	1.2 1.4 1.4 1.4	7.1 Fourier Series 7.2 Fourier Integrals & Fourier Transforms 7.3 Laplace Transforms 7.4 Other Integral transforms	5 6 6 *
8.	Partial Differential equations	6.1 6.2 6.2 7.1	8.1 PDE basics & First order PDE 8.2 Second order PDE (Heat, Wave & Laplace equations)	6 6
9.	Complex Analysis	3.1 3.2 1.1	9.1 Complex algebra and special functions 9.2 Differentiability & Analytic functions 9.3 Complex Integration 9.4 Bilinear Transformation & conformal mapping 9.5 Series expansions & Calculus of Residue 9.6 Evaluation of Improper integrals	3 6 6 5 5 5
10.	Numerical Methods	5.1 6.1, 6.2 5.1 5.1	10.1 Motivation & Errors in numerical computations 10.2 Algebraic & Transcendental equations 10.3 Interpolation & Approximation 10.4 Systems of linear equations 10.5 Numerical differentiation & Integration 10.6 Numerical methods for solving ODE 10.7 Finite difference methods 10.8 Finite Element methods	3 4 4 4 4 4 4 6
11.	Statistical methods		11.1 Descriptive Statistics 11.2 Probability & Probability distributions 11.3 Bivariate distributions & Bayes' Theorem 11.4 Regression & correlation 11.5 Sampling & Sampling distributions 11.6 Estimation & Confidence Interval 11.7 Testing of Hypothesis (single population) 11.8 Statistical Quality control	5 5 4 3 4 4 5 6

12.	Discrete Mathematics		12.1 Sets, Relation & Functions 12.2 Posets, Boolean Algebra & Lattices 12.3 Algebraic structures & Morphisms 12.4 Propositional & Predicate Logic 12.5 Inference theory and methods of proof 12.6 Logic gates & Switching circuits 12.7 Basics of graph Theory 12.8 Trees, Circuits & related Algorithms 12.9 Special graphs, Connectivity and Isomorphism 12.10 Graph Theoretic Algorithms & Applications	4 6 6 6 4 4 8 8 8 6
13.	Operations Research	11.2 14.1 11.2 5.1, 11.2	13.1 Introduction & Inventory control 13.2 Queuing theory 13.3 Game Theory 13.4 Reliability Theory 13.5 Network Analysis (PERT & CPM) 13.6 Markov Processes 13.7 Decision Theory	6 6 6 6 6 6 4
14.	Numerical Optimization Techniques	5.1 3.1 3.1 11.2 11.2	14.1 Linear Programming 14.2 Transportation & Assignment problems 14.3 Single variable Unconstrained Optimization methods 14.4 Multivariable Unconstrained Optimization methods 14.5 Constrained Non-linear Optimization methods 14.5 Dynamic programming 14.6 Genetic Algorithms 14.7 Evolutionary computing	6 6 4 7 7 6 6 6
15.	Mathematical Modelling & Simulation.		15.1 Modelling basics 15.2 Models of engineering systems 15.3 Models of Population dynamics 15.4 Probabilistic models 15.5 Traffic flow models 15.6 Monte-Carlo Simulation	3 6 6 5 4 8
16.	Fuzzy sets and Fuzzy Logic	12.4, 1 2.5	16.1 Fuzzy sets and set theoretic operations 16.2 Fuzzy numbers & Fuzzy arithmetic 16.3 Fuzzy relations & Fuzzy Implications 16.4 Fuzzy Logic 16.5 Fuzzy Controllers 16.6 Fuzzy decision making	4 4 5 8 4 5

must also be given. A sample list of probable streams and corresponding modules is given in Table 2.1. The syllabi for the proposed Mathematics courses can be recommended by selecting appropriate modules from the list amounting to the required 60 or 30 hours of teaching. The entire document can then be circulated to academic boards of various branches. These boards then either adopt the recommended course or reassemble the syllabus by choosing the desired modules.

Another important issue is as to when the Mathematics courses could be taught. Currently, all the Mathematics courses are taught in the first 3 to 4 semesters. The problem with this arrangement is that in these semesters students get a very little idea about their branch and hence are not able to realize the need for such courses. By the time they realize this they lose touch with Mathematics. So we may think of spreading Mathematics teaching over at least 6 semesters by replacing 4 credits course by more number of two credit courses. In later semesters students' preparedness for learning Mathematics courses will be better and will also help students to realise the importance of such courses to their branch. Thus we propose that replace Mathematics I and Mathematics II courses (common to all branches), which are currently taught in first two semesters by four 2-credit courses be taught in first four semesters. This will help students to digest the mathematical concepts at proper pace and provide room for introducing core engineering courses right from the first semester onward. Mathematics III and Mathematics IV courses can also be divided in four 2 credit courses and the branches will have a flexibility to choose 2, 3 or 4 of these courses and modules to be included in their courses as per their requirements. Such courses can be taught in 3 to 6 semesters. Students may also be allowed to optionally choose additional courses in the later semesters, to earn more credits if they opt for honours degree as proposed by AICTE. The list of

probable core Mathematics courses and Elective courses is given in Tables 2 and 3 respectively.

Table-2: List of Core Mathematics courses

Sem	Discipline	Course	Cr	Modules (as per table 2) to be included
1	All (C, M, E, El, EC, Ch, Met, CSE)	Mathematics-I (a): Calculus (1 variable) & Matrix Algebra	2	1.1, 1.2, 1.3, 1.4, 5.1, 5.2
2	All	Mathematics-I (b): Multivariable Calculus & ODE	2	2.1, 3.1, 3.2, 6.1, 6.2, 6.3
3	All	Mathematics-II (a): Integral Transforms & PDE	2	7.1, 7.2, 7.3, 8.1 8.2
4	All	Mathematics II (b): Analytic geometry & Complex Analysis	2	2.2, 2.3, 9.1 to 9.4
5	C, E, El, EC, Ch CSE	Mathematics-III (a): Linear Algebra, Vector Calculus & Special functions	2	5.3, 5.4, 4.1, 4.2, 6.4, 6.5
		Mathematics III (c): Discrete Mathematics	2	12.1 to 12.6
		Mathematics-IV (a): Statistical Methods.	2	11.1 to 11.7 & Software R (relevant tools)
6	C, M, E, El, EC, Ch, Met CSE	Mathematics-III (b) Num. Methods & software based comp	2	10.1 to 10.6 and relevant Math. Software
		Mathematics III (d): Graph Theory	2	12.7 to 12.10

Table-3: List of Optional Mathematics courses

Sr. No.	Course	Cr	Modules (as per table 2) to be included
1	Operations Research	2	13.1 to 13.5
2	Numerical Optimization Techniques	2	14.1 to 14.5
3	Mathematical Modelling and Simulation	2	15.1 to 15.7
4	Fuzzy Sets, Fuzzy Logic & Fuzzy Controllers	2	16.1 to 16.6
5	Evolutionary Computing	2	
6	Mathematical Theory of Controls	2	
7	Wavelet Transforms	2	
8	Advanced Numerical Analysis	2	
9	Finite Element Analysis	2	
10	Theory of Algorithms & Computational Complexity	2	

