The Mathematics Consortium

BULLETIN

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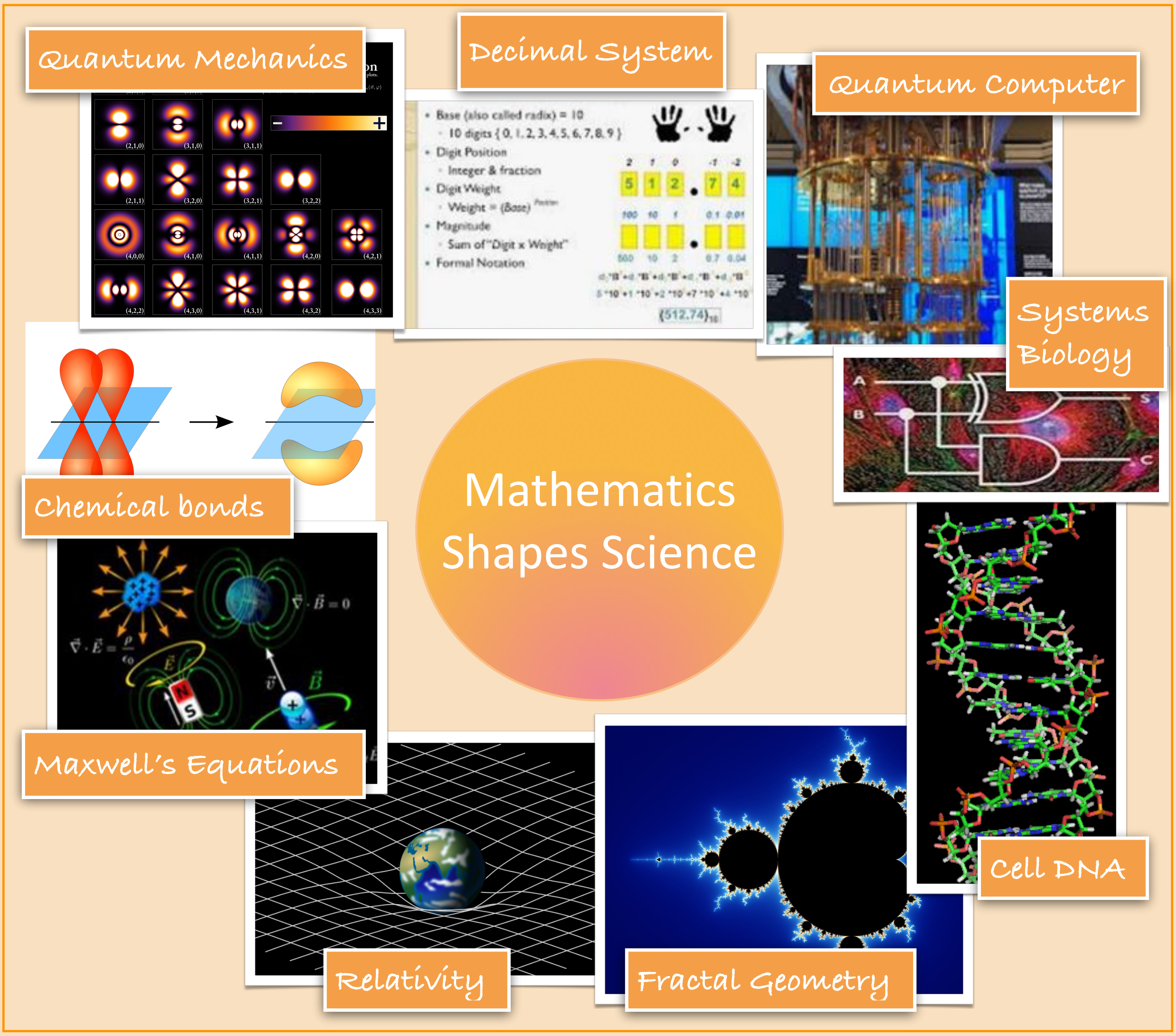

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Contents

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Typeset and designed by: Praanjala Typesettings, Contact: +91 99603 03874, Email: mr.texdoc@gmail.com.

From Editors' Desk

Even a passing view of History of Science reveals a deep connection between progress in Science with better understanding and development of mathematical ideas and tools, going back all the way to counting as the basic mathematical ability, to the role played in modern physical theories by revolutionary developments in geometry, analysis, probability, to name a few branches. Other scientific disciplines such as biology, and even areas of humanities such as economics and sociology have not been lagging far too behind. Mathematics has not been an equipage or facilitator in the course of the build of the scientific edifice, though such has been the popular perception for various reasons. It has also in many, and crucial, instances an inspiring guide for being able to peep into the future, and take steps towards it. Some of these aspects of the role of mathematics in scientific theory-building were discussed in my article "Definition of Mathematics", which appeared as the first article in the first issue of TMCB.

In the opening article of this issue, Prof. K. N. Joshipura illustrates the above developmental process in Sciences and Mathematics through an overview of some remarkable ideas in Science and their Mathematical perspective as perceived by late Professor Subhash Bhatt. The article is based on a lecture delivered by Professor Bhatt a few months before his demise in February 2020.

The second article by Prof. Jugal Verma gives an exposition on the theme of Singular value decomposition (SVD) of real Matrices. He lucidly presents basic facts about the SVD, from scratch, making the article accessible even to the undergraduate students in Mathematics as well as in other allied disciplines.

Recently the Indian Mathematics Community received with grief and shock the news of *Conjeevaram Srirangachari Seshadri*, FRS, a world-renowned mathematician and Padma Bhushan awardee, passing away, in Chennai on 17 July, 2020, at the age of 88. Aside from his profound contributions in the field of algebraic geometry, with many breakthroughs in the subject to his credit, Prof. Seshadri is known for having founded the Chennai Mathematical Institute (CMI), in 1989, which has given a great impetus to higher education in mathematics in India.

Prime Minister Narendra Modi condoled his demise with the words "India has lost an intellectual stalwart who did outstanding work in mathematics." President Ram Nath Kovind and Tamil Nadu Governor Banwarilal Purohit, too, expressed their condolences. These gestures manifest the high regard in which Prof. Seshadri is held, as a Face of Indian Mathematics in India.

We include in this issue some spontaneous tributes to Prof. Seshadri. A more comprehensive tribute is planned in the form of a Special issue of the Bulletin dedicated to Prof. Seshadri, to be scheduled in place of our April 2021 issue. Prof. S. G. Dani has accepted to be the editor in charge of bringing out the special issue. Those who wish to pay tributes or contribute an article related to Prof. Seshadri's work may get in touch with Prof. Dani (`shrigodani@cbs.ac.in`).

Prof. Harish-Chandra, another outstanding Indian Mathematician of an earlier era, was born on 11 October 1923 and passed away on 16 October 1983. We offer special tributes to him in this October issue.

In the regular feature on current developments Dr. D. V. Shah writes about various important events that occurred in the world of Mathematics recently. Tributes are also paid to those Mathematicians who are no longer amongst us.

In the Problem Corner, Dr. Udayan Prajapati presents a solution to the problem posed in the last issue by Mr. Dhruv Bhasin, a student of Integrated Ph. D. program at IISER, Pune, and poses a problem from combinatorics for our readers. Dr. Ramesh Kasilingam gives a calendar of Academic events, planned during December 2020 and February 2021.

We are very happy to bring out the second issue of volume 2 of TMC Bulletin in October, 2020. We thank all the authors who have contributed articles for this issue, all the editors, our designers Mrs. Prajkta Holkar and Dr. R. D. Holkar and all those who have directly or indirectly helped us in bringing out this issue on time.

Chief Editor, TMC bulletin.

1. Some remarkable ideas in Science: A Mathematician's Perspective

(Based on a lecture delivered by late Prof. S. J. Bhatt)
K. N. Joshipura
Ex-Professor, Department of Physics, Sardar Patel University
Vallabh Vidyanagar - 388120 (Gujarat)

Preamble: *On February 26, 2020 I was really shocked to learn about the sad and sudden demise of Prof. Subhash J. Bhatt, Ex Professor and Head, Department of Mathematics, S. P. University, Vallabh Vidyanagar. During his funeral on the same day, I met Prof. Joshipura who remarked that Subhashbhai had delivered a very interesting lecture on "Remarkable Ideas in Science" and how well he connected the various branches of science during the course of his lecture. Knowing well the stature of Subhashbhai as a Mathematician and the reputation of Prof. Joshipura as a well-known Physicist, I requested Prof. Joshipura to write an article based on the referred lecture for TMC bulletin. Out of love and regards for Subnashbhai, Prof. Joshipura immediately accepted my request and this article is a result of that. In fact, Prof. M. H. Vasavada has offered a tribute to Subhashbhai in April, 2020 issue of our bulletin. However, this article is a sort of Mathematical tributes to dear Subhashbhai.*

Fellow of Gujarat Science Academy Prof. K. N. Joshipura, a retired Professor and Ex-head, Dept. of Physics, SPU, Vallabh Vidyanagar, is an accomplished researcher in the field of theoretical atomic-molecular Physics and actively involved for over 45 years, in educational and outreach programmes on Physics and astronomy. He has guided several M. Phil./Ph. D. students; had academic visits to USA, UK, Germany, Italy, Spain and Austria; and co-authored a research monograph entitled "Atomic-molecular ionization by electron scattering: theory and applications" with Prof. Nigel Mason (UK), which is published by Cambridge University Press (2019). He has been on the editorial board of the annual Gujarati magazine-journal Pragaami Tarang being published by IAPT (Gujarat region), Prajnaa, Research journal on Sciences, and Sheel Shrutam, SPU Newsletter and contributed several articles in these publications. – Managing Editor.

1.1 Introduction

In the sad demise of Prof. S. J. Bhatt (Ex-Professor of Mathematics, Sardar Patel University Vallabh Vidyanagar) a few months ago, we lost a hard-core mathematician. He was a man of single-minded pursuit in his subject, but his keen interest had been also to look for mathematics as an essential background in various sciences. The title of this article is actually the title of a remarkable lecture he delivered at GCET Engineering College, V V nagar, in the National Conference on Science, Technology, Engineering and Mathematics, held during September 27-28, 2019. I was very fortunate to have a long standing acquaintance with Prof. Subhashbhai Bhatt (SJB), and we used to exchange views on different topics common to Physics and Mathematics. I recall him telling me once, "Physicists have a typical sixth sense, a physical intuition which leads to breakthrough discoveries and inventions." Once he expressed the view that, classical mechanics - just like quantum mechanics - should be axiomatic and should start with postulates (say on space, time, matter, inertia etc). In the said lecture, the learned speaker (SJB), focussed on some of the remarkable scientific ideas developed over the past two or three centuries, and brought forth the underlying mathematical themes. Listed below are the ideas he discussed in the lecture.

1. Counting: mathematics
2. Probability and inference: statistical science
3. Space, time and matter: classical dynamics
4. Relativity theory: geometrization of Physics
5. Quantum mechanics and quantization: non-commutativity
6. Quantum computing: quantization of computing
7. Atoms, molecules and chemical bonding: Science of materials

8. Thermodynamics and entropy
9. Boltzmann and Gibbs: equilibrium and non-equilibrium statistical mechanics
10. Chaos and fractals: non-linear dynamics
11. Chemical foundation of life: cells, DNA and Genetics
12. Systems biology
13. Evolution by natural selection
14. No arbitrage principle: the BSM theory of financial derivatives


Prof. Bhatt at the centre

In a sweeping coverage, the fundamental concepts of classical and modern physics, along with those of chemistry and biology were discussed in this elegant exposition. In (14) above, one also notes mathematics entering the domains of economics and finance. What was the mathematician's perspective in all these ideas?! Whereas a detailed answer to this question, with mathematical rigour, was presented in the said lecture, an informal outline of its contents is attempted in this article. For convenience, some of the above themes are clubbed together appropriately in the following discussion.

1.2 Remarkable scientific ideas

Counting, which is one of our first practical or hands-on experiences is as old as the earliest human civilizations. Even while having a casual look at the surrounding, we notice that there is either nothing or something, and then counting begins. It is through our day to day observations and experiences that the ideas of natural numbers have grown. The place value system for expressing numbers was an epoch-making achievement that turned the corner. The number system evolved further into rational and irrational numbers and science was gifted with all pervading irrational numbers π and e. Eventually negative numbers were introduced in the system, and today we have even negative marking in examinations! Algebra explores real numbers further through equations, while geometry presents real numbers as points on a straight line. The amazing equation, $x^2 + 1 = 0$, introduces the marvellous entity $\sqrt{(-1)}$ the imaginary unit, and thereby the complex number $z = x + iy$. This tiny 'i' has hugely impacted classical and modern physics, and hence other sciences. Complex numbers occur in connection with waves and other periodic phenomena. The special relativistic space-time coordinates (x, y, z, ict) involve 'i', with 'c' as the speed of light in vacuum. The Schrödinger equation - the gateway to quantum mechanics - also involves 'i' in general. SJB notes that the mathematical implications of the advent of complex numbers are in terms of the Gauss' Fundamental Theorem of Algebra and the Cauchy-Abel theorem of non-Solvability by Radicals. An intriguing concept in connection with numbers is 'infinity'. Questions like, *'Is infinity a number? Can there be different types of infinity??'* aroused keen interest. Infinitesimal, the opposite of infinity, though bizarre, forms the basis of calculus.

Probability and *inference* are two of the well-known ideas extensively used in various scientific and other disciplines. The words like possibility, chance and uncertainty are common in our daily life, and they indicate an incompleteness of certain knowledge. Randomness indicates a lack of a pattern or predictability. Probability is a well-defined mathematical entity. In this connection one also speaks of stochastic behaviour. In Physics and in other experimental sciences, different sources of error give rise to uncertainties in the data obtained. Therefore to arrive at an inference, concepts like probable error, systematic error etc have been developed and employed suitably. Actually measurements are considered to be incomplete without error statements. There is another way in which randomness or probability enters Science in general and Physics in particular, and that is exhibited by a macro system (that is, a large system of our common experience) consisting of myriads of tiny building blocks or constituent units. A typical example of this is a gas consisting of a very large number of atoms or molecules, characterized by a well-known constant called

Avogadro number, $N_A = 6.023 \times 10^{23}$. As for example, 18 grams of water contains N_A number of H_2O molecules. In a gas the atoms/molecules are continuously colliding and it would be an impossible task to solve $\sim 10^{23}$ equations of motion to describe the state of the gas system. Therefore to arrive at a reasonable description of the properties of a macro-system that arise from its micro-components, the ideas of probability were employed in the form of *statistical mechanics* by Maxwell, Boltzmann and Gibbs in the second half of 19^{th} century. If somehow, all the molecules of a gas were moving with the same speed in the same direction, there would be just a single description of the 'micro-state' of the system, and there would be perfect order. This does not happen, and there are myriads of micro-states corresponding to a macro-state, i. e. an observable state with measurable properties, and disorder (lack of exact knowledge) is inevitable. Physicists have introduced 'entropy' as a quantity representing disorder. In thermodynamic equilibrium, the entropy of the system is at maximum. The concept of canonical ensemble, the statistical ensemble that represents the possible states of a system in thermal equilibrium with a constant temperature heat bath, was initially propounded by Boltzmann and later reformulated by Gibbs. While statistical mechanics explores the 'macro' in terms of 'micro', in an empirical approach called thermodynamics, fundamental statements or laws are propounded to explain the large-scale properties of a system. There are four laws of thermodynamics, and their numbering begins from zero! A mathematician like SJB would like to call them axioms! The first law is essentially a law of conservation of energy, connecting heat energy, mechanical work and internal energy of a system. The concept of energy was generalized and the law of conservation of total energy, an article of faith in Physics, was established in the mid - 19^{th} century. The first and the second laws were proposed almost simultaneously in that period, while the zeroth law came up much later!

The second law governs the direction of natural processes. One of the statements of the second law says that, no cyclic process exists whose sole effect is to extract heat from a system and convert it entirely to work. In the natural irreversible processes the entropy of an isolated system increases. In fact, Rudolf Clausius, one of the founding fathers of thermodynamics had made a bold statement, "the total energy of the universe is constant, the total entropy is continually increasing. . . !"

Science per se is sacrosanct and exact, and at the very fundamental level there should be no scope of any uncertainty in scientific knowledge. However the rise of modern physics in the 20^{th} century brought in its wake, the uncertainty principle propounded by Heisenberg in 1927. The principle effectively operates in the micro-world i. e. at subatomic levels, and says that, even for a single particle like an electron an exact simultaneous determination of position and momentum is impossible. This kind of uncertainty is inherent effectively in the realm of atomic/subatomic systems, and it has nothing to do with errors in measurements etc. The uncertainty principle is expressed in terms of the non-commutativity of the operators representing dynamical variables in quantum mechanics. That apart, probability concepts are fundamental to the large scale data analysis, and bioinformatics, which is an emerging multidisciplinary science. These ideas are also extended to human systems, and in social sciences attempts are made to understand the group behaviour of people. A modern trend is to focus on 'information' theory, a study of quantification, storage and communication of information. The theory was proposed by a mathematician and an electrical engineer Claude Shannon in 1948, and it finds applications in diverse fields like ecology and biodiversity.