2.4 ROLE OF MATHEMATICS DEPARTMENTS IN ENGINEERING INSTITUTES

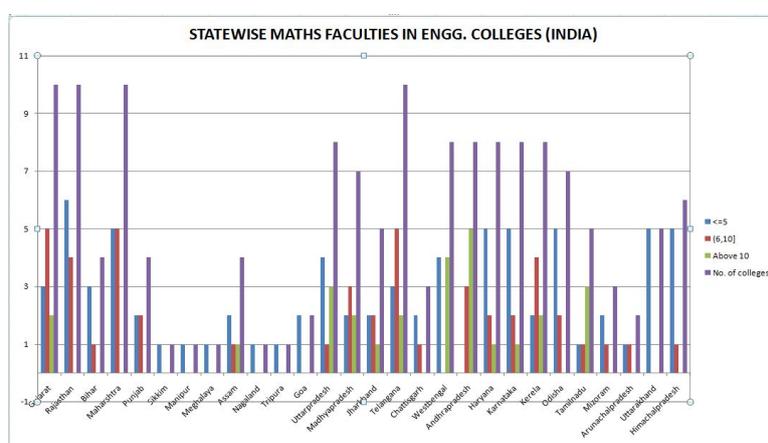
Looking at the current status of the Mathematics departments, it appears that most of the engineering colleges have not been able to realise the potential of these departments to play a significant role in engineering education. Though most of the Mathematics courses are currently taught explicitly in first four semesters, the concepts and techniques are used throughout the eight semesters of UG engineering courses. The objective of Mathematics education cannot be merely passing on information regarding useful techniques but to inculcate among the students mathematical way of thinking, analysing and understanding the engineering concepts and processes. Students must be trained to find out the mathematical essence in engineering problems and accordingly make the right choice of the techniques to solve them. It is in this context we feel that Mathematics departments have much bigger role to play in engineering Institutions.

Hence, it is essential to recognize Mathematics departments as independent entities with added responsibilities. Primarily, these departments must be responsible for formulation, teaching and evaluation (overall conduct) of routine Mathematics courses. They should also be made responsible for development of new elective courses, short term training programs on some specialized topics, provide core support for students' final year projects or UG research projects. In addition,

Mathematics departments should be encouraged to team up with engineering departments to generate revenue by consultancy projects, and sponsored projects from government agencies like DST, BRNS, CSIR and other R & D organizations across India. It is advisable that the college management plays a decisive role in creating such separate entities by providing with adequate administrative and financial support.

It is also important to look at the size of Mathematics faculty along with its quality in terms of faculty qualifications. According to our survey of 150 colleges, 76 colleges have less than or equal to 5 faculties (there are instances with just 1 or 2 faculties), 47 colleges are having more than 5 and upto 10 faculties and 27 are having more than 10 faculties. In all there are 1033 faculties (averaging to around 7 per institute) out of that about 50% are Ph.D. holders. The following Figure 2 gives the state wise Mathematics faculties in surveyed Engineering colleges.

If Mathematics departments are to efficiently play their role as indicated above this scenario has to change. Hence, the management should focus on maintaining adequate student-teacher ratio and on appointing more qualified teachers in the department.



tency of units of measurements, expand their vision, acquire preliminary knowledge of various engineering disciplines, and contribute significantly in UG research projects and make efforts to get sponsored/ collaborative projects from Funding agencies / Industries.

2.6 ASSESSMENT

Assessment is one of the most important components in teaching learning process. Several examination reforms like continuous evaluation, Bloom's taxonomy suggesting format of question paper, proper division of objective and subjective questions, are suggested. However, examining students from allied disciplines in Mathematics is an altogether different ball game. Frequently the mathematics subject is considered responsible for poor results in the first year. This leads to an undue pressure on Mathematics teachers to improve the results. I have seen many engineering colleges affiliated to the state technological universities readjusting the internal marks to improve their results. These facts are also known to the students and hence the seriousness of continuous evaluation / internal examinations is at stake. Though the examinations increase the seriousness of the students in studying a subject, poor results may lead to reduction in their confidence and interest in the subject.

One of the measures to improve the situation in my opinion could be to give simple home assignments after completion of each unit in the syllabi, having direct questions as well as some objective questions, testing conceptual understanding. Assignment submission should be mandatory and 50% of internal marks can be allotted to these assignments and other 50% to the mid semester examination. People may argue that students may copy the answers from fellow students but I believe that the questions should be so simple that an average student should be able to solve them without any help from others and if required hints also can be given in the assignment itself. While setting the papers for internal or external examinations the teacher must make it a point to test what students know and not to find out what they do not know, in addition to other guidelines suggested by AICTE / NAAC. I believe that this may help to increase confidence of students in doing Mathematics and to retain their interest.

Experts in education say that preparedness of students for learning and their confidence that "I will be able to do it", are much more important than any other factors in a teaching learning process. This is more evident in the context of Mathematics education in allied disciplines. Most of the measures suggested here, I believe, may help in addressing these issues.

This article is intended to trigger a discussion among teachers and academic administrators in Engineering colleges. I will be looking forward to readers' feedback and comments on this important issue. I am very grateful to Prof. M. C. Joshi from IIT Gandhinagar, Prof. S. Katre from SPPU and Prof. S. Ramamohan from MSU for their valuable suggestions in finalizing this article.

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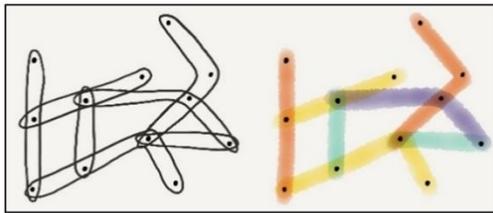
3. What is happening in the Mathematical world?

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3.1 MATHEMATICIANS SETTLE ERDŐS COLORING CONJECTURE

Fifty years ago, in September 1972, Paul Erdős, together with Vance Faber, and László Lovász, came up with a graph theory problem, and though they thought that they might solve on the spot, it is only recently that a team of mathematicians has finally settled it, proving that to color the edges of a linear hypergraph, you never need more colors than the number of vertices in the graph.



The difficulty with the problem was realized only slowly. Paul Erdős, known for offering prizes for solving mathematical problems, originally offered US \$50 for proving the conjecture, and later raised the reward to US \$500, and often advertised it as one of his three favorite problems. Ordinary graphs are built from points, called vertices, linked by exactly two vertices. But the

hypergraphs Erdős, Faber and Lovász considered are less restrictive: Their edges can corral any number of vertices—like shared membership in a group.

The Erdős-Faber-Lovász conjecture is a coloring question about a specific type of hypergraph where the edges overlap minimally. In these structures, known as linear hypergraphs, no two edges are allowed to overlap at more than one vertex. The conjecture predicts that the chromatic index, namely the number of colors needed to color the graph so that no overlapping edges have the same color, of a linear hypergraph is at most its number of vertices.

The extreme generality of the Erdős-Faber-Lovász conjecture makes it challenging to prove. But now, nearly 50 years later, a team of five mathematicians has finally proved this conjecture.

It was earlier proved by Jeff Kahn of Rutgers University, in 1992, that the chromatic number of the graphs in the conjecture is at most $k + o(k)$. Last November, Daniela Kühn, and Deryk Osthus (both senior mathematicians) and their team of three postdocs -Dong Yeap Kang, Tom Kelly and Abhishek Methuku- set out to improve Kahn's result, Their ideas turned out to be more powerful than they expected and as they moved on they realized that they might be able to prove the conjecture exactly.

They started by sorting the edges of a given hypergraph into several different categories based on edge size (the number of vertices an edge connects). After this sorting they turned to the hardest-to-color edges first: edges with many vertices. Their strategy for coloring the large edges relied on reconfiguring these edges as the vertices of an ordinary graph. They colored them using established results from standard graph theory and then transported that coloring back to the original hypergraph.