Classical dynamics, a very important foundation of Physics has a proven track record with a reasonably complete mathematical foundation; one would like to call it music of nature! Isaac Newton put forward what we now call Newtonian mechanics. He said in his classic epic Philosophiæ Naturalis Principia Mathematica, "Absolute space in its own nature, without regard to anything external remains always similar and immovable. . . . Absolute, true and mathematical time, of itself, and from its own nature flows equably without regard to anything external." Space, time, matter and energy (radiation) are the basic attributes of the universe. The Classical space is continuous, 3-dimensional, homogeneous, isotropic and Euclidean. It is independent of the matter, radiation or physical processes taking place within. Time flows evenly and independently and is irreversible.

The notion of thermodynamical 'arrow of time' was added later on. Classical mechanics gained a stronger mathematical foundation with the seminal contributions of Lagrange, Hamilton and others. For a single particle of mass m, Newton's second law of motion $\mathbf{F} = m\mathbf{a} = md^2\mathbf{x}/dt^2$ is essentially a second order differential equation in displacement $\mathbf{x}$. The mechanics is extended to a system of discrete particles, and then to a continuous mass distribution by invoking the concept of a rigid body. A fundamental treatment of rotation of a rigid body was given by Euler. Fluid dynamics was also developed eventually. Mathematics also offers a differential equation to describe the motion of a wave, which is a periodic disturbance that propagates in space and time. A wave is characterized by frequency ν, wavelength λ and wave speed v, and we have the basic relation $v = \lambda.\nu$. Another quantity called the amplitude 'A' measures its strength. Waves are essentially represented by sine and cosine functions, and a wave equation is a differential equation in space coordinates and time. The classical notions of particles and waves are quite exact, clear and distinct.

Let us recall at this stage an elegant and a remarkable application of mathematics in Physics, viz. Maxwell's equations of electrodynamics established around 1865. Maxwell synthesized the basic laws of electricity and magnetism known already, added his intuitive idea and arrived at a set of four coupled differential equations describing the space and time variation of alternating electrical and magnetic fields. He interpreted light along with invisible radiations as electromagnetic waves travelling in free space (vacuum) with an identical speed c, and brought about a kind of unity in diversity.

Classical mechanics, along with thermodynamics, statistical mechanics, electrodynamics and optics is collectively called *classical Physics*. Towards the end 19^{th} century, classical Physics enjoyed a great success in interpreting a variety of observable and laboratory phenomena. In those days, some of the physicists even started believing that there was hardly anything that remained to be known! They saw the end of Physics in sight, but that was not to be...!!!

Theories of relativity put forward by Albert Einstein (Special theory in 1905 and General theory in 1915) resulted into a kind of renaissance in Physics. In fact the year 1905 was *annus mirabilis* - a miracle year for Einstein. In mechanics we describe the motion of an object relative to a particular observer in a coordinate system or a 'frame of reference'. Observations from two observers i. e. from two reference frames mutually in a uniform relative motion can be simply connected by (classical) Galilean transformation equations. Although the description changes as per the frames of reference, the basic laws of Physics should not change. Einstein tried to reconcile this assertion with the laws of electrodynamics, and arrived at a basic formulation for the special case of two reference frames in constant relative motion, and hence the name special theory of relativity (STR). Of the two postulates of this theory, the first one is the principle of relativity, which says that, 'the laws of physics are identical in form in all inertial frames of reference.' The second postulate says that, 'the speed of light in free space has the same value c in all inertial frames of reference, irrespective of the motion of the source or the observer.' With these postulates, a deep connection is established between space and time, and the Lorentz transformation equations relating the space-time coordinates between two reference frames in uniform relative speed v, hold a fundamental significance. These equations reduce to the classical Galilean transformation if the relative speed is not too high i. e. $(v/c) \ll 1$. The physical interpretations of STR are appalling. They amount to a major paradigm shift in the classical notions of absolute and independent space and time. In fact, absolute motion has no meaning now. Speed of light in vacuum or free space assumes a very significant role of a fundamental constant of the universe. Besides this, the other important consequence of the theory is that, no object can move with speed of light which is almost 3×10^8 meters/sec. A mathematician finds it interesting to note that the STR involves non-Euclidean geometry. A special gift from the special relativity is one of the most well-known equations of modern Physics, viz. $E = mc^2$, implying equivalence of mass (m) and energy (E). Mass and energy are no more separate physical quantities; they are different manifestations of one and the same entity, and are mutually convertible. Accordingly the law of conservation of total 'energy' is expanded to include mass also as a form of energy.

General theory of relativity (GTR) a unique creation of the genius of Einstein, leads to geometrization of Physics. Geometry which is a human perception of space and shapes of objects now enters the realm of physics in an unusual way. Since space-time is filled with matter, mass has to be incorporated in relativistic space-time, and that is done in GTR. As a consequence, the space-time is not flat. It has curvature due to the presence of matter. With this, the classical picture of gravitation as force acting at a distance is totally altered. A free particle travels along the geodesics of this curved space-time. The space-time is not passive; it is an actor in Nature. Space and matter interact with each other. The General relativity plays a key role in understanding astronomical phenomena like the evolution of stars and black holes etc. An important prediction of the GTR was on gravitational waves. The prediction has now come true, and the gravitational waves have been detected through elaborate experiments in 2016.

Quantum mechanics and quantization were the truly remarkable ideas that brought about revolutionary changes in Physics and in our classical perception of Nature. Quantum (plural quanta) means a unit or a discrete amount of a quantity; quantization indicates a kind of unitization. Max Planck (1900) in his attempt to give a mathematical formula to describe correctly the distribution of energy emitted by a perfectly black body as a function of the wavelength of radiation, was required to assume that emission/absorption of radiation occurs through discrete packets of energy. Einstein built up the idea further and proposed that electromagnetic radiation is corpuscular; it exists in the form of discrete units or quanta of energy, called *photons*. The energy content E of a photon in the radiation of frequency ν is given by the basic equation $E = h\nu$, where 'h' known as Planck's constant is a fundamental constant of the universe. But then what about electromagnetic 'waves' already used successfully to explain a large number of optical phenomena?? Well, the picture that emerges finally is the dual nature of electromagnetic radiation, wave or particle. Many of the observed phenomena are explained by the wave picture, while certain phenomena like the photoelectric effect reveal the corpuscular or quantum picture of radiation. A quantum is a discrete unit, and a fraction of quantum has no meaning, just as a fraction of one paisa or a fraction of a biological cell has no meaning. With Bohr's atom model (1913), quantization entered the atomic regime. Electrons in incessant motion in atoms and molecules possess discrete energies and angular momenta. Now, says SJB, it requires a leap of faith to envisage that the space-time is also quantized, geometry is quantized and length, area and volumes are quantized at an appropriate extremely minute scale, named as the Planck scale. The whole of Mathematics has to be quantized it seems...!

A major step forward in theoretical physics was taken by de Broglie with his hypothesis of the wave nature of particles, like electrons. A particle having momentum p is endowed with wavelength λ_{dB}, expressed as $\lambda_{dB} = h/p$. This equation along with $E = h\nu$ represents a remarkable shift from ideas of classical physics. The argument turns full circle now; radiation exhibits dual nature wave and particle, matter exhibits dual nature particle and wave. The entire picture is very well established experimentally.

The gateway to *quantum mechanics* is Schrödinger's wave equation. Represented symbolically as $H\Psi = E\Psi$, the Schrödinger equation has a general dependence on space coordinates and time. In quantum mechanics the dynamical variables (physical quantities) are represented by operators obeying appropriate commutation relations. The Hamiltonian operator H is the total energy operator, and in a time independent (or stationary) situation, the equation $H\Psi = E\Psi$, is the energy Eigen value equation, with 'E' as the total energy of the particle or the system. Further, we have $H = T + V$, where T is the kinetic energy operator and V is the potential energy operator. In a 3-dimensional case, T involves the Laplacian operator ∇^2, and hence the Schrödinger equation is a (linear) second order differential equation. It can be solved under appropriate boundary conditions, which result into quantization of energy. The solution called the wave function of the particle in question is such that $|\Psi|^2$ represents the probability of finding the particle in a unit volume around its position $\mathbf{r}$. In general, $\Psi = \Psi(\mathbf{r}, t)$, and the Schrödinger equation represents time evolution of the quantum mechanical system. In 1928, Dirac introduced the special theory of relativity in this formalism and developed relativistic quantum mechanics.

As noted by SJB, philosophical and mathematical implications of quantum mechanics are as follows.

1. Non-commutativity at the fundamental microscopic level leads to commutativity at classical level.
2. Inherent randomness leads to causality at classical level.
3. Quantization is inherent in Nature.

Quantum mechanics was soon put to a firm axiomatic mathematical foundation, when the wave function $\Psi = \Psi(x, y, z, t)$ representing the physical state of a system, was elevated to the status of an abstract 'state vector' in an appropriate infinite dimensional Hilbert space, with the dynamical operators also defined accordingly. Apart from the fundamental aspects, there is an immense significance of quantum mechanics in terms of atomic- molecular and nuclear physics and also in terms of chemistry and biology.

Quantum computing is an emerging marvel of the modern age and is rooted in quantum mechanical concepts. In his lecture, SJB outlined the preliminary mathematical aspects of our present or 'classical' computing, like the Boolean algebra. The classical computer - the machine - contains a complicated circuitry of electronic devices and logic gates. In this context, a *bit* i. e. binary digit is a unit of information - it has only two possible states 0 and 1, corresponding to 'off' and 'on' condition of electric current through the circuit component. In the mathematical logic, a statement can be either 'true' or 'false', and this fact corresponds respectively to '1' or '0' in the present context. Thus, a computation means transformation (in one or several steps) of an input sequence of 0 and 1 into an output sequence of 0 and 1. Complexity of a computational problem lies in terms of (i) size of a problem, i. e. the number of bits required to state the problem, and (ii) cost of a problem, which includes the time taken and the amount of memory required in solving the problem.

The question is; can we exploit quantum effects somehow in computing. . . ?!

The answer is yes, and serious efforts are going on in the world today. The quantum version of a 'bit' is a quantum bit or a *'Qubit'* of quantum information, which can be, not just 0 or 1, but a superposition of the 0 and 1 states, in view of the linearity of the Schrödinger equation. Let us invoke the half-bracket notation $|x\rangle$ invented by theoretical physicist Dirac, and denote by $|0\rangle$ and $|1\rangle$ the quantum analogues of classical bit-states '0' and '1'. Then, a quantum linear superposition $s = \alpha|0\rangle + \beta|1\rangle$ also represents a possible state. Here α and β are complex coefficients, such that $|\alpha|^2 + |\beta|^2 = 1$. With infinitely many choices for α and β, a quantum computer can, in principle, operate like a massive parallel computer simultaneously working on so many classical bit string inputs at a time. A quantum system initially in a superposition of states would fall upon or collapse into a particular state, which cannot be predicted exactly. In addition to the superposition the other two quantum mechanical properties that form the basis of quantum computing are interference (resulting from superposition) and entanglement. The quantum entanglement is a yet another counter intuitive phenomenon and has no classical analogue. It entails a situation where in two or more interacting particles, initially in proximity; continue to be inter-dependent even if the separation becomes very large. Based on these quantum characteristics, an algorithm i. e. a set of rules for performing operations can be devised and much faster computers can be developed in future.

Atoms, molecules and chemical bonding constitute some of the most remarkable concepts which have connected physics, chemistry and also biology in an enduring bondage. The idea of atom as the ultimate unit or building block of all matter is age old and has roots in philosophy, both Indian and Greek. The first ever scientific concept of an atom was given in 1803 by chemist John Dalton for explaining his general observations on chemical reactions. The concept of a molecule as a system composed of atoms was put forward by Avogadro in 1811. Prior to this, almost in 1738, D. Bernoulli had put forward the kinetic theory of gases through an ideal gas model, in an axiomatic approach. The idea behind was to interpret macroscopic properties of a gas in terms of the random motions of microscopic constituents, i. e. molecules. The theory leads to an expression of the pressure of a gas as a function of $\langle \nu \rangle^2$ where, $\langle \nu \rangle$ is the average molecular speed. Boyle's law for air or a gas, well-known by then was also deduced in this simple model. Kinetic theory of gases was one of the early attempts in Physics in terms of a logical chain of reasoning, viz., to begin with a set of plausible assumptions, build up a mathematical theory and develop a formula, and employ it to arrive at the observations or laws known empirically. Almost through the 19^{th} century, atoms and molecules remained in the realm of chemistry. New chemical elements were discovered and then emerged an epoch making scientific development viz., the periodic table of elements (Mendeleev 1869). Atoms and molecules remained theoretical artefacts, practically impossible to observe directly. Kinetic theory of gases evolved into a probabilistic mathematical theory in the form of statistical mechanics or Maxwell-Boltzmann statistics, as mentioned earlier.

The word 'atom' which stands for indivisible, has been carried over even today, although the internal structure of the atom is well understood now. In the beginning of the 20^{th} century, Rutherford proposed, on the basis of his experiments, that an atom consists of the nucleus, i. e. central massive core having positive electric charge, surrounded by negatively charged electrons moving in circular orbits. Rutherford's 'planetary' atomic model soon ran into trouble, since an electron in circular motion being an accelerated charge, must emit radiation as per the theory of electrodynamics, and would spiral down soon into the nucleus. In that case an atom would not be stable! Neils Bohr started afresh by making revolutionary postulates. Firstly, an electron moving round the nucleus normally stays in a stationary non-radiating orbit. Secondly, emission or absorption of a radiation quantum takes place only when the electron jumps from one to another stationary orbit. Perhaps he had in mind a new physics, a non-classical physics that would hold in the micro regime. Based on his postulates Bohr derived a formula for the wavelength of spectral lines emitted by hydrogen atoms. His formula was found to be identical to an empirical formula given earlier by a mathematics teacher Johann Balmer in 1885. Bohr's theory thus succeeded in explaining the line spectrum emitted by hydrogen atoms. Lo and behold! This is how a theoretical physicist proceeds; he starts with plausible assumptions or postulates, employs them to derive a theory i.e. a mathematical expression relating the observed physical quantities, and finally puts it on the anvil of experimental tests. Of course, with Bohr's theory began a period of confusion, as more questions were raised than answered.

Besides, there were other issues too, for example, how is it that two H atoms combine and form a hydrogen molecule?! In 1916, a physical chemist G. N. Lewis put forward the idea of covalent bond resulting from electron sharing in molecules. Undeterred by the fact that exact mathematical solution of the Schrödinger equation is possible for a few ideal cases only, physicists started working on approximate methods aimed at understanding the electronic structure of atoms as also the chemical bonding in molecules. What amazes physicists and mathematicians alike is the symmetry of various types exhibited by different molecules. Today with the aid of theoretical and computational physics and quantum chemistry, a big knowledge base has been created not just for atoms, but also for the structure and bonding in various molecules from a common ubiquitous H_2O, to macro-molecules like proteins. Materials science, a fusion of applied physics and chemistry has grown now into a veritable interdisciplinary field of science and technology. Of geometer's delight are fullerenes, the 3-dimensional hollow micro structures made up of carbon atoms. C_{60}, a molecule of 60 carbon atoms is an icosahedron made up of 20 hexagons and 12 pentagons.

Chaos and fractals are the new frontiers of interdisciplinary science now-a-days. Chaos is defined as the property of a system in which it exhibits a great sensitivity to initial conditions, and hence a minute change in the initial state results in a large and unpredictable change in a final state. A metaphorical example of this situation is to imagine that a butterfly flutters in Mumbai, and later on there is tornado in Tokyo...! In an informal way, chaos is termed as butterfly effect. The idea that small causes may have large effects in general and in weather specifically was earlier recognized, around 1898 by French mathematician and engineer Henri Poincaré, and by others. In 1961, mathematician - meteorologist E. Lorenz was running a numerical computer model to redo a weather prediction from the middle of the previous run. He entered the initial condition 0.506 from the printout instead of entering the fully precise value 0.506127. To his surprise, the result was a completely different weather scenario. This is not a small surprise, as small errors in measurements and rounding off errors etc are quite common and manageable, and usually they do not lead to diverging outcomes in the future behaviour of the system. As a very simple example of initial condition we can consider linear simple harmonic motion, a standard topic in elementary physics. The displacement x of a simple (ideal) harmonic oscillator at instant t is given by the equation $x(t) = Asin(\omega t + \delta)$, where A is the amplitude, ω is the angular frequency and δ is the initial phase. Along with 'A' and ω, the parameter δ provides the initial condition, and with these inputs the periodic displacement x can be exactly predicted at any later instant t. Chaos on the other hand is a mathematical study of dynamical systems which are often governed by deterministic laws that are highly sensitive to initial conditions. Linearity is an ideal concept in physics, and most of the realistic systems of practical interest are nonlinear. Chaotic systems are nonlinear and complex and the examples are, a double pendulum, an atmospheric system responsible for weather and climate, and some aspects of our human body.

Fractals are infinitely complex patterns that are self-similar across different scales, and they involve a never-ending pattern. Who knows if the old Sanskrit phrase यत्पिण्डेतत्ब्रह्माण्डे points to the idea of fractals. . . ?! Have a close look at the cauliflower, our common vegetable item, and you will notice a kind of self-similarity or fractal behaviour. Fractals differ from finite geometrical figures in terms of the scales and dimensions. If the sides (or edges) of a square (or a polygon) are doubled its area is multiplied by four i. e. 2^2. If the radius of a sphere is doubled, its volume scales up by eight, i. e. 2^3. Please note the power or the index of 2, which is '2' for a square, and '3' in the case of a sphere, and these two are the standard geometrical shapes. The index may be called the scale factor. Quite interestingly, there can be an intermediate situation, wherein the scale factor may not be an integer. Indeed this is found to be so for certain geometrical patterns called fractals. In the case of a fractal if the one-dimensional lengths are all doubled, the spatial content of the fractal scales by a power that is not necessarily an integer. This power is called the fractal dimension. To understand the complexity involved here in a different way, consider the length of a meter rule or meter stick, whose length is exactly 1 meter, as it has to be. . . ! In contrast, think of the length of a coastline between two points A and B on it, separated by a straight line distance x. Now, if you observe the coastline closely, the ground reality is that it is a zigzag pattern giving the actual length to be more than x, and upon still closer examination, the actual length of a coast-line seems to increase to be much more than x, due to kinks in it. . . ! In short, unusual ideas like this have evolved into the geometrical concept of *fractals*. We omit further discussion and mention that both chaos and fractals involve a kind of iteration or recursion. It is found that the graphs of most of the chaotic processes are fractals. Some of the examples of fractals in nature are snowflakes, leaves and branches of trees, interior in our lungs etc. Fractals have entered a new era with technological applications such as fractal antennas.

Let us turn now to a mathematician's perspectives on themes pertaining to biology.

Systems biology is the study of biological entities in terms of their constituent components. Thus the biologists analyse a whole as a sum of parts, not just disjoint, but connected and interacting. This is a very general approach pursued in almost all scientific investigations. The aim in Systems biology is to see how different parts of a biological entity combine together to form a functional living system. The biological hierarchy of units starts with small molecules (formed by a group of atoms like C, H, O, N) combining into macromolecules and supra-molecules forming aggregates. The building-up sequence then expands to Organelles, biological cells and to tissues that combine into an individual living being, which finally yields a biological species. At each stage, a group displays individuality and traits not possessed by its members. A living entity is highly structural; it exchanges matter, energy and information from the environment, exhibits structural stability and ability to reproduce. It also exhibits variations resulting from natural selection. Groups of individuals exhibit properties or behaviour not displayed by individuals. Does this not remind us of thermodynamics and statistical mechanics of physical systems?! The collective behaviour is characterized by special features such as, cooperativity, self-organization, self-assembly and even entropy. Entropy. . . ? . . . yes. . . ! The evolution of life is an irreversible process.

Chemical foundation of life has been an interesting subject of study since long. Some of most abundant elements in living organisms include carbon, hydrogen, nitrogen, oxygen, along with sulphur and phosphorus. How can we forget water molecule, with its unique property of the electric dipole moment, which is very intimately involved in the processes of life?! Cells, the biological units of living organisms are made of many complex molecules (macromolecules) such as proteins, nucleic acids (RNA and DNA), carbohydrates, and lipids. Genes, made up of DNA contain the information necessary for living cells to survive and reproduce. Genetics is the study of heredity, which is a biological process whereby a parent passes certain genes onto their children or offspring. Life is a mysterious manifestation of nature, and every form of life employs both chemistry and physics in amazingly different ways. You would surely like to examine this statement with reference to our human body.

Evolution by natural selection makes Biology a dynamical discipline. Natural Selection is an optimization process via heritable variations of minor effects accumulating over a large time, resulting into large differences amongst the descendants of similar organisms. The small variations that arise from gene mutations are believed to be random. A mathematician would like to compare this with von Neumann algebra endowed with a 1-parameter modular auto-morphism group!

In the end of his lecture, mathematician SJB went beyond the boundary of basic sciences, and touched up on the *'No arbitrage principle: the BSM (Black-Scholes-Merton) theory of financial derivatives'*. From the realm of energy, momentum, quanta, atoms, molecules, entropy, chemistry and biology, he took us to the arena of stocks and share markets, dividends, profits and so on.

1.3 Summing up

In his lecture, the learned speaker SJB dwelt impressively upon the themes listed above in the introduction. The present article is a cursory glance of the mathematician's view-point implicit in these scientific ideas. The loud and clear message is, *'Mathematics is everywhere'*.

□ □ □

2. Singular value decomposition of real matrices

J. K. Verma
Department of Mathematics, IIT Bombay
jkv@iitb.ac.in

2.1 Introduction

The objective of this expository article is to present basic facts about the singular value decomposition (SVD) of real matrices accessible to students. The SVD is not normally included in a first course or introductory textbooks on linear algebra. The excellent books on linear algebra by Gilbert Strang changed this. See for example [6]. It is an accessible topic with numerous applications in data science.

Let A be an $m \times n$ real matrix. The **singular value decomposition** (SVD) of A is a factorisation of A as $A = U\Sigma V^t$ where U is an $m \times m$ orthogonal matrix, V is an $n \times n$ orthogonal matrix and Σ is an $m \times n$ diagonal matrix whose diagonal entries are non-negative. These are called **singular values** of A.

These notes are based on lectures delivered in various teachers' training programmes of the National Centre for Mathematics. The material is freely borrowed from the textbooks and papers listed in the references. No claim of originality is made.

The singular values of matrices arose in several investigations starting in the 19th century. The theory was developed for a variety of applications. We shall present a very brief sketch of the historical developments in Section 2.2. The preliminaries about orthogonal matrices are covered in Section 2.3 and positive matrices are discussed in Section 2.4.

In Section 2.5, we shall show that every real matrix has a singular value decomposition. In Section 2.6, we shall show that SVD can be used to construct best approximation to solutions of $Ax = b$ when b is not in the column space of A. We shall also see that the singular values can be used to find matrices of rank smaller than rank A which are best approximations to A. This leads to an application of SVD in image processing and data compression.

2.2 A brief history of SVD of matrices

A detailed history of the SVD is beautifully described in [5]. We refer the reader to an extensive bibliography in [5] and [4] about important works on SVD and its diverse applications.

We present a brief history. Eugenio Beltrami (1835-1899) found the SVD for invertible matrices for a simplification of bilinear forms in 1873. He showed that the eigenvalues of real symmetric matrices are real and those of AA^t for an invertible matrix A are positive. He also found an algorithm to construct an orthonormal basis of eigenvectors for real symmetric matrices.

Camille Jordan (1838-1922) found the SVD independent of Beltrami a year later. He showed that real quadratic forms can be diagonalised by means of orthogonal transformations. He obtained geometric interpretation of the largest singular value while investigating the maximum and minimum values of the bilinear form x^tAy when x and y are unit vectors.

James Joseph Sylvester (1814-1879) wrote two papers on the SVD in 1889. He found algorithms to diagonalise quadratic and bilinear forms by means of orthogonal substitutions.

Erhard Schmidt (1876-1959) was a student of David Hilbert. He is known for the Gram-Schmidt process used for construction of an orthonormal basis from a given basis. He discovered the SVD for function spaces while investigating integral equations. His aim was to find good low rank approximations to integral operators. His problem was to find the best rank k approximations to A of the form $u_1v_1^t + \cdots + u_kv_k^t$.

Hermann Weyl (1885-1955) discovered in 1912 a more elegant approach to find approximations to A using its singular values. Suppose that A has rank k and $\tilde{A} = A + E$ where E represents the error. He showed that the last $n-k$ singular values of $\tilde{A}$ satisfy

$$\tilde{\sigma}_{k+1}^2 + \cdots + \tilde{\sigma}_n^2 \leq ||E||^2.$$

L. Autonne found the SVD for complex matrices in 1913. Eckhart and Young extended SVD to rectangular matrices in 1936. Golub and Kahan introduced SVD in numerical analysis in 1965 and Golub proposed an algorithm for SVD in 1970 which is most widely used.

2.3 Orthogonal matrices and their canonical forms

In this section we review basic facts about orthogonal matrices and their canonical form. For any two column vectors $u, v \in \mathbb{R}^n$, we write their inner product as $\langle u, v\rangle = u^tv$ where u^t denotes the transpose of the column vector u. We say that u and v are perpendicular or orthogonal if $\langle u, v\rangle = 0$. The length of u is defined as $||u|| = \sqrt{\langle u, u\rangle}$. Suppose that $q_1, q_2, \ldots, q_n$ are mutually orthogonal unit vectors in $\mathbb{R}^n$. Such a set of vectors is called an **orthonormal basis** of $\mathbb{R}^n$. Let $Q = [q_1 \;\; q_2 \;\; \cdots \;\; q_n]$ be the matrix whose column vectors are $q_1, \ldots, q_n$. Then $Q^tQ = I$.

Definition 1. *A real $n \times n$ matrix Q is called orthogonal if $Q^tQ = I$.*

Remark 1. *1. If Q is orthogonal then $Q^t = Q^{-1}$. Therefore $QQ^t = I$. Hence the row vectors of Q also form an orthonormal basis of $\mathbb{R}^n$. Therefore the inverse of an orthogonal matrix is again orthogonal.*

2. *Product of orthogonal matrices is again orthogonal. Hence $n \times n$ orthogonal matrices form a group under multiplication. It is denoted by $O(n)$.*

3. *Let $\det A$ denote the determinant of the square matrix A. If Q is orthogonal then $\det Q = \pm 1$. The map $f : O(n) \to \{1, -1\}$ given by $f(Q) = \det Q$ is a group homomorphism and $\ker f = \{Q \in O(n) \mid \det Q = 1\}$. This subgroup, called the special orthogonal group of order n, is denoted by $SO(n)$, and it is clearly a normal subgroup of $O(n)$. Leonard Euler proved that rotations of the 3-space are induced by the linear maps corresponding to matrices in $SO(3)$.*

Example 2.3.1. *A 2×2 orthogonal matrix has two possibilities:*

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \text{ or } \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}.$$

The first matrix represents rotation of the plane by an angle of θ in anticlockwise direction and the second matrix represents a reflection of vectors in $\mathbb{R}^2$ with respect to the line $y = \tan(\theta/2)x$.

Definition 2. *A hyperplane in $\mathbb{R}^n$ is a subspace of dimension $n-1$. A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is called a reflection with respect to a hyperplane H if $Tu = -u$ where $u \in H^\perp$ and $Tu = u$ for all $u \in H$. For any subspace V of $\mathbb{R}^n$, define:*

$$V^\perp = \{u \in \mathbb{R}^n \mid \langle u, v\rangle = 0 \quad \forall\, v \in V\}$$

Proposition 1. *For any subspace V of $\mathbb{R}^n$, we have $\mathbb{R}^n = V \oplus V^\perp$.*

Proof. Clearly $V + V^\perp \subset \mathbb{R}^n$. Let $\{v_1, v_2, \ldots, v_m\}$ be an orthonormal basis of V. Define $T : \mathbb{R}^n \longrightarrow V$ by:

$$Tu = \sum_{i=1}^{m} \langle u, v_i\rangle v_i.$$

We write $T(u) = P_V(u)$ and call $T(u)$ as the projection of u in V. The set $\{v_1, v_2, \ldots, v_m\}$ can be extended to an orthonormal basis $\{v_1, v_2, \ldots, v_m, \ldots, v_n\}$ of $\mathbb{R}^n$. Note that T is clearly onto and

$$\ker(T) = \{u \in \mathbb{R}^n \mid \langle u, v_i\rangle = 0 \quad \forall\, i = 1, 2, \ldots, m\} = V^\perp.$$

Hence by the rank-nullity theorem, $n = \dim V + \dim V^\perp$. Note that $u = Tu + (u - Tu)$. Note that $u - Tu \in V^\perp$. Indeed,

$$\begin{aligned} \langle u - Tu, Tu\rangle &= \langle u, Tu\rangle - \langle Tu, Tu\rangle \\ &= \left(\sum_{i=1}^{n} \langle u, v_i\rangle v_i\right) \cdot \left(\sum_{i=1}^{m} \langle u, v_i\rangle v_i\right) - \sum_{i=1}^{m} \langle u, v_i\rangle^2 = 0 \end{aligned}$$

Hence $(u - Tu) \perp Tu$. Hence $\mathbb{R}^n \subset V + V^\perp$. Therefore $\mathbb{R}^n = V \oplus V^\perp$. □

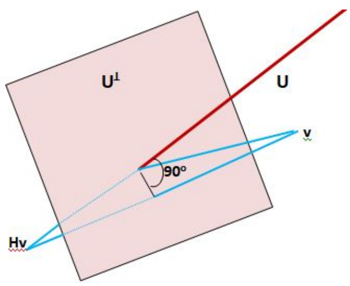

Figure 1: Reflection in $\mathbb{R}^3$

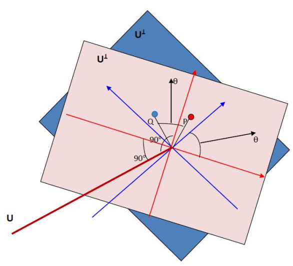

Figure 2: Rotation in $\mathbb{R}^3$

Let $L(u)$ denote the linear span of u. Then by proposition 1, $\mathbb{R}^n = L(u) \oplus L(u)^\perp$. Hence $L(u)^\perp$ is the $(n-1)$- dimensional subspace of $\mathbb{R}^n$ which is perpendicular to $L(u)$.

Example 2.3.2. *Let u be a unit vector in $\mathbb{R}^n$. Consider the* **Householder matrix** *of u, defined as*

$$H = I - 2uu^t.$$

Then $Hu = u - 2u(u^t u) = -u$. If $w \perp u$ then $Hw = w - 2uu^t w = w$. Thus $H = I - 2uu^t$ induces a reflection with respect to the hyperplane $L(u)^\perp$. Moreover H is an orthogonal symmetric matrix with $H^2 = I$. Indeed

$$H^t H = \left(I - 2uu^t\right)\left(I - 2uu^t\right) = I - 2uu^t - 2uu^t + 4uu^t uu^t = I.$$

We recall the concept of an inner product and an inner product space. Let $\mathbb{F}$ denote either $\mathbb{R}$ or $\mathbb{C}$.

Definition 3. *Let V be a vector space over $\mathbb{F}$. An inner product on V is a rule which to any ordered pair (u, v) of elements V associates a scalar, denoted by $\langle u, v\rangle$ satisfying the following axioms. For all u, v, w in V and $c \in \mathbb{F}$,*

(1) $\langle u, v\rangle = \overline{\langle v, u\rangle}$ ***(conjugate symmetry)***

(2) $\langle u, v + w\rangle = \langle u, v\rangle + \langle u, w\rangle$ ***(additivity)***

(3) $\langle u, cv\rangle = c\langle u, v\rangle$ ***(homogeneity)***

(4) $\langle v, v\rangle \geq 0$ *with* $\langle v, v\rangle = 0 \iff v = 0$ ***(positive definite)****.*

An ***inner product space*** *is a vector space with an inner product.*

Example 2.3.3. *(1) Let $v = (x_1, \ldots, x_n)^t, w = (y_1, \ldots, y_n)^t \in \mathbb{R}^n$. The* **standard inner product** *on $\mathbb{R}^n$ is defined as*

$$\langle v, w\rangle = v^t w = \sum_{i=1}^{n} x_i y_i.$$

(2) Let $v = (x_1, \ldots, x_n)^t$ and $w = (y_1, \ldots, y_n)^t \in \mathbb{C}^n$. The **standard inner product** *on $\mathbb{C}^n$ is defined as*

$$\langle v, w\rangle = v^* w = \sum_{i=1}^{n} \overline{x_i} y_i.$$

Definition 4. *Let V be an inner product space. We say that $u, v \in V$ are orthogonal if $\langle u, v\rangle = 0$. The norm of a vector $u \in V$ is defined as $||u|| = \sqrt{\langle u, u\rangle}$. If $||u|| = 1$ then u is called a unit vector. A basis of V consisting of unit vectors which are mutually orthogonal is called an orthonormal basis of V.*

Definition 5. *(**Orthogonal Transformation**). Let V be a vector space with an inner product. A linear transformation $T : V \to V$ is called orthogonal if $\|Tu\| = \|u\|$ for all $u \in V$.*

Theorem 1. *Let V be a finite dimensional inner product space and $T : V \to V$ be a linear transformation. Then the following are equivalent:*

(1) T is an orthogonal transformation.