After coloring the largest edges, they worked their way downward, saving the smallest edges of a graph for last. Since small edges touch fewer vertices, they are easier to color. But saving them for last also makes the coloring harder in one way: By the time the researchers got to the small edges, many of the available colors had already been used on other adjacent edges. To address this, the researchers took advantage of a new technique in combinatorics called absorption that they and others have been using recently to settle a range of questions. This technique contains a combination of ideas that researchers have exercised in recent years to settle a number of long-standing open problems. It wasn't available to Erdős, Faber and Lovász when they dreamed up the problem. But now, staring at its resolution, the two surviving mathematicians from the original trio can take pleasure in the mathematical innovations their curiosity provoked.

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3.2 MATHEMATICIANS ANSWER AN OLD QUESTION ABOUT ODD GRAPHS

For decades, mathematicians have debated a simple question about graphs and the number of connections they have. Now, using simple arguments, Asaf Ferber of the University of California, Irvine and Michael Krivelevich of Tel Aviv University have finally provided the answer in the form of a proof posted in March, 2021.

Over the last century mathematicians have proved a number of basic results related to the “parity” of a graph’s vertices, meaning whether they are connected to an odd or even number of other vertices. In the 1960s Tibor Gallai proved that it is always possible to split the vertices of a graph into two groups, or subgraphs, such that all the vertices within each subgraph have an even number of edges through them (ignoring the edges joining to vertices outside the group) - a property called even “degree”. Around the same time, he observed that it is also always possible to split the vertices in a graph into two subgraphs such that the vertices in one all have even degree and the vertices in the other all have odd degree. Observe that in Figure 1, the highlighted subgraph has all the vertices with an odd degree.

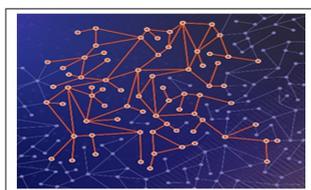


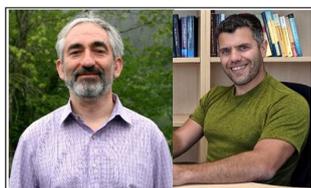
Figure 1

The final option, however, is impossible: There is no way to split every graph into two subgraphs such that all the vertices within each have odd degree. We know this because in the 1730s Leonhard Euler proved that if a group of vertices all have odd degree, then the group must have an even number of vertices. If you have split the vertices of a graph into two subgraphs, and all the vertices within each subgraph have odd degree, then each subgraph must have an even number. This means that if the original graph has an odd number of vertices, there is

no way to do such a splitting.

Given the fact that you cannot always split a graph into two subgraphs of odd degree, the next natural question becomes: What is the largest proportion of the vertices in a graph that you can always be assured will have odd degree? About 50 years ago, mathematicians predicted that for graphs of a given size, there is always a subgraph with all odd degree containing at least a constant proportion of the total number of vertices in the overall graph, independently of the size of the graph.

Progress was made in the early 1990s towards the conjecture by Caro who proved that if you have N vertices in a graph, there is a subgraph containing at least $\frac{1}{\sqrt{N}}$ of them in which all the vertices have odd degree. Two years later, Alex Scott improved that result to $\frac{1}{\log N}$, which is a considerable improvement over Caro’s result, but yet away from the conjecture. Progress on the problem languished for almost 30 years until February 2020.



Now, Michael Krivelevich of Tel Aviv University (left) and Asaf Ferber of the University of California (right) have shown that at least $1/10,000$ of the total vertices of any graph form a subgraph in which all vertices have an odd number of edges through them. (The actual proportion they arrived at was slightly larger, but they rounded it to $1/10,000$ for aesthetic reasons.) Their basic approach - which others before them had also followed - was to sort graphs into three types: “sparse” graphs, where there are lots of vertices that are connected to few other vertices, “dense” graphs with a single vertex connected to many others, and graphs in the middle, with neither of these qualities. Previous work from the 1990s made the sparse and dense cases easy to understand. The hardest part was to understand the middle ground.

The authors came up with a procedure that has allowed them to prove that if a graph is neither sparse nor dense, it must have another quality: Many small subgraphs that are dense within themselves and which are completely disconnected from each other. Proving, that the many small dense subgraphs are not connected to each other, was one of the trickiest parts of the project.

Ferber and Krivelevich established that these many small, dense subgraphs can be joined

together to create a larger subgraph in which all vertices have odd degree. Now they had covered all the possibilities - sparse graphs, dense graphs and graphs in between - and showed that they all necessarily contain odd subgraphs of a certain minimum size. They proved that every graph contains a subgraph, accounting for at least $1/10,000$ of its vertices, in which all the vertices are of odd degree.

Sources: <https://www.quantamagazine.org/mathematicians-answer-old-question-about-odd-graphs-20210519/>

3.3 SCIENTISTS INVENT A MACHINE THAT GENERATES MATHEMATICS WE HAVE NEVER SEEN BEFORE

A new algorithmic invention is developed by researchers from Technion – Israel Institute of Technology, Israel which could help us mechanize the discovery of mathematical formulas relating to the fundamental constants such as π , e , Catalan’s constant, and values of the Riemann zeta function which would help reveal the underlying structure of the constants, which has so far largely eluded the mathematical community. The authors, Gal Raayoni, Shahar Gottlieb, Yahel Manor, George Pisha, Yoav Harris, Uri Mendlovic, Doron Haviv, Yaron Hadad and Ido Kaminer, present two algorithms that have proved to be useful in finding conjectures: a variant of the meet-in-the-middle algorithm and a gradient descent optimization algorithm tailored to the recurrent structure of continued fractions. The approach has been aptly named by the authors as the ‘Ramanujan Machine’ after the genius Srinivasa Ramanujan who left a treasure trove of mathematical formulas, the task of proving which has engaged generations of mathematicians. The algorithms are based on matching numerical values and as such they conjecture formulas without providing proofs, while on the other hand they do not require any prior knowledge of the underlying mathematical structure relating to the constants. The system of algorithms powered by a conglomeration of cloud-connected computers, is capable of producing a large number of conjectural formulas involving various constants, and while several conjectures churned out so far have been simple to prove, there are others which remain unproven.

It is important to point out that the algorithm itself is incapable of proving the conjectures it generates; the task is left to be resolved by human mathematicians. The researchers observe that there are limitations to what the Ramanujan Machine can produce; notably, in some instances, what appear to be previously unknown conjectures generated by the algorithms may be merely mathematical coincidences that break down once finer calculations are made involving matching of more digits.

So far, however, there are reasons to get excited about what these algorithms are enabling – especially the discovery of a new algebraic structure concealed within Catalan’s constant, which hints that the machine might be capable of generating actual breakthroughs the mathematics world has never seen before.

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3.4 LONG-SOUGHT BUILDING BLOCKS FOR SPECIAL POLYNOMIALS FOUND

Hilbert’s 12th problem, from his celebrated list of 23 problems proposed in 1900 that have been a driving force for a good deal of mathematics since then, asked for novel analogues of the roots of unity, as building blocks for certain number systems. Now, over 100 years later, two mathematicians have produced them, for a large class of such systems.

In the 1800s, prior to Hilbert, mathematicians discovered that the roots of unity could serve as “building blocks” for a class of numbers: the roots of abelian polynomials with rational coefficients, namely, the numbers such that the corresponding Galois group is abelian; for example, $x^2 - 2$ is

abelian, but $x^3 - 2$ is not. The roots of unity form building blocks for this class, in the sense that all the roots of abelian polynomials can be expressed as rational linear combinations of them, and conversely any rational linear combination of roots of unity is a root of an abelian polynomial.