(2) $(Tu, Tv) = (u, v)$ for all $u, v \in V$ where (u, v) represents the inner product of u and v.

(3) If $\{u_1, u_2, \ldots, u_n\}$ is an orthonormal basis of V then $\{Tu_1, Tu_2, \ldots, Tu_n\}$ is an orthonormal basis of V.

(4) The matrix A of T with respect to any orthonormal basis of V is orthogonal.

Proof. (1)⇒(2). Let T be an orthogonal transformation. Since $\|T(u+v)\| = \|u+v\|$ for all $u, v \in V$, we have

$$\begin{aligned}(Tu+Tv, Tu+Tv) &= (Tu,Tu) + 2(Tu,Tv) + (Tv,Tv) \\ &= (u,u) + 2(u,v) + (v,v).\end{aligned}$$

Hence $(Tu, Tv) = (u, v)$.
(2)⇒(3). Let $\{u_1, u_2, \ldots, u_n\}$ be an orthonormal basis of V. Then $\{Tu_1, Tu_2, \ldots, Tu_n\}$ is also an orthonormal basis of V since T preserves inner product.
(3)⇒(1). Suppose for some orthonormal basis $\{u_1, u_2, \ldots, u_n\}$, $\{Tu_1, Tu_2, \ldots, Tu_n\}$ is also an orthonormal basis. Let $u = x_1u_1 + \cdots + x_nu_n$ for some $x_1, \ldots, x_n \in \mathbb{R}$. Then

$$\begin{aligned}Tu &= x_1Tu_1 + \cdots + x_nTu_n, \\ (Tu, Tu) &= x_1^2 + \cdots + x_n^2 = \|u\|^2.\end{aligned}$$

(3)⇒(4). Let $B = \{u_1, \ldots, u_n\}$ be an orthonormal basis of V and $Tu_j = \sum_{i=1}^n a_{ij}u_i$ for $j = 1, 2, \ldots, n$. We show that $A = (a_{ij})$ is an orthogonal matrix. Since $Tu_1, \ldots, Tu_n$ is an orthonormal basis, we get

$$1 = (Tu_j, Tu_j) = \left(\sum_{i=1}^n a_{ij}u_i,\ \sum_{i=1}^n a_{ij}u_i\right) = \sum a_{ij}^2.$$

For $j \neq k$, we have

$$(Tu_j, Tu_k) = \left(\sum_{i=1}^n a_{ij}u_i,\ \sum_{i=1}^n a_{ik}u_i\right) = \sum_{i=1}^n a_{ij}a_{ik} = 0.$$

(4)⇒(3). If A is orthogonal then the above equations show that T maps an orthonormal basis to an orthonormal basis. □

2.3.1 Canonical form of orthogonal matrices

Recall that if A is an $n \times n$ matrix with entries in a field $\mathbb{F}$ then a nonzero vector $v \in \mathbb{F}^n$ is called an eigenvector of A if there exists $\lambda \in \mathbb{F}$ such that $Av = \lambda v$. The eigenspace of A for the eigenvalue λ is defined as $E(\lambda) = \{v \in \mathbb{F}^n \mid Av = \lambda v\}$. Let U be an $n \times n$ orthogonal matrix and $T = U + U^t$. Then T is an $n \times n$ real symmetric matrix. Thus T has real eigenvalues, say, $\lambda_1, \lambda_1, \ldots, \lambda_n$. Let the distinct ones among them be $\lambda_1, \lambda_1, \ldots, \lambda_k$. Let $E(\lambda)$ denote the eigenspace of λ. Then

$$V = E(\lambda_1) \oplus E(\lambda_2) \oplus \cdots \oplus E(\lambda_k).$$

Moreover these eigenspaces are mutually orthogonal.

Proposition 2. *(1) The eigenspaces of $T = U + U^t$ are invariant under U.*

(2) If $0 \neq v \in E(\lambda)$ then $(U^2 - \lambda U + I)v = 0$ and the determinant of $T_U : E(\lambda) \to E(\lambda)$ is 1.

(3) If $\lambda = 2$ then $T_U : E(\lambda) \to E(\lambda)$ is given by $T_U(v) = v$. If $\lambda = -2$ then $T_U(v) = -v$ for all $v \in E(\lambda)$.

(4) If $\lambda \neq \pm 2$, then $W = L(v, Uv)$ is a two dimensional T_U-invariant subspace of $E(\lambda)$.

Proof. (1) Let $Tv = \lambda v$ where λ is an eigenvalue and v is an eignevector. Then

$$T(U(v)) = (U + U^t)U(v) = U^2(v) + v.$$

Since $Tv = (U + U^t)v = \lambda v$, we have $U^2v + v = \lambda Uv$. Hence $T(Uv) = \lambda Uv$. This implies that each eigen space of T is an invariant subspace of U.

(2) Let $v \in E(\lambda)$, then $Tv = (U + U^t)v = \lambda v$. Thus $(U^2 + I)v = \lambda v$. Hence $(U^2 - \lambda U + I)v = 0$. Therefore the characteristic polynomial of T_U is $x^2 - \lambda x + 1$. Hence the determinant of $T_U : E(\lambda) \to E(\lambda)$ is 1.

(3) If $\lambda = 2$, then $(U-I)^2 v = 0$. Thus 1 is the only eigenvalue of T_U. Hence $T_U = I$. Similarly when $\lambda = -2$, then $T_U = -I$.

(4) Suppose that $\lambda \neq \pm 2$. Then U has no eigenvectors in $E(\lambda)$. Thus for any nonzero vector $v \in E(\lambda)$, the vectors v, Uv are linearly independent. Let $W = L(v, Uv)$. As $(U^2 + I)v = \lambda v$, W is a U invariant subspace of $E(\lambda)$.

□

Recall that an $n \times n$ real matrix A is called orthogonally similar to an $n \times n$ real matrix B if there exists an orthogonal matrix U such that $B = U^t A U$.

Theorem 2. *Let I_p denote the $p \times p$ identity matrix. An $n \times n$ orthogonal matrix U is orthogonally similar to a matrix of the form*

$$\begin{bmatrix} I_r & & & & & & & \\ & -I_s & & & & & & \\ & & \ddots & \cos\theta_1 & -\sin\theta_1 & & & \\ & & & \sin\theta_1 & \cos\theta_1 & & & \\ & & & & & \ddots & & \\ & & & & & & \cos\theta_k & -\sin\theta_k \\ & & & & & & \sin\theta_k & \cos\theta_k \end{bmatrix}$$

Proof. We have observed above that T_U acting on each eigenspace $E(\lambda)$ of T, is either I or $-I$ or a two dimensional T_U -invariant subspace W splits out of $E(\lambda)$. The map T_U is an orthogonal map of W with determinant 1. Hence it is a rotation of W. Hence its matrix is the matrix of rotation by θ. If $E(\lambda) \neq W$, then we have another two dimensional subspace W_2 such that $E(\lambda) = W \oplus W_2 \oplus V$ where V is orthogonal to both W and W_1. Repeating this process, we see that $E(\lambda)$ is a direct sum of 2-dimensional invariant subspaces and on each one of them, T_U induces a rotation. □

2.4 Positive definite matrices

Definition 6. *A real symmetric matrix A is called positive definite (resp. positive semi-definite) if $x^t Ax > 0$ (resp. $x^t Ax \geq 0$) for every nonzero vector x.*

Theorem 3. *Let A be an $n \times n$ real symmetric matrix. Then A is positive definite if and only if each eigenvalue of A is positive.*

Proof. Let A be positive definite and x be an eigenvector with eigenvalue λ. Then $Ax = \lambda x$. Hence $x^t Ax = \lambda ||x||^2$. Thus $\lambda > 0$. Conversely, let each eigenvalue of A be positive. Suppose that $v_1, v_2, \ldots, v_n$ is an orthonormal basis of eigenvectors with positive eigenvalues $\lambda_1, \cdots, \lambda_n$. Then any nonzero vector x can be written as $x = a_1 v_1 + \cdots + a_n v_n$ where at least one $a_i \neq 0$. Then

$$x^t Ax = x^t(a_1\lambda_1 v_1 + \cdots + a_n \lambda_n v_n) = \sum_{i=1}^{n} \lambda_i a_i^2 > 0.$$

□

Remark 2. *It is clear that an analogous statement holds true for positive semi-definite matrices.*

We can test positive-definiteness without computing the eigenvalues. This is done by computing the determinants of sub-matrices A_k of A obtained by deleting the last $n-k$ column vectors and last $n-k$ row vectors for each $k = 1, 2 \ldots, n-1, n$.

Definition 7. *A* **principal submatrix** *of a square matrix is a square matrix formed by a set of rows and corresponding set of columns. A* **principal minor** *of A is the determinant of a principal submatrix.*

Proposition 3. *Let A be an $n \times n$ matrix. Let s_i be the sum of all $i \times i$ principal minors of A. Then for any λ*

$$\det(\lambda I - A) = \lambda^n - s_1\lambda^{n-1} + s_2\lambda^{n-2} + \cdots + (-1)^{n-1}s_{n-1}\lambda + (-1)^n s_n.$$

Proof. Let X be the diagonal matrix with the variable x in every diagonal entry. Then the characteristic polynomial of A is $\det(X - A)$.

Think of every column of $X - A$ as a sum of two columns: one the column of X and the other the corresponding column of $-A$. By multilinearity $\det(X - A)$ is the sum of the determinants of 2^n matrices, where each matrix is obtained by replacing a subset of columns of X by the corresponding columns of A. If the set is empty we get the term x^n.

If the set has one element, say column i, we get the term $-a_{ii}x^{n-1}$. Summing over i we get the coefficient of x^{n-1} to be -trace(A).

If the set has two elements, say column i and j, we get the term x^{n-2} (determinant of the principal minor of A with rows and columns i, j). Summing over all i, j we get the coefficient of x^{n-2}, and so on. □

Theorem 4. *Let A be an $n \times n$ real symmetric matrix. Then A is positive definite if and only if all principal minors of A are positive.*

Proof. If $\mathbf{x} = (x_1, x_2, \ldots, x_k, x_{k+1}, \ldots, x_n)^t$ then we write $\mathbf{x_k} = (x_1, x_2, \ldots, x_k)^t$. Let A be an $n \times n$ positive definite matrix. Let $\mathbf{x}$ be such that $\mathbf{x_k} \neq \mathbf{0}$. Then

$$\mathbf{x^t} A\mathbf{x} = \begin{bmatrix} \mathbf{x_k^t} & 0^t \end{bmatrix} \begin{bmatrix} A_k & B \\ B^t & C \end{bmatrix} \begin{bmatrix} \mathbf{x_k} \\ 0 \end{bmatrix} = \mathbf{x_k^t} A_k \mathbf{x_k} > 0.$$

Hence A_k is positive definite. Therefore each eigenvalue of A_k is positive. Hence $\det A_k > 0$. One can similarly prove that the other principal minors are positive.

Conversely let all the principal minors of A be positive. Then by the proposition 3, $f(x) = \det(xI - A)$ has exactly n sign changes in its coefficients. Let $f(x)$ have r positive roots. Note that $f(-x)$ has no variation in signs of its coefficients. Hence it has no positive roots by Descartés rule of signs. Hence $f(x)$ has no negative root. Thus all roots of $f(x)$ are positive. Therefore all eigenvalues of A are positive. Hence A is positive definite. □

Theorem 5. *A real symmetric $n \times n$ matrix A is positive definite if and only if $A = BB^t$ for some invertible real matrix B.*

Proof. Let $A = BB^t$ for an invertible real matrix B. Then for any nonzero $x \in \mathbb{R}^n$,

$$x^t Ax = x^t BB^t x = ||B^t x||^2 > 0.$$

Hence A is positive-definite. Conversely let A be positive-definite. Then there is an orthogonal matrix U and positive real numbers $d_1, d_2, \ldots, d_n$ so that $U^t AU = D := \operatorname{diag}(d_1, d_2, \ldots, d_n)$. Put $\sqrt{D} = \operatorname{diag}(\sqrt{d_1}, \ldots, \sqrt{d_n})$. Hence

$$A = U\sqrt{D}\sqrt{D}U^t = BB^t \text{ where } B = \sqrt{D}U^t.$$

□

2.5 The Singular Value Decomposition

Let A be an $m \times n$ real matrix. Suppose there are orthogonal matrices $U \in \mathbb{R}^{m\times m}, V \in \mathbb{R}^{n\times n}$ and a diagonal matrix $\Sigma \in \mathbb{R}^{m\times n}$ with nonnegative diagonal entries $\sigma_1, \sigma_2, \ldots,$ such that

$$A = U\Sigma V^t$$

then we say that it is a singular value decomposition of A. To understand how to find U and V, observe that

$$A^tA = V\Sigma^tU^tU\Sigma V^t = V\Sigma^t\Sigma V^t \text{ and } AA^t = U\Sigma V^tV\Sigma^tU^t = U\Sigma\Sigma^tU^t.$$

Hence the column vectors of U form an orthonormal basis of eigenvectors of AA^t and those of V form an orthonormal basis of eigenvectors of A^tA. These matrices are positive semidefinite and symmetric. Hence they are orthogonally diagonalisable and the eigenvalues are nonnegative. Let $V = [v_1, v_2, \dots, v_n]$ and $U = [u_1, u_2, \dots, u_m]$ where we have displayed the column vectors of U and V. Now consider the equation

$$AV = U\Sigma, \text{ equivalently } Av_1 = \sigma_1u_1, Av_2 = \sigma_2u_2, \dots, Av_n = \sigma_nu_n.$$

Therefore the eigenvectors $v_1, v_2, \dots, v_n$ of A^tA are mapped to scalar multiples of the eigenvectors of AA^t. Since the matrices A, AA^t, A^tA have equal rank, say r, we may assume that the nonzero **singular values of** A, are $\sigma_1, \sigma_2, \dots, \sigma_r$.

Theorem 6. *Let A be an $m \times n$ real matrix of rank r. Then there exist orthogonal matrices $U \in \mathbb{R}^{m\times m}, V \in \mathbb{R}^{n\times n}$ and a diagonal matrix $\Sigma \in \mathbb{R}^{m\times n}$ with nonnegative diagonal entries $\sigma_1, \sigma_2, \dots,$ such that*

$$A = U\Sigma V^t.$$

Proof. Since A^tA is symmetric and positive semi-definite, there exists an $n \times n$ orthogonal matrix V whose column vectors are the eigenvectors of A^tA with non-negative eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$. Hence $A^tAv_i = \lambda_iv_i$ for $i = 1, 2, \dots, n$. Let $r = \operatorname{rank} A$. Arrange the eigenvalues so that $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r > 0$ and $\lambda_j = 0$ for $j = r+1, r+2, \dots, n$. Set $\sigma_i = \sqrt{\lambda_i}$ for all $i = 1, 2, \dots, n$. Note that

$$v_i^tA^tAv_i = \lambda_iv_i^tv_i = \lambda_i \geq 0.$$

Then $||Av_i|| = \sigma_i$ for $i = 1, 2, \dots, n$. Set $Av_i/\sigma_i = u_i$. The set $u_1, u_2, \dots, u_r$ is an orthonormal basis of the column space $C(A)$. Indeed,

$$u_i^tu_j = \frac{(Av_i)^tAv_j}{\sigma_i\sigma_j} = \frac{v_i^tv_j\lambda_j}{\sigma_i\sigma_j}.$$

We can extend it to an orthonormal basis $\{u_1, u_2, \dots, u_m\}$ of $\mathbb{R}^m$. Since $Av_i = \sigma_iu_i$ for all i, we have the singular value decomposition $A = U\Sigma V^t$ where $\Sigma = (\sigma_{ij})$ is a diagonal matrix with $\Sigma_{ii} = \sigma_i$ for all i. □

Theorem 7. *Let A be an $m \times n$ real matrix. Then the largest singular value of A is given by*

$$\sigma_1 = \max\{||Ax|| : ||x|| = 1\}.$$

Proof. Let $v_1, \dots, v_n$ be an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors of A^tA with eigenvalues

$$\sigma_1^2 \geq \sigma_2^2 \geq \dots \geq \sigma_r^2 > 0 \geq \dots \geq 0.$$

Write $x \in S^{n-1}$ as $x = c_1v_1 + c_2v_2 + \dots + c_nv_n$ for some real numbers $c_1, \dots, c_n$. Hence

$$||Ax||^2 = x^tA^tAx = x.A^tAx = x.(c_1\sigma_1^2v_1 + c_2\sigma_2^2v_2 + \dots + c_r\sigma_r^2v_r) = c_1^2\sigma_1^2 + c_2^2\sigma_2^2 + \dots + c_r^2\sigma_r^2.$$

Hence $||Ax||^2 \leq \sigma_1^2(c_1^2 + c_2^2 + \dots + c_n^2) \leq \sigma_1^2$. The equality holds if $x = v_1$. □

Excercise 2.5.1. *Find similar interpretation of other singular values of A.*

2.6 Applications of SVD

There are numerous applications of SVD. We shall discuss five applications: (1) construction of bases of four fundamental subspaces associated to a matrix, (2) best approximation of a vector by a vector in the column space of a matrix which is accomplished by solving the normal equations, (3) approximation of a matrix by matrices of lower rank (4) data compression and image processing and finally (5) construction of the polar decomposition of a square matrix.

2.6.1 The four fundamental subspaces of a matrix

Let F be a field and $A \in F^{m\times n}$. Let $R_1, R_2, \ldots, R_m$ be the row vectors of A and $C_1, C_2, \ldots, C_n$ be its column vectors. The subspace

$$C(A) = \{Au = u_1C_1 + u_2C_2 + \cdots + u_nC_n \mid u = (u_1, u_2, \ldots, u_n)^t \in V = F^n\}$$

is called the column space of A. It is the image in $W = F^m$ of the linear map $T_A : V \to W,\ T_A(u) = Au$. The null space of A is the subspace

$$N(A) = \{u \in V \mid Au = 0\}.$$

The row space of A denoted by $R(A)$ is the subspace of V spanned by the row vectors of A. The fourth fundamental subspace of A is $N(A^t)$, the null space of A^t. It is easy to see that

$$R(A) \oplus N(A) = V \text{ and } C(A) \oplus N(A^t) = W.$$

Moreover $N(A) = R(A)^\perp$ and $N(A^t) = C(A)^\perp$. If $Ax = b$ where A is an $m \times n$ matrix and x is an $n \times 1$ column vector and b is an $m \times 1$ column vector then $x = x_n + x_r$ where $x_n \in N(A)$ and $x_r \in R(A)$. Hence $Ax = Ax_r = b$. This means that the solutions live in the row space of A.

Remark 3. *The orthogonal matrices U and V can be used to find bases of the four fundamental subspaces of A.*

1. *Let $\operatorname{rank} A = r$. Then the first r column vectors of U correspond to the nonzero eigenvalues $\sigma_1^2, \ldots, \sigma_r^2$ of AA^t and $Av_i = \sigma_i u_i$ for all $i = 1, 2, \ldots, r$. Hence $u_1, \ldots, u_r$ is an orthonormal basis of $C(A)$.*
2. *The last $m - r$ columns of U are eigenvectors of AA^t with eigenvalue 0. Therefore these column vectors form an orthonormal basis of the left nullspace $N(A^t)$ since $N(A^t) = C(A)^\perp$.*
3. *Let $v_1, \ldots, v_n$ be the column vectors of V. The first r columns of V give an orthonormal basis of the row-space of A.*
4. *The last $n - r$ column vectors of V form an orthonormal basis of the nullspace of A.*

2.6.2 Best approximation of a vector in a subspace via SVD

Suppose that U is a subspace of a finite-dimensional real inner product space V and $w \in V$. We want to find a vector in $u \in U$ which is the best approximation to w in the sense that $||w - u||$ is the smallest. We show that u is the projection of w onto U. We know that $V = U \oplus U^\perp$. Hence $w = u + (w - u)$ where $u \in U$ is uniquely determined. We say that u is the projection of w and write this as $u = P_U(w)$.

Theorem 8. *For any $w \in V$ and $u \in U$,*

$$||w - P_U(w)|| \le ||w - u||$$

and equality holds if and only if $u = P_U(w)$.

Proof. Since $w = P_U(w) + (w - P_U(w))$, $P_U(w) \perp w - P_U(w)$. Therefore by Pythagoras' Theorem,

$$||w - P_U(w)||^2 \leq ||w - P_U(w)||^2 + ||P_U(w) - u||^2 = ||w - u||^2.$$

It is clear that equality holds if and only if $P_U(w) = u$. $\square$

Consider a system of linear equations $Ax = b$ where A is an $m \times n$ real matrix of rank r, x is an unknown vector and $b \in \mathbb{R}^m$. If $b \in C(A)$ then using Gauss elimination, we can find the exact solution. If $b \notin C(A)$, we try to find the vector x so that $||b - Ax||$ is smallest possible. If we take the projection of b in $C(A)$, we have the best approximation Ax to b. This means that $Ax - b$ is in the perpendicular subspace $C(A)^\perp$. Thus $A^t(Ax - b) = 0$. Thus for Ax to be the best approximation to b we need to solve the **normal equations:**

$$A^t Ax = A^t b.$$

The SVD of A helps us to solve the normal equations. Let $A = U\Sigma V^t$ be an SVD for A. Then

$$Ax - b = U\Sigma V^t x - b = U\Sigma V^t x - UU^t b = U(\Sigma V^t x - U^t b).$$

Set $y = V^t x, c = U^t b$. As U is orthogonal $||Ax - b|| = ||\Sigma y - c||$. Let $y = (y_1, y_2, \ldots, y_m)^t$, $c = U^t\, b = (c_1, c_2, \ldots, c_m)^t$ and $\sum = \operatorname{diag}(\sigma_1, \ldots, \sigma_r, 0, \ldots, 0)$. Then

$$\Sigma y - c = (\sigma_1 y_1 - c_1, \sigma_2 y_2 - c_2, \ldots \sigma_r y_r - c_r, -c_{r+1}, \ldots, -c_m)^{\cdot}$$

It is now clear that we have the best approximation if and only if $\sigma_i y_i = c_i$ for all $i = 1, 2, \ldots, r$.

2.6.3 Approximation of a matrix by lower rank matrices

Distance between matrices is measured using matrix norms. A matrix norm on the space $V = \mathbb{R}^{m\times n}$ is a function $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ which satisfies the following conditions for all $A, B \in V$ and $r \in R$,

1. $f(A) \geq 0$ and $f(A) = 0$ if and only if $A = 0$.
2. $f(A + B) \leq f(A) + f(B)$
3. $f(rA) = |r|f(A)$.

Matrix norms are constructed using vector norms. If $v = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$ then the p norm of v is defined as $||v||_p = \sqrt[p]{|v_1|^p + \cdots + |v_n|^p}$. The infinity norm is defined as $||v||_\infty = \max\{|v_1|, |v_2|, \ldots, |v_n|\}$.

Example 2.6.1. *1. The Frobenius norm of $A \in \mathbb{R}^{m\times n}$ is defined as $||A||_F = \sqrt{\sum_{i,j} |a_{ij}|^2}$. One can show that $||A||_F = \sqrt{Tr(AA^t)} = \sqrt{\sigma_1^2 + \cdots + \sigma_r^2}$.*

2. Let p be a positive integer. Then the p-norm of A is defined as

$$||A||_p = \sup_{x\neq 0} \frac{||Ax||_p}{||x||_p}.$$

We denote the 2-norm of A simply by $||A||$. The 2-norm of A is also called the Euclidean norm of A.

Theorem 9. *(Eckhart-Young, 1936) Let $A \in \mathbb{R}^{m\times n}$ and $\operatorname{rank}(A) = r$. Let $A = U\Sigma V^t$ be a singular value decomposition of A with singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$. Let $A_k = \sum_{i=1}^k \sigma_i u_i v_i^t$. Then*

$$\min_{\operatorname{rank}(B)=k} ||A - B|| = ||A - A_k|| = \sigma_{k+1}.$$

Proof. Since $A_k = U\,\mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_k, 0, \ldots, 0)V^t$, $\mathrm{rank}(A_k) = k$. Note that

$$U^t AV - U^t A_k V = \mathrm{diag}(0, \ldots, 0, \sigma_{k+1}, \ldots, \sigma_r, 0, \ldots, 0).$$

Hence

$$||A - A_k|| = ||U^t(A - A_k)V|| = \sigma_{k+1}.$$

Let $B \in \mathbb{R}^{m\times n}$ be a rank k matrix. Since $\dim N(B) = n - k$, we can choose an orthonormal basis $\{x_1, x_2, \ldots, x_{n-k}\}$ of $N(B)$. The subspace W spanned by $v_1, v_2, \ldots, v_{k+1}$ has a nonzero intersection with $N(A)$. Let z be a unit vector in $W \cap N(B)$. Then $Bz = 0$ and

$$Az = \sum_{i=1}^{r} \sigma_i u_i v_i^t z = \sum_{i=1}^{k+1} \sigma_i (v_i^z) u_i.$$

Hence

$$||A - B||^2 \geq ||Az - Bz||^2 = ||Az||^2 = \sum_{i=1}^{k+1} \sigma_i^2 (v_i^t z)^2 \geq \sigma_{k+1}^2.$$

Thus A_k is closest to A among rank k matrices. □

2.6.4 Use of SVD in image processing

Suppose that a picture consists of 1000×1000 array of pixels. This can be thought of a 1000×1000 matrix A of numbers which represent colors. Note that A has a million entries. Suppose we have an SVD of A. Then $A = U\Sigma V^t$ can be written as a sum of rank one matrices:

$$A = \sigma_1 u_1 v_1^t + \sigma_2 u_2 v_2^t + \cdots + \sigma_r u_r v_r^t.$$

Suppose that we take 20 singular values. Then we send $20 \times 2001 = 40002$ numbers rather than a million numbers. This represents a compression of 25 : 1.

2.6.5 Polar decomposition of matrices

Corollary 10. *Let $A \in \mathbb{R}^{n\times n}$. Then there exists an orthogonal matrix W and a positive semi-definite matrix P such that $A = WP$.*

Proof. Let $A = U\Sigma V^t$ be a singular value decomposition of A. Here all the factors are $n \times n$ matrices. Then $A = UV^t(V\Sigma V^t)$. The matrix UV^t is orthogonal. Since the diagonal entries of Σ are nonnegative, the matrix $V\Sigma V^t$ is a positive semi-definite matrix. □

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Bibliography

[1] Michael Artin, Algebra, Second Edition, Prentice Hall of India, 2011.

[2] Sheldon Axler, Linear Algebra Done Right, Second Edition, Springer, 1997.

[3] Charles Curtis, Linear Algebra, an Introductory Approach, Springer, 1984.

[4] Carla D. Martin and Mason A. Porter, The extraordinary SVD, Amer. Math. Monthly, **119**(2012), 838-851.

[5] G. W. Stewart, On the early history of the singular value decomposition, SIAM Review, **35** (1993), 551-566.

[6] Gilbert Strang, Linear Algebra and its Applications, Fifth Edition, Wellesley-Cambridge Press, 2016.

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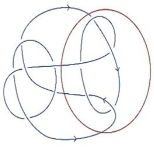
3. What is happening in the Mathematical world?

Devbhadra V. Shah
Department of Mathematics, Veer Narmad South Gujarat University, Surat;
Email: drdvshah@yahoo.com

3.1 Decades-Old Conway Knot Problem That Had Been Driving Scientists Crazy Has Been Solved

The Conway knot, a major knot theory problem, had been unsolved for 50 years. But *Lisa Piccirillo* of MIT, where she is now an assistant professor, solved it in less than a week.

Half a century ago, a brilliant young mathematician named John Horton Conway discovered a knot which was not something one would encounter in the real world. It could be certainly created out of a string, but, generally speaking, it existed only in Conway's calculations. There are of course these kind of conceptual knots in a bewildering alley of mathematics known as knot theory. The Conway knot is hardly remarkable at first glance. With just 11 crossings, or places where it overlaps itself, it is rather non descript by the standards of higher-dimensional knot theory. But the knot has one property that made it the subject of intense mathematical analysis. Conway, who died in April this year at the age of 82 due to complications from COVID-19, noted the property but was unable to find a proof for it, and it became known as the Conway knot problem.



Convey Knot

INIMS, Cambridge University, gate

The problem had to do with determining whether the Conway knot (see above figure) was something called "slice", an important concept in knot theory. Of all the many thousands of knots with 12 or fewer crossings, mathematicians had been able to determine the sliceness of all but one: the Conway knot. For more than 50 years, the knot stubbornly resisted every attempt to unravel its secret, along the way achieving a kind of mythical status. A sculpture of it even decorates the gate at the Isaac Newton Institute for Mathematical Sciences at the prestigious University of Cambridge as shown in the above figure.

Now let us take one of these knots and think for a moment about the space in which it exists. That space has a fourth dimension, such as time, and to a topologist, our knot is a kind of sphere that sits within it! Topologists see spheres everywhere, but in a specialized way: A circle is a one-dimensional sphere, while the skin surrounding an orange is a two-dimensional sphere. Just as the spherical peel of an orange can be viewed as the join of two hemispherical parts with the circular (one dimensional sphere) boundaries glued together, topologists identify two oranges glued together along their entire spherical boundaries (only conceptually!) as a three-dimensional sphere, and can think of it as the skin of a four-dimensional orange !!

Two years ago, a little-known graduate student named Lisa Piccirillo, who graduated from Boston College in 2013, learnt about the knot problem while attending a conference in 2018. A speaker mentioned the Conway knot during a discussion about the challenges of studying knot theory. The speaker showed a slide depicting the Conway knot and explained that mathematicians had long suspected that the knot was not, in fact, slice, but no one had been able to prove it. So what does it mean for a knot to be slice? Let's return for a moment to that four-dimensional orange. Inside of it there are disks - think of them as the surface of a plate. In knot theory, a three-dimensional knot, like Conway's, is an embedded circle in a 3-sphere which can be thought of as the boundary of a 4-dimensional ball. A knot is a slice if it bounds a nicely embedded 2-dimensional disk in the 4-dimensional ball. If it cannot, then it is not a slice.

Topologists use mathematical tools called invariants to try to determine sliceness, but for half a century, those tools had been unable to help them prove the prevailing belief that the Conway knot was not a slice. Sitting in that lecture hall two years ago, however, Piccirillo had sensed right away that the techniques she was using in a different area of topology might help these invariants better apply to the Conway knot problem. She started on the problem the very next day. In order to solve this problem, she relied on the fact that certain knots have "mutant siblings", whose crossings are reversed like a kind of mirror effect. However, they have the same nature: if one is a "slice", then the other has to be as well. Therefore, she redrew the knot in a method called making its trace because if this sibling knot was proved to be a slice, the Conway knot would be as well.

And as it finally turned out, this knot was not a slice. In less than a week, Piccirillo had created a knot that hit the sweet spot: It had the same 4-D properties as the Conway knot, and it was found by an algorithm to be not a slice. She had suddenly succeeded where countless mathematicians had failed for five decades. The adjoining picture depicts Piccirillo's version of the knot she created to solve the Conway knot problem. When J. H. Conway passed away last April, he had at least the consolation that one of the greatest enigmas of the knot theory he had modeled in 1970 was no longer going to be so.

Sources:

1. *https://www.bostonglobe.com/2020/08/20/magazine/ math-problem-stumped- experts- 50 - years -this-grad-student-maine-solved-it-days/*
2. *https://aldianews.com/articles/culture/ social/student-unravels-one-most- complex- knots - mathematics /58807*
3. *https://www.gentside.co.uk/science/a-mathematical-problem-that-had-been-driving-scientists-crazy-for-decades-has-finally-been-solved_art5596.html*

3.2 Landmark Proof Clears Hurdle In The First Section Of The Famous Erdös Conjecture

Mathematicians Thomas Bloom of the University of Cambridge and Olof Sisask of Stockholm University have solved the first portion of one of the most famous conjectures about the additive properties of whole numbers. Proposed more than 60 years ago by the legendary Hungarian mathematician Paul Erdös, the conjecture asks when an infinite list of whole numbers will be sure to contain patterns of at least three evenly spaced numbers i. e. numbers in an arithmetic progression. In the 1930s, Johannes van der Corput used the special structure of the primes to show that they do indeed contain infinitely many evenly spaced triples (such as 17, 23 and 29).

Erdös posed thousands of problems over the course of his career, but the question of which number lists contain evenly spaced numbers was one of his all-time favorites. The Erdös-Turan conjecture in additive combinatorics is one of the longest lasting unsolved problems.

What is the conjecture?

As a rule, a denser list of numbers has a higher chance of containing arithmetic progressions than a sparser list, so Erdös proposed a simple density test: Just add up the reciprocals of the numbers on your list. If your numbers are plentiful enough to make this sum infinite, Erdös conjectured that your list should contain infinitely many arithmetic progressions of every finite length - triples, quadruples and so forth.

Even though countless mathematicians have tried to solve this conjecture, Bloom and Sisask's method is different so far, and does not require a strong knowledge of prime numbers' unique structure in order to prove that they contain an infinitely many triples. All you need to know is that prime numbers are abundant enough for the sum of their reciprocals to be infinite - a fact which mathematicians have known for centuries.