What Hilbert was looking for, when he posed his 12th problem, was for mathematicians to find the building blocks, in an analogous sense, for the roots of abelian polynomials with coefficients from more general number fields in place of the field of rational numbers. He expected it to be possible because, as he was writing it, he had an idea about how to describe the building blocks for one other type of number system, namely imaginary quadratic fields, the fields generated by the square root of a negative integer.



In March this year, after the efforts of three years, Samit Dasgupta, a mathematician at Duke University (left) along with Mahesh Kakde of the Indian Institute of Science (right) posted a paper, that used some very modern ideas involving p-adic L-functions to answer, for the first time, Hilbert's question for a large class of number fields, namely totally real fields, the extensions of the rational numbers all whose Galois conjugates are real.

What the authors obtain is a theoretical solution, with the building blocks being conceptual objects, known as Brumer Stark units. To realize them concretely in the manner of the roots of unity involves intricate calculation, with inputs about the specific field in question. Two students working with Dasgupta, Max Fleischer and Yijia Liu, wrote a computer program to generate the building blocks in a concrete form, for number fields generated by \sqrt{D} , square free positive integer, thus baking the now-completed recipe into demonstrable output of the building blocks as sought by Hilbert.

Sources: <https://www.quantamagazine.org/mathematicians-find-polynomial-building-blocks-hilbert-sought-20210525>

3.5 A NEW KIND OF PRIME NUMBERS DISCOVERED



In new research, mathematicians have revealed a new category of “digitally delicate” prime numbers, called “widely digitally delicate primes”.

Take a look at the numbers 294001, 505447 and 584141, they're all prime and they are unusual, in that if you pick any single digit in any of those numbers and change it, the new number is no longer prime. For example, change the 1 in 294,001 to a 7, for instance, and the resulting number is divisible by 7; change it to a 9, and it's divisible by 3. Such numbers are called “digitally delicate primes,” and they are a relatively recent mathematical invention. In 1978, the mathematician and prolific problem poser Murray Klamkin wondered if any numbers like this existed. His question got a quick response from one of the most prolific problem solvers of all time, Paul Erdős. He proved not only that they do exist, but also that there are an infinite number of them — a result that holds not just for base 10, but for any base. Other mathematicians have since extended Erdős' result, including the Fields Medal winner Terence Tao, who proved in a 2011 paper that a “positive proportion” of primes are digitally delicate (again, for all bases). That means the average distance between consecutive digitally delicate primes remains fairly steady as prime numbers themselves get really big - in other words, digitally delicate primes won't become increasingly scarce among the primes.

Now, with two recent papers, Michael Filaseta of the University of South Carolina has carried the idea further, coming up with an even more rarefied class of digitally delicate prime numbers. Motivated by Erdős' and Tao's work, Filaseta wondered what would happen if you included an infinite string of leading zeros as part of the prime number. The numbers 53 and ... 0000000053 have the same value, after all; would changing any one of those infinite zeros tacked on to a digitally delicate prime automatically make it composite?

Filaseta decided to call such numbers, assuming they existed, “widely digitally delicate,” and he investigated their properties in a November 2020 paper with his former graduate student Jeremiah Southwick.

In fact, Filaseta and Southwick couldn't find one example in base 10 of a widely digitally delicate prime, despite looking through all the integers up to 1,000,000,000. But that didn't prevent them from proving some strong statements about these hypothetical numbers.

First, they showed that such numbers are indeed possible in base 10, and, what's more, an infinite number of them exist. Going a step further, they also proved that a positive proportion of prime numbers are widely digitally delicate. (In his doctoral dissertation, Southwick achieved the same results in bases 2 through 9, 11 and 31.)

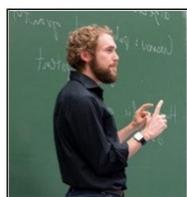
Then, in a January 2021 paper, Filaseta and his current graduate student Jacob Juillerat made an even more astonishing claim: There exist arbitrarily long sequences of consecutive primes, each of which is widely digitally delicate. It would be possible, for instance, to find 10 consecutive primes that are widely digitally delicate. But to do so, you would have to examine a huge number of primes, Filaseta said, "probably more than the number of atoms in the observable universe." He compared it to winning the lottery 10 times in a row: The odds of doing so are extraordinarily slim, but still nonzero.

Since Filaseta and Southwick published their proofs, there are more special cases of digitally delicate numbers in the works as other mathematicians use their research as a jumping off point.

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3.6 AN 80-YEAR-OLD ALGEBRA CONJECTURE DISPROVED



Inside the symmetries of a crystal shape, a postdoctoral researcher has unearthed a counterexample to a basic conjecture about multiplicative inverses. On February 22, a postdoctoral mathematician named Giles Gardam of the University of Münster, gave an hour long online talk about the unit conjecture, a basic but confounding algebra question that had stood open for more than 80 years. He claimed to have proved that the unit conjecture is false.

In 1940, an algebraist named Graham Higman conjectured that in a "group algebra," a structure that combines a number system with a group, only the simplest elements can have multiplicative inverses. Since elements with multiplicative inverses are called units, Higman's hypothesis came to be known as the unit conjecture.

Over the next few decades, Irving Kaplansky, one of the leading mathematicians of the 20th century, popularized this conjecture along with two other group algebra conjectures called the zero divisor and idempotent conjectures; the three came to be known as the Kaplansky conjectures which can be briefly stated as follows:

Let K be a field, and G a torsion-free group.

Kaplansky's *zero divisor conjecture*: The group ring $K[G]$ does not contain nontrivial zero divisors, that is, it is a domain.

Kaplansky's idempotent conjecture states: $K[G]$ does not contain any nontrivial idempotents, i.e., if $a^2 = a$, then $a = 1$ or $a = 0$. and Kaplansky's unit conjecture (which was originally made by Graham Higman and popularized by Kaplansky) states: $K[G]$ does not contain any nontrivial units, i.e., if $ab = 1$ in $K[G]$, then $a = kg$ for some k in K and g in G .

The zero divisor conjecture implies the idempotent conjecture and is implied by the unit conjecture. As of 2021, the zero divisor and idempotent conjectures are open. The unit conjecture, however, was disproved by Giles Gardam in February 2021: he posted a preprint on the arXiv that constructs a counterexample.

Mathematicians were able to prove the unit conjecture for many specific classes of groups by showing that those groups had a property similar to the notion of the highest exponent in polynomials. But researchers also knew of a handful of groups that violate this property, including

a simple one called the Hantzsche-Wendt group. This group captures the symmetries of a shape physicists have considered as a possible model for the shape of the universe, and which is built by gluing together the sides of a three-dimensional crystal. This group seemed like a fruitful place to search for a counterexample to the unit conjecture. But doing so was no straightforward task: The Hantzsche-Wendt group is infinite, so there are infinitely many possibilities even for short sums in the group algebra. In 2010, a pair of mathematicians showed that if there is a counterexample in this group, it will not be found among the simplest of these sums.

Now Gardam has disproved the unit conjecture by finding unusual “units” - elements with multiplicative inverses - inside a group algebra built out of the symmetries of a particular three-dimensional crystallographic shape. He has turned up a pair of multiplicative inverses with 21 terms each within a group algebra built from the Hantzsche-Wendt group. Finding the pair required a complex computer search, but verifying that they really are inverses is well within the realm of human computation. It is simply a matter of multiplying them together and checking that the 441 terms in the product simplify down to the number 1.