It is easy to make an infinite list with no arithmetic progressions if you make the list sparse enough. For example, consider the sequence 1, 10, 100, 1000, 10000, ...(whose reciprocals sum to the finite decimal 1.11111...). These numbers spread apart so rapidly that you can never find three numbers that are evenly spaced. One may wonder, though, if there are significantly denser number sets that still avoid arithmetic progressions. You could, for example, walk down the number line and keep every number that does not complete an arithmetic progression. This creates the sequence 1, 2, 4, 5, 10, 11, 13, 14, ..., which looks pretty dense at first. But it becomes incredibly sparse as you move into higher numbers - for instance, by the time you get to 20-digit numbers, only about 0.000009% of the whole numbers up to that point are on your list. In 1946, Felix Behrend came up with denser examples, but even these become sparse very quickly - a Behrend set that goes up to 20-digit numbers contains about 0.001% of the whole numbers.

At the other extreme, if your set includes almost all the whole numbers, it will definitely contain arithmetic progressions. However, between these extremes is a vast, largely unknown interior. How sparse can you make your set, mathematicians have wondered, and still be sure that it will have arithmetic progressions?

Erdös provided one possible answer. His condition about the sum of reciprocals is a statement about density in disguise: It turns out to be the same as saying that the density of your list up to any number N is at least approximately 1 over the number of digits in N. So, up through 20-digit numbers it should have density at least about 1/20. Provided this density condition is met, Erdös conjectured, your list should contain infinitely many arithmetic progressions of every length.

In 1953, Klaus Roth started mathematicians on a path toward proving Erdös' conjecture. In the work that helped earn him a Fields Medal five years later, he established a density function that guarantees evenly spaced triples - not a density as low as Erdös', but nevertheless one that approaches zero as you go out along the number line. Roth's theorem meant that even a list of numbers whose density drops below 1%, and then below 0.1%, and then below 0.01%, and so on, at some times, must still contain arithmetic progressions as long as there are arbitrarily large times, in between, when it is above a positive threshold.

Then in 2011, Katz and Michael Bateman figured out how to overcome this barrier in a simpler setting: the card game set, in which you search for matching triples of patterned cards. There is a precise way in which a matching set triple can be thought of as an arithmetic progression, and just as with lists of whole numbers, you can ask what fraction of the cards you must lay down to be sure of finding at least one triple.

Fortunately, Bloom and Sisask had already started thinking about the Erdös conjecture, separately at first, both captivated by the beauty of the techniques involved. Bloom and Sisask joined forces in 2014, and by 2016 they thought they had pushed through to a solution. The pair kept going, diving into the inner workings of Bateman and Katz's method and eventually figuring out what new ideas would allow them to transfer it over from the world of a finite set to the setting of all whole numbers.

Now, in a paper posted online, Bloom and Sisask have proved the conjecture when it comes to evenly spaced triples. The pair has shown that whenever a number list's sum of reciprocals is infinite, it must contain infinitely many evenly spaced triples. The two mathematicians used density to prove that a certain set must behave a certain way. In a 70 pages long new paper they have uploaded to arXiv and submitted to journals, two mathematicians say they have untangled the first part of Paul Erdös's famously thorny and unproven conjecture. The paper is currently being peer-reviewed and has been pre-published in arXiv. It will take time for mathematicians to check it carefully, but many feel optimistic that it is correct. If this is true, the next generation of researchers could start from that point with the first part finished and in hand.

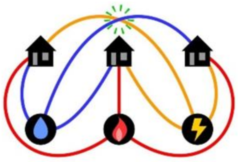
It is an exciting time for mathematicians, however, there is still a fair amount of work yet to be done before the full Erdös conjecture is proven, as this was only the first part of it. Bloom and Sisask have only proved the conjecture for evenly spaced triples, not for longer arithmetic progressions, a task that currently seems out of reach.

Sources:

1. *https://news.yahoo.com/mathematicians-solve-part-weirdest-open-151900979.html*
2. *https://interestingengineering.com/ mathematicians- solve-first- section-of- the- famous-erdos - conjecture*
3. *https://www.quantamagazine.org/ landmark- math-proof-clears-hurdle-in-top -erdos - conjecture - 20200803/*

3.3 Maths Riddle From The 1980's Finally Solved - It May Serve To Improve Phones And Computers

The two computer scientists, Assistant Professor *Jacob Holm* of University of Copenhagen and Associate Professor *Eva Rotenberg* of the Technical University of Denmark have solved a mathematical riddle from *Graph theory, outstanding since 1980's, whose resolution could be used in improving tomorrow's phones and computers.* Interestingly, it turned out that their earlier paper, submitted in the summer of 2019, which they thought was a minor result and believed that the full resolution of the riddle was still five years away, actually contained much of the solution, which they could spruce up to complete the task.



In 1913, a precursor to the now solved mathematical challenge was published in "The Strand Magazine" as "The Three Utilities Problem". It caused the magazine's readers to scratch their heads and ponder. In the problem, each of three cottages must have water, gas, and electricity, while the "lines" between the houses and water, electricity and gas may not cross each other - which is not possible. Simply put, the puzzle is about how to connect a number of points in a planar graph without allowing the lines connecting them to cross. And how, with a mathematical calculation - an algorithm - you can make changes to an extensive "graph network" to ensure that no lines intersect without having to start all over again. The property that can be used for, among other things, building immense road networks or the tiny innards of computers, where electrical circuitry on circuit boards may not cross.

Jacob Holm has been interested in the mathematical puzzle since 1998, but the answer was only revealed while the two researchers were reading through their already submitted research article. In the meantime, the researchers heard about a novel mathematical technique that they realized could be applied to the problem. This is when the two researchers got busy writing the research paper and tying up loose ends to solve the puzzle that Holm had been working on occasionally since 1998. They worked on the article non-stop, for five to six weeks and ended up filling more than 80 pages.

So, what can the solution to this mathematical puzzle be used for? When drawing wires on a circuit board, they must never intersect. Otherwise, short circuits will occur. The same applies to microchips, which contain millions of transistors and for which one must have a graph drawing. Thus the design of microchips and circuit boards, found in all electronics, could be an area where the results can be used.

Source: *https://scitechdaily.com/math-riddle-from-the-1980s-finally-solved-could-be - used-to-improve-phones-and-computers/*

3.4 Two Mathematicians Solved A Century-Old Geometry Problem

Researchers have solved a century-old math problem using cutting-edge geometry. In 1911, the German mathematician Otto Toeplitz first posed the "square peg problem", in which he predicted that "any closed curve contains four points that can be connected to form a square". For more than a century, it remains unsolved. Now, two mathematician friends Prof. Joshua Greene, at the Department of Mathematics, Boston College, MA, USA and Prof. Andrew Lobb at the Okinawa Institute of Science and Technology in Japan, have used their quarantine time to crack a special case of a close offshoot of the age-old geometry problem, called "rectangular peg problem". They analyzed a set of loopy shapes called smooth, continuous curves to prove why every one of these shapes contains four points that form a rectangle of any desired "aspect ratio", meaning the ratio of their side lengths. Note that a square is a rectangle with aspect ratio 1:1. Smooth, continuous curves contrast with curves that are merely continuous, but not smooth is the type of a curve that features in Toeplitz's square peg conjecture. This type of a curve can have corner

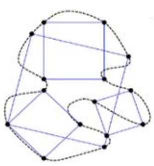


Figure-1

places where they veer suddenly in different directions. One prominent example of a curve with many corners is the fractal Koch snowflake, which in fact is made of nothing but corners. The Koch snowflake, and other curves like it, cannot be analyzed using calculus and related methods, a fact that makes them especially hard to study.

The first major progress on the rectangular peg problem was made in a proof from the late-1970s by Herbert Vaughan. The proof initiated a new way of thinking about the geometry of a rectangle and established methods that many mathematicians, including Greene and Lobb, later picked up. Instead of thinking of a rectangle as four connected points, Vaughan thought of it as two pairs of points that make line segments of equal length and also intersect at their midpoints. Vaughan also found that plotting all the pairs of points that make up a closed curve, when plotted indiscriminately with the x or y coordinates taken in any order makes the shape of a Möbius strip. Somehow, removing the $x - y$ context lets the pairs of coordinates arrange in exactly the right way. Vaughan used that fact to prove that every such curve contains at least four points that form a rectangle.

Like many other cutting-edge proofs, Greene and Lobb used newer developments in mathematics to shift their perspective and resituate the problem they were trying to solve.

One of the first big ingredients of their proof appeared in November 2019 when a Princeton graduate student named Cole Hugelmeyer posted a paper that introduced a new way of analyzing Vaughan's Möbius strip. Hugelmeyer's idea was to embed the Möbius strip in a four-dimensional space, where he could use features of four-dimensional geometry to prove the results he wanted about rectangles. To do this, he began with a given point on the Möbius strip and looked at the two points on the original closed curve it represented. Then he found the midpoint of that pair of points and determined its x and y coordinates, measured the distance d between the two points on the curve and calculated the angle ϕ formed where a line through the two original points meets the x-axis. Thus a point on the Möbius strip was uniquely identified with a point in the four-dimensional space by co-ordinates (x, y, d, ϕ).

To resolve the rectangle problem Hugelmeyer adopted a technique of turning the given curve into a Möbius strip and plotting it in a four-dimensional space. In the 4D plot, he "rotated" the Möbius strip so that the two coordinates encoding the midpoint between pairs of points remained the same, as did the coordinate encoding the distance between pairs of points. The rotation only changed the last coordinate, the one encoding information about the angle of the line segment between the pairs of points. The rotated Möbius strip was offset from the original, so the two copies intersected each other. Like a spinning top, he found that a core remained static and overlapped as the strip rotated. Where the rotated Möbius strips overlap, the corresponding pairs of points on the original curve formed the rectangle back in regular two-dimensional space. The Möbius strip can be rotated by any angle between 0 and 360 degrees, and he proved that one-third of those rotations yield an intersection between the original and the rotated copy. This fact turns out to be equivalent to saying that on a closed curve, you can find rectangles with one-third of all possible aspect ratios. But, again, it was a hook into the next step. With the knowledge of some portion of overlaps, Greene and Lobb extended Hugelmeyer's important finding into one more critical step.


Figure:2

They decided to consider a special shape called a Klein bottle(Figure-2), which is a charismatic example of a shape that overlaps itself in the right dimensionality of space. The two mathematicians found that while just one-third of curves conformed to Hugelmeyer's specific condition, it worked when they took the shape into a slightly different four-dimensional space: symplectic space, where the rules are slightly different. All the smooth, continuous curves can be represented as Klein bottles in this symplectic space, and only Klein bottles with overlaps can exist in this space by nature of the rules. From there, they concluded that every closed, smooth curve must contain sets of four points that can be joined together to form rectangles of all aspect ratios.

Something great about this solution is how researchers expand on and study each other's work over literal generations of scholars - and the way individuals with a hint of an idea can let it roll around for months or even years before the final piece falls into place.

Source:

1. *https://www.popularmechanics.com/science/math/a33002042/geometry-problem-solved/*
2. *www.quantamagazine.org new-geometric-perspective-cracks-old-problrm-about-rectangles*

3.5 New Mathematical Formula May Help 5G Wireless Networks Efficiently Share Communications Frequencies

Researchers at the National Institute of Standards and Technology (NIST) have developed a mathematical formula that, computer simulations suggest, could help 5G and other wireless networks select and share communications frequencies about 5,000 times more efficiently than trial-and-error methods. The novel formula is a form of machine learning that selects a wireless frequency range, known as a channel, based on prior experience in a specific network environment. The formula could be programmed into software on transmitters in many types of real-world networks. It is a way to help meet growing demand for wireless systems, including 5G, through the sharing of frequency ranges, also known as bands, that are unlicensed, Wi-Fi, for example. The NIST study focuses on a scenario in which Wi-Fi competes with cellular systems for specific frequencies, or subchannels. What makes this scenario challenging is that these cellular systems are raising their data-transmission rates by using a method called License Assisted Access (LAA), which combines both unlicensed and licensed bands.

This work explores the use of machine learning in making decisions about which frequency channel to transmit on. This could potentially make communications in the unlicensed bands much more efficient. It enables transmitters to rapidly select the best subchannels for successful and simultaneous operation of Wi-Fi and LAA networks in unlicensed bands. The transmitters learn to maximize the total network data rate without communicating with each other. The scheme rapidly achieves overall performance that is close to the result based on exhaustive trial-and-error channel searches.

The NIST formula produces results that are close to the optimum one but through a much simpler process. The study found that an exhaustive effort to identify the best solution would require about 45,600 trials, whereas the formula could select a similar solution by trying only 10 channels, just 0.02 percent of the effort.

The study addressed indoor scenarios, such as a building with multiple Wi-Fi access points and cellphone operations in unlicensed bands. Researchers now plan to model the method in larger-scale outdoor scenarios and conduct physical experiments to demonstrate the effect.

Source: *https://scitechdaily.com/ new-mathematical- formula-may- help-5g-wireless- networks - efficiently -share-communications-frequencies/*

3.6 The Mathematical Structure Of Particle Collisions Comes Into View

Physicists have identified an algebraic structure underlying the messy mathematics of particle collisions. When particle physicists try to model experiments, they meet an impossible calculation - an infinitely long equation that lies beyond the reach of modern mathematics. Fortunately, they can generate largely accurate predictions without seeing the hidden math all the way through. By cutting the calculation short, scientists at CERN's Large Hadron Collider in Europe make forecasts that match events they actually observe when they send subatomic particles barreling toward each other around a nearly 17-mile track. As measurements grow more precise, the predictions made by approximation schemes theorists may not be able to keep up.

But three recent papers from a group of physicists led by *Pierpaolo Mastrolia* of the University of Padua in Italy and *Sebastian Mizera* of the Institute for Advanced Study in Princeton, New Jersey, have revealed an underlying mathematical structure in the equations. The structure provides a new way of collapsing interminable terms into just dozens of essential components. Their method may help bring about new levels of predictive accuracy, which theorists desperately need if they are to move beyond the leading but incomplete model of particle physics.

Mastrolia and Mizera's work is rooted in a branch of pure mathematics called algebraic topology, which classifies shapes and spaces. Mathematicians pursue this classification with "cohomology" theories, which allow them to extract algebraic fingerprints from complicated geometric spaces. Feynman diagrams can be translated into geometric spaces that are open to analysis by cohomology. Each point within these spaces might represent one of a multitude of scenarios that could play out when two particles collide.

In 2017, Mizera was struggling to analyze how objects in string theory collide when he stumbled upon tools pioneered by Israel Gelfand and Kazuhiko Aomoto in the 1970s and 1980s as they worked with a type of cohomology called "twisted cohomology." Later that year Mizera met Mastrolia, who realized that these techniques could work for Feynman diagrams too. Last year they published three papers that used this cohomology theory to streamline calculations involving simple particle collisions.

Their method takes a family of related physical scenarios, represents it as a geometric space, and calculates the twisted cohomology of that space. This twisted cohomology has everything to say about the integrals. In particular, the twisted cohomology tells them how many master integrals to expect and what their weights should be. The weights emerge as values they call as "intersection numbers." In the end, thousands of integrals shrink to a weighted sum of dozens of master integrals.

The cohomology theories that produce these intersection numbers may do more than just ease a computational burden - they could also point to the physical significance of the most important quantities in the calculation.

If successful, the technique could help guide the next generation of theoretical predictions. And, a few researchers suspect, it may even foreshadow a new perspective on reality.

Source: *https://www.quantamagazine.org/new-particle-collision-math-may-offer- quantum - clues - 20200820/*

3.7 NeelakantabhanuPrakash, Mathematics Prodigy Emerges As 'World's Fastest Human Calculator'

NeelkanthaBhanu Prakash, a 20-year-old student from Delhi's St Stephen's College, has made history by becoming the first Indian and also the first Asian to win the gold medal in the 23-year-old history of the Mental Calculation World Championship at Mind Sports Olympiad on 15 Aug. 2020. Beating 29 competitors, up to 57 years of age, from 13 countries, he won the gold medal with a clear margin of 65 points. The judges were spellbound by his speed; they required him to perform more calculations to confirm the accuracy.

Hyderabad boy Bhanu Prakash holds four world records and 50 Limca records for being the fastest human calculator in the world. He says - "My brain calculates quicker than the speed of a calculator. Breaking these records, once held by Math maestros like Scott Flansburg and Shakuntala Devi, is a matter of national pride. I have done my bit to place India on the global level of mathematics."

Bhanu Prakash was praised by Vice President Venkaiah Naidu for winning the gold medal. "He has done India proud. My best wishes to him for all future endeavors," the Vice President's office tweeted with the hashtag 'Human Calculator'.

Mind Sports Olympiad is one of the most prestigious international competitions for games of mental skill and mind sports which is held in London every year. It was first held in 1998. This year, 30 participants from 13 countries including UK, Germany, UAE, France, Greece and Lebanon took part in the virtual Mind Sports Olympiad.

On his future plans, the Hyderabad boy said he wanted to eradicate math's phobia prevalent among students, mainly in the rural and underprivileged communities. His vision is to create "VISION Math" labs and reach out to millions of children to make them start loving math.

Source: *https://www.livemint.com/news/india/this-indian-mathematics-prodigy-holds-worlds-fastest-human-calculator-title-11598443812969.html*

3.8 C. R. Rao, a renowned Indian statistician, turns 100

Calyampudi Radhakrishna Rao or popularly C. R. Rao who shaped the dramatic growth of mathematical statistics in the 20th century, refining and restructuring it from its somewhat ad hoc origins, turned 100 on 10 Sept. 2020. He continues to guide and inspire students and practitioners of statistics, as he has for the past eight decades. C. R. Rao was born in the small town of Huvina Hadagali in the present-day Karnataka.

After completing a BA (Hons.) degree, from Andhra University, with distinction and first rank, he hoped to get a scholarship to pursue higher studies in mathematics, but this did not materialize. The Second World War had commenced by that time, and he decided to apply for a job in the survey unit of the Army and went to Calcutta for this purpose. While he did not qualify for the job, coincidentally, while in Calcutta he got introduced, through a friend, to the emerging Indian Statistical Institute (ISI). The young C. R. Rao joined a training programme offered by the ISI.

In 1943, C. R. Rao completed his MA in Statistics from Calcutta University, again with a first rank. By the age of 22, he had already published seven technical papers in mathematical statistics. Mahalanobis recognized the potential of this young man, and C. R. Rao was deputed to Cambridge University in 1946 to assist in the analysis of some anthropological data. While he was at Cambridge, he registered for a doctoral degree under the guidance of the celebrated Professor Ronald A. Fisher. After completing Ph. D. he returned to the ISI in 1948, and was made professor at the Institute the following year.

Prof. Rao spent more than three-and-a-half decades at the ISI, serving the institution in various capacities: as the Head and Director of the Research and Training School of the ISI, as the Jawaharlal Nehru Professor of the Institute and so on. In 1972, he succeeded Mahalanobis as the Director. Prof. Rao thus played a key role in building the worldwide reputation of the ISI.

He had been an exemplary teacher and has supervised 50 students in their doctoral research. He was mentor to many outstanding mathematical statisticians and mathematicians. The late Debabrata Basu (Professor at ISI and subsequently at Florida State University), S. R. Srinivasa Varadhan (Professor at the Courant Institute of Mathematical Sciences in New York and winner of the Abel Prize in 2007), R. Ranga Rao, K. R. Parthasarathy and the late V. S. Varadarajan are some among the many distinguished academics mentored by Professor Rao. In a speech at the institute, Professor Fisher once pointed out that more than half of the qualified statisticians working in the world were Indians, for quite some time. Most of them were Prof. Rao's students.

On his retirement from the ISI in 1979, he joined the University of Pittsburgh as Professor and later shifted to Pennsylvania State University as the Eberly Family Chair Professor of Statistics. He served as the Director of the Center for Multivariate Analysis at Penn State until 2008. He published over 270 journal articles since his retirement from the Indian Statistical Institute.

As India started building its institutional infrastructure after Independence, Prof. Rao was instrumental in setting up our national statistical framework. During the 1960s, he served as Chairman of the Committee on Statistics of the Government of India. He also chaired the Committee on Mathematics of the Atomic Energy Commission and was a member of the Committee on Science and Technology. He was the moving force behind the International Statistical Educational Centre at the ISI, where students and government statisticians from developing countries could learn the techniques of statistics and the methods for establishing their national statistical bureaus.

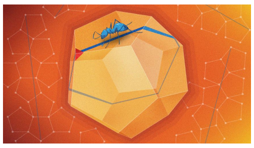
It is scarcely possible to describe the contours of Professor Rao's research output in an article such as this one. Some of his outstanding contributions include the well-known Cramér-Rao lower bound and the famous Rao-Blackwell theorem which appeared in a celebrated paper that Professor Rao published in 1945 - when he was only 24. In 1947 he introduced the statistical test used extensively in econometrics, widely known as "Lagrange multiplier test" (on account of use of Lagrange multipliers). He is also recognized for "Rao's Score Test", which has certain practical advantages over the two other commonly used tests for the same purpose - the Wald test and the likelihood-ratio test. Other than these, the orthogonal arrays, quadratic entropy are also among his main contributions.

He has been the recipient of countless awards and accolades. He was honoured with the Padma Vibhushan in 2001. He received the Shanti Swarup Bhatnagar Award from Prime Minister Nehru in 1963. The then President of the United States, George W. Bush, conferred the National Medal of Science on him in 2002. The American Statistical Association awarded him the Samuel S Wilks Memorial Award in 1989. In 2011, he became the first non-European and non-American to receive the Guy Medal in Gold from the Royal Statistical Society of the United Kingdom. Prof. Rao has been bestowed with 38 honorary doctoral degrees from universities located in 19 different countries. The inspiration he provides should, and indeed will, live for generations to come.

Source: *https://www.thehindu.com/ opinion/ open-page /a- doyen -of- statistics- turns -100 / article32563522. ece*

3.9 Mathematicians report new discovery about the Dodecahedron

Mathematicians have spent over 2,000 years dissecting the structure of the five Platonic solids - the tetrahedron, cube, octahedron, icosahedron and dodecahedron - there's still a lot we don't know about them. Recently three mathematicians have resolved one of the most fundamental questions about straight paths on the 12-sided Platonic solid - the dodecahedron.



Suppose you stand at one of the corners of a Platonic solid. Is there some straight path you could take that would eventually return you to your starting point without passing through any of the other corners? For the four Platonic solids built out of squares or equilateral triangles - the cube, tetrahedron, octahedron and icosahedron - mathematicians recently figured out that the answer is no. Any straight path starting from a corner will either hit another corner or wind around forever without returning home. But with the dodecahedron, which is formed from 12 pentagons, mathematicians didn't know what to expect. Now Jayadev Athreya, David Aulicino and Patrick Hooper have shown that an infinite number of such paths do in fact exist on the dodecahedron and these paths can be divided into 31 natural families. The solution required modern techniques and computer algorithms.

Mathematicians have indeed speculated about straight paths on the dodecahedron for over a century, but there has been a resurgence of interest in the subject in the recent years following gains in understanding what are known as "translation surfaces. "They have proved useful in studying a wide range of topics involving straight paths on shapes with corners, from billiard table trajectories to the question of when a single light can illuminate an entire mirrored room. The new result shows that even objects that have been studied for thousands of years can still hold secrets.

Source: *https://www.quantamagazine.org/ mathematicians-report- new- discovery - about- the - dodecahedron - 20200831/*

3.10 Obituary

3.10.1 *Ronald Graham, who unlocked the magic of numbers, dies at the age of 84*

Ronald L. Graham, who gained fame with wide-ranging theorems in discrete mathematics that have found uses in diverse areas, ranging from making telephone and computer networks more efficient to explaining the dynamics of juggling, died on 6 July 2020 at the age of 84.

Dr. Graham was one of the most productive American mathematicians of the past half century. Working in both applied and pure mathematics, often on the foundations of theoretical computer science and telecommunications, he spent two-thirds of his long career in industry, at Bell Labs in New Jersey, while the rest was at the University of California in San Diego. He carried out pioneering research in discrete mathematics, particularly in an area known as combinatorics, specifically in Ramsey theory, quasi-randomness, scheduling theory, and discrete and computational geometry, as well as in recreational mathematics and mathematical magic. He was also an engaging popularizer of mathematics through extensive expository talks and writings.

However, his biggest fame was due to *Graham's Number*, the largest specific positive whole number used in a mathematical proof, a fact acknowledged by the Guinness Book of Records in 1980. His fellow mathematician Martin Gardner brought it to the attention of a wide audience in a 1977 column for Scientific American. The number came out of a problem known as the Ramsey theory, which states that in large systems there can never be complete disorder, that pockets of structure will appear within the apparent chaos. Dr. Ronald was looking at cubes in which the lines between the corners were colored red or blue. In a three-dimensional cube, it is easy to color the lines so that no planar slice of the cube with four vertices has edges all of one color. But mathematicians can also imagine cubes in four dimensions and greater, and so Dr. Ronald wanted to know whether this property of being able to avoid slices of one color would persist in greater dimensions. The answer turns out to be in the negative, and while no one knows in precisely what dimension this unavoidability would kick in, Dr. Ronald calculated an upper bound for the answer - a number so huge that there is not enough space in the entire universe in which to write all of the digits.

Although he never officially completed high school, Ronald received a Ford Foundation scholarship to attend the University of Chicago at the age of 15 for three years at the start of the 1950s. When his scholarship ran out, he transferred to the University of California, Berkeley, where he majored in electrical engineering and studied number theory. He received his bachelor's degree in 1958 in physics, because the university was not accredited to award degrees in mathematics. In 1962, he received Ph. D. degree in number theory. Then he joined Bell Labs and solved problems that proved helpful for a telephone company. Over a period of 37 years, he rose to be chief scientist there, directing mathematical research during an era that saw the field transform how telecommunications was done.

Outside of his work, Dr. Graham supported Paul Erdös, the most prolific mathematician in history, who roamed the world without the hindrance of a proper job. Over time he effectively became his travel agent and banker, as well as a close research collaborator; they wrote around 30 papers together.

In all he published about 400 papers, 100 of them with his fourth wife and frequent collaborator Fan Chung, an emeritus mathematician at the University of California, San Diego, with whom he developed the idea of quasi-random graphs, which applied numerical preciseness in describing the random-like structure of networks and also wrote with her an influential book *Erdös on Graphs: His Legacy of Unsolved Problems* (1999).

Having been elected president of the International Jugglers' Association in 1972, over the years he also published a number of papers about the mathematics of juggling. At one point Dr. Graham and three other juggling mathematicians proved an equation for the number of possible ball-juggling patterns before a pattern repeats. After Erdös died in 1996, Ronald continued to administer cash prizes that Erdös had offered for particularly challenging math problems. He also mentored masses of aspiring students in math and computer science, and many in juggling and magic.

Dr. Graham was elected to the National Academy of Sciences of the United States in 1985, and was President of both the American Mathematical Society (1993-94) and the Mathematical

Association of America (2003-04). He was inducted as a fellow of the Association of Computing Machinery in 1999 "for seminal contributions to the analysis of algorithms". He received the George Pólya prize in 1971, the Euler medal in 1993 and the AMS Leroy Steele prize for lifetime achievement in 2003.

Source:

1. *https://www.nytimes.com/2020/07/23/science/ronald-l-graham-who-unlocked-the- magic -of-numbers-dies-at-84.html*
2. *https://www.msn.com/en-gb/news/world/ron-graham-obituary/ar-BB17vLpH*

3.10.2 Nathaniel Friedman, leader in ergodic theory and dynamical systems, died at the age of 82

Nathaniel A. Friedman, a mathematician, educator, and pioneer in the international art and math movement died on 2nd May 2020 at the age of 82, from complications due to COVID-19.

He received both B.A. and M.A. degrees in mathematics from Michigan and then attended Brown University where he received a Doctorate in 1964. After appointments at the University of New Mexico, and Westfield College, University of London, he settled into a tenure-track position at the State University of New York at Albany in 1968. In 1970, Prof. Friedman wrote one of the early textbooks on ergodic theory - "An Introduction to Ergodic Theory". Ergodic theory is a field of pure mathematics that studies the statistics of systems that change over time. He laid a foundation in ergodic theory and dynamical systems that continues to have a broad influence on many areas of mathematics to this day.

In 1992, Prof. Friedman started the international and interdisciplinary Art and Mathematics Conference, which convened annually. In 1998, he founded ISAMA (The International Society for the Arts, Mathematics, and Architecture) to further interdisciplinary education relating the arts, mathematics and architecture, with international conferences in the USA and Europe. He retired as full professor in 2000 and focused his energy on the art/math community and his own artistic practice. He explored ways of seeing and hyperseeing through different mathematical forms and concepts including knots, minimal surfaces, Möbius strips, fractals, and chaos. Prof. Friedman left behind a far-reaching legacy and long lasting impact with hundreds of mathematicians and artists.

Source: *https://www.legacy.com/obituaries/timesunion-albany/obituary.aspx?n=nathaniel- a - friedman&pid=196299089*

3.10.3 Field Medalist Mathematician Sir Vaughan Jones Passes Away At the Age of 67

Sir Vaughan F. R. Jones, one of the world's foremost mathematicians and a celebrated professor in Vanderbilt University's College of Arts and Science died suddenly on 6 Sept. 2020 at the age of 67 due to complications resulting from a recent severe ear infection. He was a 1990 recipient of the Fields Medal, widely regarded as the "Nobel Prize of Mathematics", among many other honors.

Sir Jones, who was born on 31 Dec. 1952, grew up in Auckland, New Zealand. He earned a bachelor of science and a master of science with first class honors from the University of Auckland. He then received a Swiss government scholarship and completed his doctorate in 1979 from the University of Geneva.

In the 1980s he worked at the University of California, Los Angeles, and the University of Pennsylvania, before being appointed Professor of Mathematics at the University of California, Berkeley. From 2011 he held the position of Stevenson Distinguished Professor of Mathematics at Vanderbilt University but remained Professor Emeritus at Berkeley.

He was a mathematician of international standing and for many years he was the only Fields Medalist from the Southern Hemisphere. He spent his career in the United States, but gave his time to the university and to New Zealand mathematics, offering courses and lectures each summer to encourage and mentor younger mathematicians. He co-founded and led the New Zealand Mathematics Research Institute (NZMRI) to promote and foster high quality mathematics.

Jones' areas of expertise included von Neumann algebra theory, subfactors and planar algebras, mathematical physics, low-dimensional topology - specifically knot theory and more. During the mid-1980s, while Jones was working on a problem in von Neumann algebra theory, which is related to the foundations of quantum mechanics, he discovered an unexpected link between that theory and knot theory, a mathematical field dating back to the 19^{th} century. His most celebrated work was on knot polynomials.

Specifically, he found a new mathematical expression - now known as the Jones polynomial - for distinguishing between different types of knots as well as links in three-dimensional space. His discoveries had real significance in an entirely different field. It had application in biological science, enabling scientists to identify whether two different types of RNA are from the same source. Jones' discovery had been missed by topologists during the previous 60 years, and this finding contributed to his being selected for the Fields Medal.

The Royal Society's Te Apārangi Jones Medal, awarded for outstanding achievements in the mathematical sciences, is named after him.

Source:

1. *https://www.odt.co.nz/news/national/celebrated-nz-mathematician-sir-vaughan-jones-dies*
2. *https://news.vanderbilt. edu/ 2020/09/09/vaughan-jones- preeminent- vanderbilt - mathematician -has -died/*

□□□

4. Tributes to Prof. C. S. Seshadri

We are really shocked to learn about the sad demise of Prof. C. S. Seshadri, a Leading Mathematician from India, on July 17^{th}, 2020 at the age of 88. Prof. Seshadri has inspired, guided and encouraged many people during his long tenure at TIFR and at his own established Institute CMI, in Chennai. Here are the tributes spontaneously offered by our chief editor Prof. Ravi Kulkarni and TMC Vice-President Prof. Jugal Verma.

4.1 Tributes by Prof. Ravi Kulkarni, Chief Editor, TMC Bulletin

C. S. Seshadri's passing away on July 17, 2020, is a very sad news for Indian Mathematics. On many occasions, all of us sought and received his wise advice, on mathematical or administrative issues.

I knew he had slowed down, but had not heard of any long illness. I came to know that in recent years he was suffering from parkinson's disease. Still, he passed away without pain. His wife, Sundari, died last year, on October 31, 2019. Seshadri had a long, loving, supportive family life. Sundari was an accomplished singer of Karnataka music. Seshadri himself was close to being a pro. My condolences to his family, and brother Rajan, who is also a mathematician, working at TIFR.

My association with Seshadri was from my student days at Harvard when around 1964 he came for an extended visit, as a young established mathematician from India. I think Mumford had invited him. I remember Tate asking me to read his paper on freeness of projective modules over a polynomial ring in 2 variables, which I did. Those were the days, when India was just picking up on the world mathematical scene. Ramanujan was an inspiring but rather solitary peak. We had started hearing the name of Harish Chandra. At Harvard, I also heard about and met Abhyankar.

But the rest of the vast landscape had little development. It was exhilarating for me to see in Seshadri, an Indian mathematician getting international recognition at a very young age.

After I finished my doctorate, I went to see him in TIFR around 1968, and then several times, in TIFR, in the Institute of Mathematical Sciences, Chennai, and later in the Chennai Mathematical Institute (CMI), which was a brainchild of Seshadri. CMI is a deemed research university patterned after the European, American model, in which an undergraduate student could meet a graduate, and postgraduate student, and attend courses and research seminars. It combines the advantages, and avoids the disadvantages, of being just a university, or just a research institute. It is better than a large university, for CMI has a small student body of 100-125 students, and 40-50 faculty, exclusively in the areas of Mathematics, Physics, and Computer Science. Every year, it invites many eminent mathematicians as visitors, which makes for a very lively research atmosphere. Developing such an institute is a tribute to Seshadri's vision, and stature in the mathematical world.