Gardam’s discovery of a counterexample to the unit conjecture is heartening. Once Gardam releases the details of his algorithm, it will be open season for other mathematicians to explore the Hantzsche-Wendt group and potentially other groups.

Sources:

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3.7 PRIME-FACTOR - FOUNDATION OF RSA CRYPTOGRAPHY ‘BROKEN’, CLAIMS CRYPTOGRAPHER

Claims by a respected German mathematician that the widely used RSA algorithm has been cracked by an advance in cryptanalysis, have received a respectful but cautious response. One-way functions that form the basis of most cryptographic algorithms rely for their security on the difficulty of solving some problems even with access to a powerful computer. The security of RSA, for example, relies on the difficulty of factoring the product of two large prime numbers. Other types of cryptography use the mathematics of elliptic curves to create a one-way function that is impractical to unravel except through a brute force attack that involves trying every possible key.

Now a mathematician and cryptographer Claus Schnorr claims that prime factorization can be reduced to a much less intractable ‘shortest vector’ problem. The paper entitled ‘Fast Factoring Integers by SVP Algorithms’, claims that this process destroys the RSA cryptosystem. If verified, the technique would work even if longer key values were deployed. Increasing the key length is the standard response to making sure algorithms stay ahead of advances in computing technology. If true, a great number of secure systems that rely on RSA would become insecure or at least vulnerable to a previously well defended vector of attack.

The finding is yet to be comprehensively demonstrated, much less proved, and cautious interest rather than alarm was the general reaction from cryptanalysis-savvy social media users.

Sources: <https://portswigger.net/daily-swig/prime-factor-mathematical-foundations-of-rsa-cryptography-broken-claims-cryptographer#:text=Deep%20Dives,Prime%20factor%20mathematical%20foundations%20of,cryptography%20'broken'%2C%20claims%20cryptographer&text=One%20way%20functions%20that%20form,access%20to%20a%20powerful%20computer>.

3.8 AWARDS

3.8.1 Abel Prize 2021 awarded for enriching link between Mathematics and Computer Science

Abel Prize is one of the most prestigious honours in the field of mathematics and is often equated to the Nobel Prize in Mathematics. This award is given by the Abel committee appointed by the Norwegian Academy of Science and Letters to recognize contributions, of extraordinary depth and

influence, to the mathematical sciences. Since its inception in 2002, this prize has been awarded annually to outstanding mathematicians.

This year the prize of 7.5 million Norwegian kroner, or about \$8,80,000 is shared by Hungarian mathematician László Lovász of the Alfréd Rényi Institute of Mathematics, Budapest, Hungary and Israeli computer scientist Avi Wigderson of the Institute for Advanced Study, Princeton, USA for their contributions to complexity theory and graph theory, which link mathematics and computer science. Lovász and Wigderson helped developing the field of computational complexity and, in fact, were among the pioneers in the field. The field of computational complexity mainly studies the speed and efficiency of computer algorithms. The announcement of the Abel Prize website states that the 2021 Abel prize is awarded to Lovász and Wigderson “ for their foundational contributions to theoretical computer science and discrete mathematics, and their leading role in shaping them into central fields of modern mathematics”.

The theme in which they have both been involved, ‘computational complexity’ which concerns itself with the speed and efficiency of algorithms and first emerged in the 1970s, is now an established field of both mathematics and theoretical computer science and provides the theoretical basis for internet security.



Both Lovász and Wigderson have made fundamental contributions to understanding randomness in computation and in exploring the boundaries of efficient computation. Thanks to the ground-breaking work of these two, discrete mathematics and the relatively young field of theoretical computer science are now firmly established as central areas of modern mathematics.

László Lovász (left), born in 1948 in Budapest, Hungary, has established connections between discrete mathematics and computer science. With Arjen and Hendrik Lenstra, he developed the LLL lattice reduction algorithm, whose applications include factorizing rational polynomials, breaking certain cryptosystems, and designing secure lattice-based cryptosystems. Among other important contributions to combinatorics and graph theory, he proved the Local Lemma, showed how to efficiently solve semidefinite programs, and solved the Kneser conjecture. His previous awards include the 1999 Wolf Prize, the 1999 Knuth Prize, the 2001 Gödel Prize and the 2010 Kyoto Prize.

Avi Wigderson (right), born in Haifa, Israel, in 1956, has conducted research into every major open problem in complexity theory. He has co-authored papers with more than 100 people and has deepened the connections between mathematics and computer science. The most important present-day application of complexity theory is internet cryptography. Early in his career, Wigderson made fundamental contributions in this area, including the zero-knowledge proof, which is now used in cryptocurrency technology. His previous awards include the 1994 Rolf Nevanlinna Prize for computer science, the 2009 Gödel Prize and the 2019 Knuth Prize.

Sources: <https://www.i-programmer.info/news/112-theory/14429-abel-prize-shared-by-math-and-computer-science.html>

3.8.2 Indian-origin Stanford professor Sourav Chatterjee wins Infosys Prize in Mathematical Sciences



Sourav Chatterjee, an Indian-origin professor of mathematics and statistics at Stanford University’s School of Humanities and Sciences has been awarded a prestigious mathematics prize - Infosys Prize of \$1,00,000 in Mathematical Sciences. The award aims to recognize outstanding researchers and scientists around the world. Through the award, the Foundation aims to encourage the spread of science in India, particularly among young people.

Sourav Chatterjee went to Stanford as a doctoral student in 2002 after earning bachelor’s and master’s degrees from the Indian Statistical Institute in Kolkata. He later joined University of California, Berkeley, as a Visiting Assistant Professor, then received a tenure-track Assistant Professor position in 2006. Deeply embedded in probability and statistics, Chatterjee’s work has had significant impacts not only in mathematics but also broadly in physics,

technology and other fields. Across his many papers, Chatterjee has devised novel mathematical approaches for scientists to apply in their own research.

Topics that have benefitted from his mathematical insights include occurrences of rare events, the dynamics of social as well as technological networks, the behavior of magnets and efforts to further solidify a mathematical basis for quantum mechanics. Chatterjee enjoys the challenge of breaking down a problem to its tiniest form and figuring out a fresh perspective. Reflecting both, this range of applications and the helpfulness of Chatterjee's work, the jury of the Infosys Science Foundation described Chatterjee as "one of the most versatile probabilists of his generation" and praised his "formidable problem-solving powers."

Source: <https://indicanews.com/2021/01/06/indian-origin-stanford-professor-wins-prestigious-infosys-prize-in-mathematical-sciences/>

3.8.3 Indian mathematician Nikhil Srivastava wins Michael and Sheila Held Prize



Nikhil Srivastava, a young Indian mathematician and currently an Associate Professor of Mathematics at the University of California, Berkeley has been named winner of the prestigious 2021 Michael and Sheila Held Prize along with two others for solving long-standing questions on the Kadison-Singer problem and on Ramanujan graphs. Srivastava, Adam Marcus from Ecole polytechnique federale de Lausanne (EPFL), and Daniel Alan Spielman from Yale University will receive the 2021 Michael and Sheila Held Prize. The prize consists of a medal and \$100,000.

The prize was established in 2017 by the bequest of Michael and Sheila Held.

Srivastava, Marcus and Spielman solved long-standing questions on the Kadison-Singer problem and on Ramanujan graphs, and in the process uncovered a deep new connection between linear algebra, geometry of polynomials, and graph theory that has inspired the next generation of theoretical computer scientists. They published new constructions of Ramanujan graphs, that describe sparse, but highly-connected networks, and a solution to what is known as the Kadison-Singer problem, a decades-old problem that asks whether unique information can be collected from a system in which only some of the features can be observed or measured. Their groundbreaking papers on the questions were published in 2015 which solved problems that mathematicians had been working on for several decades. Their proofs provided new tools to address numerous other problems, which have been embraced by other computer scientists seeking to apply the geometry of polynomials to solve discrete optimisation problems.

The Michael and Sheila Held Prize is presented annually and honours outstanding, innovative, creative, and influential research in the areas of combinatorial and discrete optimisation, or related parts of computer science, such as the design and analysis of algorithms and complexity theory.

Source: <https://www.thehindubusinessline.com/news/science/indian-mathematician-nikhil-srivastava-named-joint-winner-of-michael-and-sheila-held-prize/article33632378.ece>

3.8.4 Atul Dixit from IIT Gandhinagar wins Gábor Szegő Prize 2021

Prof. Atul Dixit, Assistant Professor of Mathematics at the Indian Institute of Technology Gandhinagar, has become the first Indian mathematician to win the prestigious Gábor Szegő Prize 2021 awarded by the Society of Industrial and Applied Mathematics (SIAM), USA.



The SIAM Activity Group on Orthogonal Polynomials and Special Functions awards the Gábor Szegő Prize every two years to one early-career researcher for outstanding research contributions in the area of orthogonal polynomials and special functions. Prof. Atul Dixit has been selected for this award for his "impressive scientific work in solving problems related to number theory using special functions, in particular related to the work of Ramanujan."

The prize includes a certificate containing the citation. The award was originally supposed to be presented at the 2021 International Symposium on Orthogonal Polynomials, Special Functions, and Applications (OPSFA16). However, due to the current global scenario of the COVID-19 pandemic, the event has been postponed from July 2021 to July 2022. As a

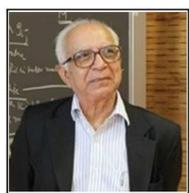
part of the award, Prof. Dixit will also be invited at the OPSFA16, to be held at the Centre de Recherches Mathematiques (CRM), Universite de Montreal, Canada, to deliver a plenary lecture at the prestigious event.

Prof. Atul Dixit's research in mathematics is at the interface of analytic number theory and special functions. His work in number theory has led him to discover new interesting special functions such as generalized modified Bessel and Hurwitz zeta functions. Likewise, his work on special functions has frequently had implications in number theory, such as the one on generalized Lambert series or the Voronoï summation formulas. Prof. Dixit's research work has been impacted to a large extent by Srinivasa Ramanujan, who has been the main source of inspiration for him.

Source: <https://economictimes.indiatimes.com/industry/services/education/prof-atul-dixit-from-iit-gandhinagar-wins-gbor-szeg-prize-2021/articleshow/81431106.cms?from=mdr>

3.9 OBITUARY

3.9.1 Distinguished mathematician Padma Bhushan Prof. M. S. Narasimhan passes away at 88



Distinguished mathematician Padma Bhushan Prof. M. S. Narasimhan FRS passed away on May 15, 2021 in Bangalore at the age of 88. Born on June 7, 1932 in a family of agriculturalists in Tandarai village in northern Tamil Nadu, Narasimhan had a keen interest in mathematics from his school days. The village did not have a high school and he used to ride a bullock cart every day to attend a school in a nearby village.

Prof. Narasimhan studied in Loyola College, Madras (Chennai) and then joined the newly established TIFR for his Ph.D. in 1953. Among his co-students was C.S. Seshadri, with whom he went on to collaborate closely. Narasimhan, together with Seshadri, shot to fame in 1965 with the publication of the Narasimhan-Seshadri theorem, which makes a deep and unexpected connection between two different areas of modern mathematics - differential and algebraic geometry. In Algebraic Geometry he has made pioneering contributions to the development of the theory of moduli spaces of vector bundles on curves and higher dimensional projective varieties. His work with Seshadri relating stable vector bundles on a curve with representations of the fundamental group has been a model for an enormous range of results relating algebraic geometry, differential geometry and non-linear partial differential equations. This and the Harder-Narasimhan filtration (which was discovered later with German mathematician G. Harder) have been generalized and stand as fundamental examples of paradigms with wide applicability.

Prof. Narasimhan was a world-renowned mathematician of extraordinary breadth and depth, who made fundamental contributions to diverse fields in mathematics such as number theory, algebraic geometry, differential geometry, representation theory of Lie groups and partial differential equations. He has done pioneering work in the study of moduli spaces of holomorphic vector bundles on projective varieties.

His papers with S. Ramanan on Universal Connections and moduli of vector bundles, his work with G. Harder which introduced canonical filtrations on algebraic vector bundles and his paper with K. Okamoto on the concrete realizations of discrete series representations of Harish-Chandra have been highly influential. Several distinguished Indian mathematicians were his former research students.

Following an illustrious career at the Tata Institute until retiring from there in 1992, Prof. Narasimhan joined the International Centre for Theoretical Physics, Trieste, Italy, operating under a tripartite agreement between the Italian government, UNESCO, and the International Atomic Energy Agency (IAEA), with a mission to foster the growth of advanced studies and research in physical and mathematical sciences, especially in support of excellence in developing countries. Serving as the head of the Mathematics group of the Centre for a decade, he contributed eminently to fulfilling the objectives of the Centre through a variety of activities. Following the stint at ICTP he returned to Bengaluru, where he continued to be associated with the TIFR Centre for Applicable Mathematics, the International Centre for Theoretical Sciences and the Indian Institute of Science.

Prof. Narasimhan has played a key role in the development of Mathematics in India and in developing countries by influencing several mathematicians and by creating structures for promoting research in Mathematics. He was the founding-Chairman of the National Board of Higher Mathematics of the Government of India. He was a member of the Executive Committee of International Mathematical Union (IMU) from 1982 to 1986 and President of the Commission on Development and Exchange of IMU (1986-94). He was an INSA Council Member (1977-79) and Vice-President, Indian Academy of Sciences (1980-82).

Narasimhan continued to be a good friend of Seshadri, and the former was a great support when Seshadri established the Chennai Mathematical Institute.

Prof. Narasimhan was a Fellow of The Royal Society, London, Indian Academy of Sciences, and the Academy of Sciences for the Developing World (TWAS). He is a recipient of Srinivasa Ramanujan Medal of INSA (1988). Among the other distinctions and awards he received are: King Faisal International Prize for Science (2006), Third World Academy award for mathematics (1987), C. V. Raman Birth Centenary Award of Indian Science Congress Association (1994), Shanti Swarup Bhatnagar award Prize (1975) and Honorary Fellowship of TIFR. He was an elected fellow of the Royal Society of London (2006). He is the only Indian to have been awarded the King Faisal International Prize for Science.

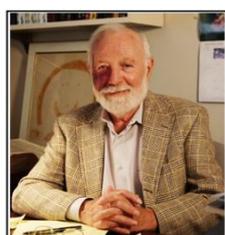
He was awarded Padma Bhushan by the President of India and “Chevalier de l’ordre du Merite” by the President of France. With all his mathematical talent and achievements, Prof. Narasimhan was a simple man, unassuming and humble. In his demise, the world has lost a great mathematician and a noble human being.

Prime Minister Narendra Modi has expressed condolences following the demise of Professor M. S. Narasimhan.