I remember a delicious dinner at his place when I went to see him in Princeton, around 1970, when he was visiting the Institute for Advanced Study. I remember a personal compliment, very rare in my life, that Sundari liked the dress I was wearing on that day, and recommended Seshadri that he should dress well too!

At his recommendation, E. C. G. Sudarshan offered me a faculty position in 1985 at IMSc. Although our active mathematical interests had a rather small intersection, we understood well what each of us was doing. He had an elder-brotherly affection and interest in my mathematical development. He graciously accepted my invitations on two occasions, when around 1983, at Indiana University (Bloomington), I had invited him for a week-long series of lectures on his Monomial Theory, which has attracted many contributions from India– and then again in December 2007 to China, when Yau had asked me to bring some Indian Mathematicians for lectures at his International Congress of Chinese Mathematicians. I remember many leisurely, insightful, mathematical conversations with him in a very pleasant atmosphere. . . . I have many sweet memories about my association with him. Perhaps the last but one was his lecture at Bhaskaracharya Pratishthan in Pune in October 2008, when he was given the prestigious Firodia award. The last one was when I went to see him at CMI about 5 years ago.

We also served together on a DAE Committee.

What impressed everybody, was that despite his enormous stature in the international mathematical world, he was always "on the ground", humble, open, gentle, and egoless, – a Brahmarshi, without a beard! More than once he mentioned to me that even his FRS, was just a useful "brand-name", which allowed him to do a few exceptional things, like establishing CMI, for Higher Mathematics in India.

I think, he had already attained Bhagvad Geeta's ideal of Jeevan-Mukti. Yet, I share the grief with his family and many friends. India has indeed lost a major leader-figure in Mathematics.

4.2 Tributes by Prof. Jugal Verma, Vice-President, TMC

We have lost a mathematician who had many other talents such as an accomplished classical music singer and a top administrator.

His mathematics was of the highest quality. It combined geometric insight with mastery over whatever mathematical tools were needed to solve problems of contemporary interest. He could connect very easily to people of all ages. He was eager to discuss mathematics anytime with anyone. He remained active in research and music until the end.

He built an Institute in a PPP model that was unheard of in those days. His aim was to attract the best undergraduate students to take up careers in academics. For many decades education at CMI was almost free. It attracted top mathematicians from all over the world. Whenever I met him at first at the SPIC Science Foundation Institute in T-Nagar and later at the Chennai Mathematical Institute, he would welcome me with a big smile and talk about whatever mathematics I was pursuing. Later most of my students went on to do Post-doctoral work at CMI.

Whenever I gave a seminar at CMI, he had the first question. He always sat in the first row and asked very penetrating questions which often led to further research. I met him in the last decade as a member of the Academic Council of CMI. His philosophy of running an institute was very simple; attract the best faculty, best students, and offer very demanding courses. Students of B. Sc. at CMI usually know whatever we teach at the M. Sc. level. No wonder so many of them have gone to the best graduate schools in Europe and USA.

Seshadri solved Serre's conjecture in dimension 2 which was a major problem in commutative algebra. He was still a graduate student at that time (1958). This work inspired several generations of mathematicians to work on projective modules and affine geometry. But Seshadri's real love was algebraic geometry a subject which is much more demanding but well connected with all the fields in mathematics. While he was at TIFR, the school of mathematics at TIFR became a leading centre of research in algebraic geometry and related fields. Seshadri constants are an active area of research in algebraic geometry.

He was a recipient of the Bhatnagar award, FRS, Padma Bhusan, TWAX, fellowship of the AMS and a fellowship of the US National academy of sciences.

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5. Problem Corner

Udayan Prajapati
Head, Mathematics Department, St. Xavier's College, Ahmedabad
Email: udayan.prajapati@gmail.com

In the earlier issues we posed some challenging problems from Geometry, Number Theory, Combinatorics and functional equation. However, so far we have received a Solution only for a problem from Number theory form one of our readers. We have received two incomplete solutions of two problems of Comnbinatorics and we are yet to get the complete solution. Now, Mr. Dhruv Bhasin, a student of Integrated Ph. D. program at IISER, Pune, has provided a solution to the problem based on functional equation proposed in the last issue of our bulletin. We congratulate Dhruv for the problem solving skills demonstrated by him.

In this issue, we present a solution of the problem posed in the last issue and pose a problem form combinatorics for our readers. Readers are invited to email their solutions to Udayan Prajapati (udayan64@yahoo.com), Coordinator, Problem Corner before 10^{th} December, 2020. Most innovative solution will be published in the subsequent issue of the bulletin.

1. **Problem:** Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(f(x)) = f(x) - 2x$.

   Solution: (Solution by Dhruv Bhasin, a student of Integrated Ph. D. program at IISER, Pune). Let us begin by noting that $f(f(0)) = f(0)$ which means that $f(f(f(0))) = f(f(0))$. Now, putting $x = f(0)$, we get that $f(f(f(0))) = f(f(0)) - 2f(0)$. This gives $2f(0) = 0$. So, $f(0) = 0$.

   Now, note that $f(x) = f(y) \Rightarrow f(f(x)) = f(f(y)) \Rightarrow f(x) - 2x = f(y) - 2y \Rightarrow 2x = 2y$ which, in turn, gives that $x = y$. Therefore, f is one-one. But it is given that f is continuous and we know that all continuous, one-one functions $f : \mathbb{R} \to \mathbb{R}$ are strictly monotone (immediately follows from intermediate value theorem). Now, we have two cases:

   Case1: f is strictly increasing. It means that $f(1) > 0$. Now, $f(1) \geq 1 \Rightarrow f(f(1)) \geq f(1) \Rightarrow f(1) - 2 \geq f(1)$ which means that $0 \geq 2$ which is a contradiction. On the other hand, if $0 < f(1) < 1$, we obtain that $0 < f(f(1)) < f(1) \Rightarrow 0 < f(1) - 2 < f(1)$ but $0 < f(1) < 1$ clearly ensures that this does not happen. So this case is not possible.

   Case 2: f is strictly decreasing. It means that $f(1) < 0$. But this gives that $f(f(1)) > 0$, so, $f(1) - 2 > 0$ which gives $f(1) > 2 > 0$ which is a contraction.

Therefore, no such functions f exist. As a side remark, we would like to note that no particular role was played by "2". It could as well be replaced by any positive number a (and replacing 1 by $a/2$ in the proof). Therefore, no continuous functions f exist such that $f(f(x)) = f(x) - ax$ where $a > 0$. It would be interesting to study the problem for $a < 0$.

Problem for this issue

For a positive integer n, a permutation $P = (p_1, p_2, \ldots, p_{n-1}, p_n)$ of $(1, 2, 3, \ldots, n-1, n)$ is said to have a peak at k^{th} position $(1 < k < n)$ if $p_{k-1} < p_k$ and $p_k > p_{k+1}$. P is said to have a bottom at l^{th} position $(1 < l < n)$ if $p_{l-1} > p_l$ and $p_l < p_{l+1}$. For example the permutation (1, 2, 7, 5, 4, 3, 6) of (1, 2, 3, 4, 5, 6, 7) is said to have exactly one peak at 3^{rd} position and exactly one bottom at 6^{th} position. Find the number of permutations of $(1, 2, 3, \ldots, n-1, n)$ having exactly one peak and one bottom.

6. Feedback on July-2020 Issue

One of our advisors, Prof. Michel Waldschmidt, from Institut de Mathématiques de Jussieu, Sorbonne University, Paris, France, brought to our notice that the Photograph of Prof. Adrien-Marie Legendre on the back cover of July 2020 issue is not the right photograph and referred to the paper by Peter Duren: Changing Faces: The Mistaken Portrait of Legendre, Notices of AMS, Vol. 56, No. 11, December 2009 [*http://www.ams.org/notices/200911/rtx091101440p.pdf*].

In the paper author writes: "For over a century one familiar portrait (as appeared on the back cover of the July 2020 issue of the TMC Bulletin) has been displayed wherever Legendre's work has been discussed in historical writing. But the portrait has nothing to do with Adrien Marie Legendre. Instead it portrays a politician named Louis Legendre, an active participant in the French Revolution, no relation to the mathematician.

Yet no one seemed to notice the anomaly until the advent of the computer age, when powerful search engines transformed the art of information retrieval. Sometime during the year 2005, two students at the University of Strasbourg were astonished to find a single portrait for two different men. Their discovery was posted on the mathematics department website les-mathematiques.u-strasbg.fr. The error was then confirmed and actively discussed by bloggers in France.

This caricature by J-L Boilly is the only known portrait of Adrien-Marie Legendre.

After the error was confirmed, the first task was to determine the identity of the shared portrait. Was it a likeness of Adrien-Marie or Louis Legendre, or neither? A mystery to be solved. The website of Jean-Bernard François [http://infofiltrage.com⟶Internet ⟶Adrien-Marie LE GENDRE] was central to that investigation and still shows a record of the chain of discoveries which identified the portrait as that of Louis Legendre. In particular, it shows that bloggers traced the portrait to its source in a book [2] of lithographs published in 1833.

Once the traditional portrait was known to be false, a feverish search began for a true portrait of Adrien-Marie Legendre. Miraculously, an authentic portrait was discovered during the year 2008 in the library of the Institut de France in Paris, among a rare collection of seventy-three caricatures of members of the Institute [Album de 73 portraits-charge aquarelles des membres de l'Institut (1820) par Julien-Léopold Boilly, Manuscrit 7749, Bibliothéque de l'Institut de France, Paris]. One of the watercolor sketches shows the heads of Legendre and Fourier, with bodies lightly sketched in pencil. Their names "Legendre" and "Fourier" are written below the sketch. Fourier is easily recognized from existing portraits, but Legendre takes on a totally new appearance (as shown in the above figure). This is the only image of Adrien-Marie Legendre known to exist."

In the paper the author further narrates the sequence of events that brought the true portrait to light.

We, therefore, request our readers to note the correct portrait of Adrien-Marie Legendre. We are really grateful to Prof. Michel Waldschmidt, for bringing this important point to our notice.

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7. International Calendar of Mathematics events

Ramesh Kasilingam

Ramesh Kasilingam, Department of Mathematics, IITM, Chennai; Email: rameshk@iitm.ac.in

December 2020

1. December 3-4, 2020, Integrable Systems Workshop 2020, The University of Sydney, Sydney, NSW 2006. *https://wp.maths.usyd.edu.au/igs/workshops/integrable-systems-2020/*
2. December 7-11, 2020, 26th International Domain Decomposition Conference 07, Hong Kong-online, China. *https://www.math.cuhk.edu.hk/conference/dd26/?Conference-Home*
3. December 7-11, 2020, SanDAL Winter School on Mathematical Statistics, University of Luxembourg, Belval Campus, Maison du Savoir 2, avenue de l'Université, L-4365 Esch-sur-Alzette LUXEMBOURG. *https://sandal.uni.lu/winter-school*

January 2021

1. January 11-15, 20201, AIM Workshop: Arithmetic intersection theory on Shimura varieties, 2021 American Institute of Mathematics, San Jose, CA. *https://aimath.org/workshops/upcoming / intersectshimura*
2. January 25-29, 2021, Actions of Tensor Categories on C*-algebras, Institute for Pure and applied mathematics. *http://www.ipam.ucla.edu/programs/workshops/actions-of-tensor-categories-on-c-algebras/*

February 2021

1. February 1-May 7, Combinatorial Algebraic Geometry, Providence, RI, USA, *https:// icerm .brown .edu/programs/sp-s21/*
2. February 15-19, 2021, Sage/Oscar Days for Combinatorial Algebraic Geometry, ICERM, Providence, RI. *https://icerm.brown.edu/programs/sp-s21/w2/*
3. February 22-26, 2021, Deep Learning and Combinatorial Optimization, Institute of Pure and Applied Mathematics (IPAM), UCLA. *https://www.ipam.ucla.edu/dlc2021*

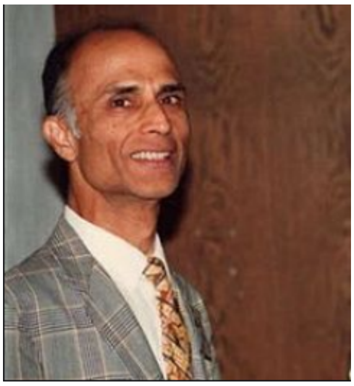
TMC Distinguished Lecture Series

The Mathematics Consortium (India) (TMC) has recently launched the Distinguished Lecture Series (DLS), which will feature virtual colloquia by some of the best researchers and expositors around the world. This activity is co-hosted by IIT Bombay and ICTS-TIFR, Bengaluru. For more details and to register for participation in TMC DLS, please visit the following webpage:
https://sites.google.com/view/distinguishedlectureseries

A Special Tribute to

A great Indian Mathematician

Harish-Chandra

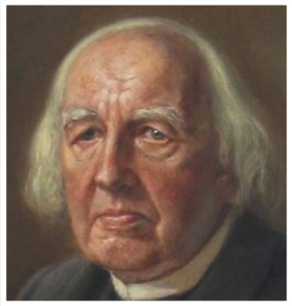


(11 October 1923 – 16 October 1983)

- **Harish-Chandra,** arguably the greatest Mathematician of Indian Origin after Ramanujan, was born on 11 October 1923 in Kanpur, in the state of Uttar Pradesh in India. He did his U.G. studies in Kanpur and Master's degree in Physics from University of Allahabad, at Prayagraj. In 1943 he joined IISc, Bangalore, for pursuing research in Theoretical Physics, and worked with Dr. Homi Bhabha. In 1945, Bhabha arranged for Harish-Chandra to go to University of Cambridge to work as a research student of Prof. Paul Dirac. After completing Ph. D. in 1947 Harish-Chandra visited Princeton along with Dirac for a year.
- Soon after words he got greatly influenced by the famous French Mathematician Claude Chevalley, and as a result he shifted from Physics to Mathematics. He did profound work in representation theory and harmonic analysis on Semisimple Lie Groups, for which he became world renowned.
- In 1948 he joined the Columbia University. In 1963 he was invited to be a permanent member of Institute of Advanced Study, Princeton and in 1968 he was appointed by IBM as Von Neumann Professor.
- After Ramanujan he was the next Indian mathematician to be elected a Fellow of the Royal Society, London, in 1973. Prior to this, Harish-Chandra had received the Cole Prize of the American Mathematical Society in 1954. It is believed that in 1958 he was also close to receiving the Fields Medal, but missed it narrowly. In 1973 the University of Delhi conferred on him an Honorary Doctorate, and in 1974 he was-honoured by the Indian National Science Academy (INSA) with the Srinivasa Ramanujan Medal. In 1977 he was decorated with the national award Padma Bhushan. He was granted membership of (United States) National Academy of Sciences in 1981 and in the same year he received an honorary doctorate from Yale University. He was also elected Fellow of the Indian Academy of Sciences and the Indian National Science Academy.
- He passed away on 16th October 1983, at Princeton, following a massive heart attack soon after participating in a conference.
- The Harish-Chandra Research Institute at Allahabad (Prayagraj) was (re)named after him, on 11th October 2000.

Robert Langlands once remarked that some parts of Harish-Chandra's work are so advanced that even Mathematicians of very high caliber found it difficult to follow.

I have often pondered over the roles of knowledge or experience, on the one hand, and imagination or intuition, on the other, in the process of discovery. I believe that there is a certain fundamental conflict between the two, and knowledge, by advocating caution, tends to inhibit the flight of imagination. – Harish Chandra



Karl Weierstrass (31 Oct. 1815 - 19 Feb. 1897)

A German mathematician.
Known as the father of modern analysis, Weierstrass devised tests for the convergence of series and contributed to the theory of periodic functions, functions of real variables, elliptic functions, Abelian functions, converging infinite products, and the calculus of variations.

Jean le Rond d'Alembert (16 Nov. 1717 - 29 Oct. 1783)

A French mathematician, mechanician, physicist, philosopher.
Best known for his principle in Mechanics, his solutions to the wave equation, the Fundamental Theorem of Algebra, his ratio test and his work on Philosophy and Music Theory. He wrote over 1000 articles in Diderot's Encyclopédie.

Johannes Kepler (27 Dec. 1571 - 15 Nov 1630)

A German mathematician, physicist, astronomer, philosopher.
Best known for his laws of planetary motion, and his books Astronomia nova, Harmonices Mundi, and Epitome Astronomies Copernicanae. He did fundamental work in the field of Optics, invented an improved version of the refracting (Keplerian) telescope.

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Email: tmathconsort@gmail.com   Website: themathconsortium.in

Contact Persons

Prof. Vijay Pathak   Prof. S. A. Katre
vdpmsu@gmail.com (9426324267)   sakatre@gmail.com (9890001215)

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