Sources:

1. <http://newsonair.com/Main-News-Details.aspx?id=416976>
2. <https://www.thehindu.com/news/national/mathematician-ms-narasimhan-passes-away/article34620145.ece>

3.9.2 Isadore Singer, who built a bridge from Mathematics to Physics, passes away at 96



Isadore Manuel Singer, an enormously influential figure in 20th century science whose work united mathematics and physics, died on February 11, 2021 at the age of 96. A longtime MIT professor who laid the foundations for the development of index theory was a recipient of both the National Medal of Science and the Abel Prize. In a career that spanned more than 50 years, Singer not only profoundly affected the development of mathematics, but also created a bridge between two seemingly unrelated areas of mathematics, and then used it to build a further bridge, into theoretical physics that led to the creation of a new field, index theory. The achievement created the foundation for a blossoming of mathematical physics unseen since the time of Isaac Newton and Gottfried Wilhelm Leibniz, when calculus first provided tools to understand how objects moved or changed. He changed the way people viewed mathematics, by showing that seemingly different areas can have deep connections.

Isadore Singer was born on May 3, 1924, in Detroit. He studied physics at the University of Michigan, graduating in two and a half years in order to join the Army as a radar officer during World War II. His drive for understanding physics through mathematics was already formed by this time. Stationed in the Philippines, he ran a communications school for the Filipino Army during the day. At night, he filled in the gaps of his abbreviated education, studying mathematics in correspondence courses to learn the prerequisites for relativity and quantum mechanics.

After leaving the Army, he spent a year studying mathematics at the University of Chicago. Though he had planned to return to physics, he fell in love with mathematics and stayed to earn his doctorate. He did a postdoctoral fellowship at the Massachusetts Institute of Technology, where he ended up teaching for almost his entire career.

During an interlude at the University of California, Berkeley, he helped in establishing the Mathematical Sciences Research Institute. He also began proving a number of important theorems, leaving mathematics literature peppered with his name: the Kadison-Singer conjecture (formulated in 1959 and proved only in 2013), the Ambrose-Singer theorem, the McKean-Singer formula and Ray-Singer torsion. But all those were dwarfed by his singular contribution, the Atiyah-Singer Index theorem. Together with the British mathematician Michael Atiyah, he created an unimagined link between the mathematical subfields of analysis and topology - and then united those fields with theoretical physics. Dr. Singer's work with Atiyah allowed for the development of critical areas of physics, like gauge theory and string theory, that have the potential to revolutionize our understanding of the most basic structure of the universe.

His work garnered a tremendous number of distinctions. Singer was a Sloan Research Fellow from 1959 to 1962, and twice received a John D. Guggenheim fellowship in 1968 and 1975. In 1969 he received the Bôcher Memorial Prize from the American Mathematical Society. In 1985 he was awarded the National Medal of Science. The following year, Singer was selected for MIT's highest faculty appointment as Institute Professor. In 1988 he received the Eugene Wigner Medal. In 1993, he was honored with the Distinguished Public Service Award, and in 2000 the Leroy P. Steele Prize for Lifetime Achievement, both by the American Mathematical Society. In 2004, Singer, jointly with Sir Michael Atiyah, received the Abel Prize from the Norwegian Academy of Science and Letters, and he was selected for the 2005 MIT James Rhyne Killian Faculty Achievement Award.

Singer's service to the mathematical and broader scientific community was also remarkable. He chaired the Committee of Science and Public Policy of the National Academy of Sciences from 1973 to 1979. He co-founded The Mathematical Sciences Research Institute (MSRI) in 1982 with Shiing-Shen Chern and Calvin Moore. From 1982 to 1988, Singer served on the White House Science Council. He was a member of the President's Committee for selection of the National Medal of Science, ending in 1989. Between 1995 and 1999, he was on the governing board of the National Research Council.

Singer was equally committed to teaching. Singer advised many graduate student advisees - 31 from MIT and 2 from the University of California at Berkeley - and had nearly 200 mathematical descendants. Even late into his career, Singer insisted on teaching undergraduates, volunteering to work for several semesters as a teaching assistant for the introductory calculus course for first-year students at MIT.

Sources:

1. <https://news.mit.edu/2021/institute-professor-emeritus-isadore-singer-renowned-mathematician-who-united-math-physics-dies-0217>
2. <https://www.nytimes.com/2021/02/12/science/isadore-singer-dead.html#:~:text=Isadore%20Singer%2C%20who%20unified%20large,by%20his%20daughter%20Natasha%20Singer.>

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4. International Calendar of Mathematics events

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August 2021

- August 2-4, 2021, PRIMA 2021 Summer School: Rational curves and moduli spaces in arithmetic geometry. <https://www.pims.math.ca/scientific-event/210802-pssrcmsag>
- August 2-4, 2021, Workshop on Advances in Theory and Algorithms for Deep Reinforcement Learning Virtually, ICERM. <https://icerm.brown.edu/events/htw-21-drl/#workshopoverview>

- August 2-6, 2021, 9th International Conference on Research and Education in Mathematics (ICREM9) Langkawi, Malaysia. <https://einspem.upm.edu.my/icrem9/>
- August 2-7, 2021, Program on Elliptic curves and the special values of L-functions (ONLINE), ICTS, TIFR. <https://www.icts.res.in/sites/default/files/media/program/poster/269>
- August 9-11, 2021, International Conference on Discrete Groups, Geometry and Arithmetic: A virtual event in honour of M. S. Raghunathan's 80th Birthday, jointly organized by NCM, India, TIFR, Mumbai and UM-DAE CEBS, Mumbai. <https://sites.google.com/view/msr80>.
- August 16-20, 2021, SIAM Conference on Applied Algebraic Geometry (AG21) Virtual Conference. <https://www.siam.org/conferences/cm/conference/ag21>
- August 18-20, 2021, The third Young Researchers in Algebraic Number Theory meeting (YRANT III) (delayed from last year) has been moved online. <https://web-eur.cvent.com/event/08d43894-c4fc-4f4d-825a-5cc6f03afeed>
- August 30 - September 3, 2021, Differential Equations and Applications Brno University of Technology, Faculty of Mechanical Engineering, Brno, Czech Republic. <http://diffeqapp.fme.vutbr.cz/2020/main.php>
- August 31 - September 3, 2021, CALCO 2021: 9th International Conference on Algebra and Coalgebra in Computer Science, Salzburg, Austria / online. <https://www.coalg.org/calco-mfps2021/>

September 2021

- September 1, 2021-August 31, 2022, A Year Long Seminar Series on Triangle Groups in Topology, Geometry and Arithmetic (Weekly Seminars), Bhaskaracharya Pratishthana, Pune, www.bprim.org
- September 6-10, 2021, Arithmetic Geometry - Takeshi 60 Graduate School of Mathematical Sciences, The University of Tokyo, Tokyo, Japan. <https://www.ms.u-tokyo.ac.jp/shiho/takeshi60/index.html>
- September 7-10, 2021, XXIX International Fall Workshop in Geometry and Physics, Universidade da Beira Interior (Online). www.ifwgp2020.ubi.pt/
- September 8-10, 2021, Online Event: 2nd IMA Conference on Mathematics of Robotics. <https://ima.org.uk/11468/ima-conference-on-mathematics-of-robotics/>
- September 20-22, 2021, Women in automorphic forms Hybrid meeting: ZiF Bielefeld, Germany, and Online. www.claudia-alfes.de/home/women-in-automorphic-forms
- September 20-24, 2021, Summer School and Conference on Six Functor Formalism and Motivic Homotopy Theory, Università degli Studi di Milano Dipartimento di Matematica "Federigo Enriques" Milano, Italy. <https://sites.google.com/view/summer-school-motivic/home>
- September 27-29, 2021, SIAM Conference on Geometric and Physical Modeling (GD/SPM21), Virtual Conference. www.siam.org/conferences/cm/conference/gdsprm21

October 2021

- Lattices and Cohomology of Arithmetic Groups: Geometric and Computational Viewpoints, Online (21w5205), October 3 to October 8, 2021. <https://www.birs.ca/events/2021/5-day-workshops/21w5205>
- October 10-15, 2021, Cohomology of Arithmetic Groups: Duality, Stability, and Computation. <https://www.birs.ca/events/2021/5-day-workshops/21w5011>
- October 19-21, 2021, 8th International Congress on Fundamental and Applied Sciences 2021 (ICFAS2021) Antalya, Turkey. <http://icfas2021.intsa.org>
- October 23-24, 2021, The Tenth Annual Upstate Number Theory Conference, In person, Union College in Schenectady, NY. <http://www.math.union.edu/hatleyj/Upstate2021/Upstate2021.html>

- October 27-29, 2021, 34th Midwest Conference on Combinatorics and Combinatorial Computing, The University of Nevada Las Vegas Las Vegas, Nevada, USA. <http://mcccc.sites.unlv.edu>
- October 27-30, 2021, International Online Conference on Differential Equations, Control and Optimization / DECOP ' 202. <http://decop.org>

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5. Problem Corner

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In the previous issues, we had posed some challenging problems from Geometry, Number Theory Combinatorics, Functional Equations and Analysis. We have so far received solutions of only two problems, one from Number theory and the other from Geometry from our readers. We have also received incomplete solutions of two problems from Combinatorics, however, we are waiting for their complete solutions.

In this issue, we present a problem based on the greatest integer function along with its solution and pose a problem from Graph Theory for our readers. Readers are invited to email their solutions to Udayan Prajapati (udayan64@yahoo.com), Coordinator, Problem Corner, before 10th December, 2021. The most innovative solution will be published in the subsequent issue of the bulletin.

Problem: Solve $[x^2] - [x] - 2 = 0$ (1)

Solution:

Let us begin by noting that $[x] = n$, where n is the integer satisfying $n \leq x < n + 1$ (2).

Also, from the given equation, $[x^2] = [x] + 2 = n + 2 \geq 0$. So, $x \geq n \geq -2$.

If $-2 \leq x < -1$ then $[x] = -2$, $[x^2] \geq 1$ then $[x^2] \neq [x]$, so if x is in this range it cannot be a solution of Equation (1).

$x = -1$, is clearly a solution of the Equation (1).

Similarly, If $-1 < x < 0$ then $[x] = -1$, $[x^2] = 0$ then $[x^2] \neq [x] + 2$, so if x is in this range it cannot be a solution of Equation (1).

If $x \geq 0$, then $n \geq 0$ and by (2), $n^2 \leq x^2 < (n + 1)^2$ (3).

Also, $n + 2 \leq x^2 < n + 3$ (4)

So from (3) and (4) we have two quadratic inequalities $n + 2 < (n + 1)^2$ and $n^2 < n + 3$ (5).

We have $n \geq 0$ and $n + 2 < (n + 1)^2$, $n = 0$ is not possible.

If $n \geq 3$, then $n^2 \geq 3n \geq n + n \geq n + 3$ which contradicts (5).

Hence, $0 < n < 3$. Thus $n = 1$ or 2 .

Case 1: For $n = 1$, $[x^2] = [x] + 2 = n + 2 = 3$. Thus $3 \leq x^2 < 4$ and $1 \leq x < 2$. Hence $\sqrt{3} \leq x < 2$. In this case the given equation can be verified.

Case 2: For $n = 2$, $2 \leq x < 3$ and so $4 \leq x^2 < 9$. Thus $[x] = 2$ and $[x^2] = [x] + 2 = 2 + 2 = 4$. So $4 \leq x^2 < 5$. Thus $2 \leq x < \sqrt{5}$. In this case the given equation can be verified.

So the solution set of the given equation is $S = \{-1\} \cup \{x \in \mathbb{R} | \sqrt{3} \leq x < \sqrt{5}\}$.

Problem for this issue

A simple connected graph has 2020 vertices and each vertex has degree 3. Find the largest positive integer k satisfying: There exist k vertices, no two of them joined by an edge, such that if they are deleted, the resulting subgraph is still a connected graph.

Continued Progress of TMC Distinguished Lecture Series

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The last issue of the Bulletin contained a brief write-up on the TMC Distinguished Lecture Series, or in short, TMC DLS, and the first few lectures in this series by Professors Yves Benoist (Paris), Bernd Sturmfels (Leipzig/Berkeley), Nalini Anantharaman (Strasbourg), and Alex Lubotzky (Jerusalem). This activity, which began in October 2020, continues to grow strong and is able to feature virtual talks almost every month by some of the best researchers and expositors around the world, thanks to the efforts of the Scientific Committee and the Organizing Committee.

In the recent past, the TMC DLS has featured talks by **Prof. Scott Sheffield** (Massachusetts Institute of Technology, Cambridge, USA, on 10 February, 2021) on *Pandemics and paradox*, **Prof. Karen Smith** (University of Michigan, Ann Arbor, USA, on 8 April 2021) on *External singularities in prime characteristic*, **Prof. Daniel Wise** (McGill University, Montreal, Canada, on 30 April, 2021) on *A cubical root to understanding groups*, **Prof. Mikhail Lyubich** (Institute for Mathematical Sciences, Stony Brook, USA, on 4 June, 2021) entitled *On the MLC conjecture*. The latest talk in TMC DLS was by **Prof. Tadeshi Tokieda** (Stanford University, USA) on *Applying Physics to Mathematics*. The video of this talk was released on 7th July 2021 and the interactive session is scheduled on 20th July 2021.

The TMC DLS is organized by The (Indian) Mathematics Consortium, and it is co-hosted by IIT Bombay and ICTS-TIFR Bengaluru. More information about the TMC DLS is available at: <https://sites.google.com/view/distinguishedlectureseries/>

The videos of the talks held thus far are available on the TMC YouTube Channel at: https://www.youtube.com/channel/UCoarOpo_-9fgzFWDap6dFFw/



George Green (14 July 1793 - 31 May 1841)

A British mathematical physicist.

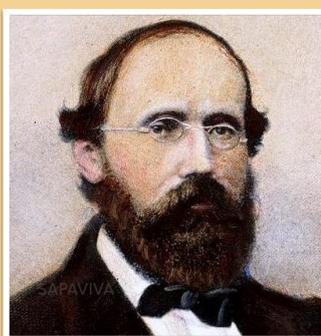
The first person to create a mathematical theory of electricity and magnetism. Introduced several innovations like: A theorem similar to the modern Green's theorem, potential functions, Green's functions. His work on potential theory ran parallel to that of Carl Friedrich Gauss. He was almost entirely self-taught.



Niels Henrik Abel (25 Aug. 1802 - 6 April 1829)

A Norwegian mathematician.

Gave the first complete proof demonstrating the impossibility of solving the general quintic equation in radicals. An innovator in the field of elliptic functions, discoverer of Abelian functions. The Abel Prize in mathematics, is named in his honor.



George Friedrich Riemann (17 Sept. 1826 - 20 July 1866)

A German mathematician, physicist.

Made contributions to analysis, number theory, and differential geometry. Known for rigorous formulation of the Riemann Integral, his work on Fourier series, introduction of Riemann surfaces and the Riemann hypothesis. Laid the foundations of the mathematics of general relativity.

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